# $\mathbb{Z}$-actions on AH algebras and $\mathbb{Z}^2$-actions on AF algebras

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**Abstract**

We consider $\mathbb{Z}$-actions (single automorphisms) on a unital simple AH algebra with real rank zero and slow dimension growth and show that the uniform outerness implies the Rohlin property under some technical assumptions. Moreover, two $\mathbb{Z}$-actions with the Rohlin property on such a $C^*$-algebra are shown to be cocycle conjugate if they are asymptotically unitarily equivalent. We also prove that locally approximately inner and uniformly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra with a unique trace have the Rohlin property and classify them up to cocycle conjugacy employing the OrderExt group as classification invariants.

## 1 Introduction

Classification of group actions is one of the most fundamental subjects in the theory of operator algebras. For AFD factors, a complete classification is known for actions of countable amenable groups. However, classification of automorphisms or group actions on $C^*$-algebras is still a far less developed subject, partly because of $K$-theoretical difficulties. For AF and AT algebras, A. Kishimoto [9, 10, 11, 12] showed the Rohlin property for a certain class of automorphisms and obtained a cocycle conjugacy result. Following the strategy developed by Kishimoto, H. Nakamura [24] showed that aperiodic automorphisms on unital Kirchberg algebras are completely classified by their $KK$-classes up to $KK$-trivial cocycle conjugacy. As for $\mathbb{Z}^N$-actions, Nakamura [23] introduced the notion of the Rohlin property and classified product type actions of $\mathbb{Z}^2$ on UHF algebras. T. Katsura and the author [8] gave a complete classification of uniformly outer $\mathbb{Z}^2$-actions on UHF algebras by using the Rohlin property. For Kirchberg algebras, M. Izumi and the author [7] classified a large class of $\mathbb{Z}^2$-actions and also showed the uniqueness of $\mathbb{Z}^N$-actions on the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$. The present article is a continuation of these works.

In the first half of this paper, we study single automorphisms (i.e. $\mathbb{Z}$-actions) on a unital simple classifiable AH algebra. More precisely, we prove the Rohlin type theorem (Theorem 4.8) for an automorphism $\alpha$ of a unital simple AH algebra $A$ with real rank zero and slow dimension growth under the assumption that $\alpha^r$ is approximately inner for some $r \in \mathbb{N}$ and $A$ has finitely many extremal tracial states. Furthermore, we also prove that if two automorphisms $\alpha$ and $\beta$ of a unital simple AH algebra with real rank zero and slow dimension growth have the Rohlin property and $\alpha \circ \beta^{-1}$ is asymptotically inner, then the two $\mathbb{Z}$-actions generated by $\alpha$ and $\beta$ are cocycle conjugate (Theorem 4.9). These results are generalizations of Kishimoto's work for AF and AT algebras ([9, 10, 11, 12]). For the proofs, we need to improve some of the arguments in [11, 12] concerning projections and unitaries in central sequence algebras. As a byproduct, it will be also shown that for $A$ as above, the central sequence algebra $A_\omega$ satisfies Blackadar's second fundamental comparability question (Proposition 3.8).

In the latter half of the paper, we study $\mathbb{Z}^2$-actions on a unital simple AF algebra $A$ with a unique trace. We first show a $\mathbb{Z}$-equivariant version of the Rohlin type theorem for single automorphisms (Theorem 5.5), and as its corollary we obtain the Rohlin type theorem for a $\mathbb{Z}^2$-action $\varphi : \mathbb{Z}^2 \curvearrowright A$ such that $\varphi_{(r,0)}$ and $\varphi_{(0,s)}$ are approximately inner for some $r, s \in \mathbb{N}$ (Corollary 5.6). This is a generalization of [23, Theorem 3]. Next, by using a $\mathbb{Z}$-equivariant version of the Evans-Kishimoto intertwining argument [4], we classify uniformly outer locally $KK$-trivial $\mathbb{Z}^2$-actions on $A$ up to $KK$-trivial cocycle conjugacy (Theorem 6.6). This is a generalization of [8, Theorem 6.5]. We remark that $KK$-triviality of $\alpha \in \mathrm{Aut}(A)$ is equivalent to $\alpha$ being approximately inner, because $A$ is assumed to be AF. For classification invariants, we employ the OrderExt group introduced in [13]. The crossed product of $A$ by the first generator $\varphi_{(1,0)}$ is known to be a unital simple AT algebra with real rank zero. The second generator $\varphi_{(0,1)}$ naturally extends to an automorphism of the crossed product and its OrderExt class gives the invariant of the $\mathbb{Z}^2$-action $\varphi$. However we do not know the precise range of the invariant in general.

