SOME GEOMETRIC PROPERTIES in ψ-DIRECT SUMS OF BANACH SPACES $X \oplus_\psi Y$

Takayuki Tamura

Graduate School of Social Sciences and Humanities
Chiba University
Chiba 263-8522, Japan
e-mail: tamura@le.chiba-u.ac.jp

1. Introduction and preliminaries

A norm on $\mathbb{C}^2$ is said to be absolute if $\|(z,w)\| = \|(|z|,|w|)\|$ and normalized if $\|(1,0)\| = \|(0,1)\| = 1$. In [2], they proved that the set N_a of all absolute normalized norms on $\mathbb{C}^2$ corresponds to the family Ψ of all convex functions on [0, 1] satisfying that $\psi(0) = \psi(1) = 1$ and $\max\{1-t, t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$ in one-to-one fashion under the following equations

$$\psi(t) = \|(1-t,t)\| \quad (0 \le t \le 1) \tag{1}$$

and

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{2}$$

for every $(z,w) \in \mathbb{C}^2$.

In [16], by using these convex functions , they introduced the ψ-direct sum $X \oplus_\psi Y$ of Banach spaces X and Y as their direct sum $X \oplus Y$ equipped with the norm

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\frac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \neq (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{3}$$

This is the generalized notion of ℓ_p-sum $X \oplus_p Y$ and this notion gives us many examples of (non ℓ_p-type) norms on $X \oplus Y$.

In this papar, we shall deal with the necessary and sufficient conditions cocerning the convexity and non-squareness. of $X \oplus_\psi Y$, which described with these properties of X and Y and with the property of the convex function ψ.

Banach space X is said to be *strictly convex* provided, if $\|x\| = \|y\| = 1$, $x \neq y$, then $\|(x+y)/2\| < 1$. X is called *uniformly convex* if any $\epsilon > 0$ there is a δ $(0 < \delta < 1)$ such that, whenever $\|x-y\| \geq \epsilon$, $\|x\| = \|y\| = 1$, one has $\|(x+y)/2\| < 1-\delta$. X is called *uniformly non-square* ([7]; cf. [1, 11]) if there exists a $\delta\,(0<\delta<1)$ such that, whenever $\|(x-y)/2\| > 1-\delta$, $\|x\|, \|y\| \leq 1$, one has $\|(x+y)/2\| \leq 1-\delta$. A function ψ on $[0,1]$ is called *strictly convex* if, for any s, $t \in [0,1]$, $s \neq t$, and for any c $(0 < c < 1)$, one has $\psi((1-c)s+ct) < (1-c)\psi(s) + c\psi(t)$.

2. Monotonicity property of absolute norms

we shall state the following monotonicity properties of absolute norms on $\mathbb{C}^2$, as they play crucial roles to prove the main theorems.

LEMMA 1 ([2, p.36, Lemma 2]). *Let* $\psi \in \Psi$.
(i) *If* $|p| \leq |r|$ *and* $|q| \leq |s|$, *then* $\|(p,q)\|_\psi \leq \|(r,s)\|_\psi$.
(ii) *If* $|p| < |r|$ *and* $|q| < |s|$, *then* $\|(p,q)\|_\psi < \|(r,s)\|_\psi$.

PROPOSITION 1 (Takahashi, Kato and Saito [16]). *Let* $\psi \in \Psi$. *Then the following assertions are equivalent*:
(i) *If* $|z| \leq |u|$ *and* $|w| < |v|$, *or* $|z| < |u|$ *and* $|w| \leq |v|$, *then* $\|(z,w)\|_\psi < \|(u,v)\|_\psi$.
(ii) $\psi(t) > \psi_\infty(t)$ *for all* $t \in (0,1)$.
In particular, if ψ *is strictly convex, the assertion* (i) *holds true.*

PROPOSITION 2. ([10]*Let* $\psi \in \Psi$ *and let* (z,w), $(u,v) \in \mathbb{C}^2$.
(i) *Let* $|z| < |u|$ *and* $|w| = |v|$. *Then* $\|(z,w)\|_\psi = \|(u,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |w|$.
(ii) *Let* $|z| = |u|$ *and* $|w| < |v|$. *Then* $\|(z,w)\|_\psi = \|(z,v)\|_\psi$ *if and only if* $\|(z,w)\|_\psi = |z|$.

3. Strict convexity, uniform convexity and uniform non-squareness of $X \oplus_\psi Y$

We need the following lemma to prove the main theorems.

LEMMA 2. *Let $\{x_n\}$ and $\{y_n\}$ be sequences in a Banach space X such that $\{\|x_n\|\}$ and $\{\|y_n\|\}$ converge to non-zero limits, respectively. Then the following are equivalent.*

(i) $\lim_{n\to\infty} \|x_n + y_n\| = \lim_{n\to\infty}(\|x_n\| + \|y_n\|)$.