## 2 Preliminaries

We collect notations and terminologies relevant to this paper. For a Lipschitz continuous map $f$ between metric spaces, $\mathrm{Lip}(f)$ denotes the Lipschitz constant of $f$. Let $A$ be a unital $C^*$-algebra. For $a, b \in A$, we mean by $[a, b]$ the commutator $ab - ba$. For a unitary $u \in A$, the inner automorphism induced by $u$ is written by $\mathrm{Ad}\, u$. An automorphism $\alpha \in \mathrm{Aut}(A)$ is called outer, when it is not inner. A single automorphism $\alpha$ is often identified with the $\mathbb{Z}$-action induced by $\alpha$. An automorphism $\alpha$ of $A$ is said to be asymptotically inner, if there exists a continuous family of unitaries $\{u_t\}_{t\in[0,\infty)}$ in $A$ such that

$$\alpha(a) = \lim_{t\to\infty} \mathrm{Ad}\, u_t(a)$$

for all $a \in A$. When there exists a sequence of unitaries $\{u_n\}_{n\in\mathbb{N}}$ in $A$ such that

$$\alpha(a) = \lim_{n\to\infty} \mathrm{Ad}\, u_n(a)$$

for all $a \in A$, $\alpha$ is said to be approximately inner. The set of approximately inner automorphisms of $A$ is denoted by $\overline{\mathrm{Inn}}(A)$. Two automorphisms $\alpha$ and $\beta$ are said to be asymptotically (resp. approximately) unitarily equivalent if $\alpha \circ \beta^{-1}$ is asymptotically (resp. approximately) inner. The set of tracial states on $A$ is denoted by $T(A)$. We mean by $\mathrm{Aff}(T(A))$ the space of $\mathbb{R}$-valued affine continuous functions on $T(A)$. The dimension map $D_A : K_0(A) \to \mathrm{Aff}(T(A))$ is defined by $D_A([p])(\tau) = \tau(p)$. For a homomorphism $\rho$ between $C^*$-algebras, $K_0(\rho)$ and $K_1(\rho)$ mean the induced homomorphisms on $K$-groups.

As for group actions on $C^*$-algebras, we freely use the terminology and notation described in [7, Definition 2.1].

Let $A$ be a separable $C^*$-algebra and let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. We set

$$c_0(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\}, \quad A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(A),$$

$$c_\omega(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|a_n\| = 0\}, \quad A^\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(A).$$

We identify $A$ with the $C^*$-subalgebra of $A^\infty$ (resp. $A^\omega$) consisting of equivalence classes of constant sequences. We let

$$A_\infty = A^\infty \cap A', \quad A_\omega = A^\omega \cap A'$$

and call them the central sequence algebras of $A$. A sequence $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ is called a central sequence if $\|[a, x_n]\| \to 0$ as $n \to \infty$ for all $a \in A$. A central sequence is a representative of an element in $A_\infty$. An $\omega$-central sequence is defined in a similar way. When $\alpha$ is an automorphism on $A$ or an action of a discrete group on $A$, we can consider its natural extension on $A^\infty$, $A^\omega$, $A_\infty$ and $A_\omega$. We denote it by the same symbol $\alpha$.

Next, we would like to recall the definition of uniform outerness introduced by Kishimoto and the definition of the Rohlin property of $\mathbb{Z}^N$-actions introduced by Nakamura.

**Definition 2.1** ([9, Definition 1.2]). An automorphism $\alpha$ of a unital $C^*$-algebra $A$ is said to be uniformly outer if for any $a \in A$, any non-zero projection $p \in A$ and any $\varepsilon > 0$, there exist projections $p_1, p_2, \dots, p_n$ in $A$ such that $p = \sum p_i$ and $\|p_i a \alpha(p_i)\| < \varepsilon$ for all $i = 1, 2, \dots, n$.

We say that an action $\alpha$ of a discrete group on $A$ is uniformly outer if $\alpha_g$ is uniformly outer for every element $g$ of the group other than the identity element.

Let $N$ be a natural number. Let $\xi_1, \xi_2, \dots, \xi_N$ be the canonical basis of $\mathbb{Z}^N$, that is,

$$\xi_i = (0, 0, \dots, 1, \dots, 0, 0),$$

where 1 is in the $i$-th component. For $m = (m_1, m_2, \dots, m_N)$ and $n = (n_1, n_2, \dots, n_N)$ in $\mathbb{Z}^N$, $m \le n$ means $m_i \le n_i$ for all $i = 1, 2, \dots, N$. For $m = (m_1, m_2, \dots, m_N) \in \mathbb{N}^N$, we let

$$m\mathbb{Z}^N = \{(m_1 n_1, m_2 n_2, \dots, m_N n_N) \in \mathbb{Z}^N \mid (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N\}.$$