(ii) $\displaystyle\lim_{n\to\infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2.$

Now we present the main theorems.

THEOREM A ([16, 13]). *Let X and Y be Banach spaces and let $\psi \in \Psi$. Then*

(i) *$X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex, and ψ is strictly convex* ([16, Theorem 1]).

(ii) *$X \oplus_\psi Y$ is uniformly convex if and only if X and Y are uniformly convex, and ψ is strictly convex* ([13, Theorem 1]).

THEOREM B ([14]). *Let $\psi \in \Psi$. Then $(\mathbb{C}^2, \|\cdot\|_\psi)$ is uniformly non-square if and only if $\psi \neq \psi_1$ and $\psi \neq \psi_\infty$.*

THEOREM C([10]). *Let X and Y be Banach spaces and $\psi \in \Psi$. Then the following are equivalent.*

(i) *$X \oplus_\psi Y$ is uniformly non-square.*

(ii) *X and Y are uniformly non-square and $\psi \neq \psi_1, \psi_\infty$.*

EXAMPLE (cf. [13]). Let $1 \leq q < p \leq \infty$ and $2^{1/p-1/q} < \lambda < 1$. Let $\psi_{p,q,\lambda} = \max\{\psi_p, \lambda\psi_q\}$, where ψ_p is as in (6). Then $\psi_{p,q,\lambda} \in \Psi$ and, as is easily seen, the norm of $X \oplus_{\psi_{p,q,\lambda}} Y$ is given by

$$\|(x,y)\|_{\psi_{p,q,\lambda}} = \max\{\|(x,y)\|_p, \lambda\|(x,y)\|_q\}. \tag{4}$$

Indeed, ψ_p and $\lambda\psi_q$ meet in $(0,1)$ (note that $\psi_p < \psi_q$, and ψ_p and ψ_q have thier minimums $2^{1/p-1}$ and $2^{1/q-1}$ respectively), and $\psi_{p,q,\lambda}$ is convex, so $\psi_{p,q,\lambda} \in \Psi$.

According to Theorem 1, $X \oplus_{\psi_{p,q,\lambda}} Y$ *is uniformly non-square if and only if X and Y are uniformly non-square.*

References

[1] B. Beauzamy, *Introduction to Banach Spaces and their Geometry, 2nd ed.*, North-Holland, 1985.

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. **10** (1973).

[3] R. Bhatia, *Matrix Analysis*, Springer, 1997.

[4] S. Dhompongsa, A. Kaewkhao and S. Saejung, Preservation of uniform smoothness and U-convexity by ψ-direct sums, submitted.

[5] P. N. Dowling, On convexity properties of ψ-direct sums of Banach spaces, J. Math. Anal. Appl. **288** (2003), 540–543.

[6] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, Cambridge Univ. Press, 1967.

[7] C. James, *Uniformly non-square Banach spaces*, Ann. of Math. **80** (1964), 542-550.

[8] M. Kato, *On Lorentz spaces* $\ell_{p,q}\{E\}$, Hiroshima Math. J. **6** (1976), 73-93.

[9] M. Kato, K.-S. Saito and T. Tamura, On the ψ-direct sums of Banach spaces and convexity, J. Aust. Math. Soc. **75** (2003), 413–422

[10] M. Kato, K.-S. Saito and T. Tamura, Uniform non-squareness of the ψ-direct sums of Banach spaces, Math. Inequal. Appl. **7**(2004), 429-437.

[11] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer, 1998.

[12] K. Mitani, K.-S. Saito and T. Suzuki, *Smoothness of absolute norms on* $\mathbb{C}^n$, *J. Conv. Anal.*, **10** (2003), 89–107.

[13] K.-S. Saito and M. Kato, *Uniform convexity of* ψ*-direct sums of Banach spaces*, J. Math. Anal. Appl. **277** (2003), 1-11.

[14] K.-S. Saito, M. Kato and Y. Takahashi, *Von Neumann-Jordan constant of absolute normalized norms on* $\mathbb{C}^2$, J. Math. Anal. Appl. **244** (2000), 515-532.

[15] K.-S. Saito, M. Kato and Y. Takahashi, *On absolute norms on* $\mathbb{C}^n$, J. Math. Anal. Appl. **252** (2000), 879-905.

[16] Y. Takahashi, M. Kato and K.-S. Saito, *Strict convexity of absolute norms on* $\mathbb{C}^2$ *and direct sums of Banach spaces*, J. Inequal. Appl. **7** (2002), 179-186.

[17] H. Triebel, *Intepolation Theory, Function spaces, Differential Operators*, North-Holland, 1978.