For simplicity, we denote $\mathbb{Z}^N/m\mathbb{Z}^N$ by $\mathbb{Z}_m$. Moreover, we may identify $\mathbb{Z}_m = \mathbb{Z}^N/m\mathbb{Z}^N$ with

$$\{(n_1, n_2, \dots, n_N) \in \mathbb{Z}^N \mid 0 \le n_i \le m_i - 1 \quad \forall i = 1, 2, \dots, N\}.$$

**Definition 2.2** ([23, Definition 1]). Let $\varphi$ be an action of $\mathbb{Z}^N$ on a unital $C^*$-algebra $A$. Then $\varphi$ is said to have the Rohlin property, if for any $m \in \mathbb{N}$ there exist $R \in \mathbb{N}$ and $m^{(1)}, m^{(2)}, \dots, m^{(R)} \in \mathbb{N}^N$ with $m^{(1)}, \dots, m^{(R)} \ge (m, m, \dots, m)$ satisfying the following: For any finite subset $F$ of $A$ and $\varepsilon > 0$, there exists a family of projections

$$e_g^{(r)} \qquad (r = 1, 2, \dots, R, \ g \in \mathbb{Z}_{m^{(r)}})$$

in $A$ such that

$$\sum_{r=1}^{R} \sum_{g \in \mathbb{Z}_{m^{(r)}}} e_g^{(r)} = 1, \quad \|[a, e_g^{(r)}]\| < \varepsilon, \quad \|\varphi_{\xi_i}(e_g^{(r)}) - e_{g+\xi_i}^{(r)}\| < \varepsilon$$

for any $a \in F$, $r = 1, 2, \dots, R$, $i = 1, 2, \dots, N$ and $g \in \mathbb{Z}_{m^{(r)}}$, where $g + \xi_i$ is understood modulo $m^{(r)}\mathbb{Z}^N$.

It is clear that if $\varphi : \mathbb{Z}^N \curvearrowright A$ has the Rohlin property, then $\varphi$ is uniformly outer.

We recall the definition of tracial rank zero introduced by H. Lin.

**Definition 2.3** ([14, 15])**.** A unital simple $C^*$-algebra $A$ is said to have tracial rank zero if for any finite subset $F \subset A$, any $\varepsilon > 0$ and any non-zero positive element $x \in A$ there exists a finite dimensional subalgebra $B \subset A$ with $p = 1_B$ satisfying the following.

(1) $\|[a, p]\| < \varepsilon$ for all $a \in F$.

(2) The distance from $pap$ to $B$ is less than $\varepsilon$ for all $a \in F$.

(3) $1 - p$ is Murray-von Neumann equivalent to a projection in $xAx$.

In [16], H. Lin gave a classification theorem for unital separable simple nuclear $C^*$-algebras with tracial rank zero which satisfy the UCT. ([16, Theorem 5.2]). Indeed, the class of such $C^*$-algebras agrees with the class of all unital simple AH algebras with real rank zero and slow dimension growth.

# 3 Central sequences

**Lemma 3.1.** *Let $A$ be a unital separable approximately divisible $C^*$-algebra. Then for any $m \in \mathbb{N}$, there exists a unital embedding of $M_m \oplus M_{m+1}$ into $A_\infty$.*

*Proof.* Let $l = m^2 - 1$. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exists a unital finite dimensional subalgebra $B \subset A$ such that every central summand of $B$ is at least $l \times l$ and $\|[a, b]\| < \varepsilon$ for any $a \in F$ and $b \in B$ with $\|b\| \leq 1$. It is easy to find a unital subalgebra $C$ of $B$ such that $C \cong M_m \oplus M_{m+1}$, which completes the proof. □

The following is Lemma 3.6 of [11].

**Lemma 3.2.** *Let $A$ be a unital simple AT algebra with real rank zero. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exist a finite subset $G \subset A$, $\delta > 0$ and $k \in \mathbb{N}$ such that the following holds. If $p, q \in A$ are projections satisfying $k[p] \leq [q]$, $\|[a, p]\| < \delta$ and $\|[a, q]\| < \delta$ for all $a \in G$, then there exists a partial isometry $v \in A$ such that $v^*v = p$, $vv^* \leq q$ and $\|[a, v]\| < \varepsilon$ for all $a \in F$.*

We generalize the lemma above to AH algebras.

**Lemma 3.3.** *The above lemma also holds for any unital simple AH algebra with slow dimension growth and real rank zero.*

*Proof.* Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero and let $Q$ be the UHF algebra such that $K_0(Q) \cong \mathbb{Q}$. By the classification theorem ([2, 1, 5]), $A \otimes Q$ is a unital simple AT algebra with real rank zero. Let $F \subset A$ be a finite subset and $\varepsilon > 0$. Applying the lemma above to $\{a \otimes 1 \mid a \in F\} \subset A \otimes Q$ and $\varepsilon/2 > 0$, we obtain a finite subset $G \subset A \otimes Q$, $\delta > 0$ and $k \in \mathbb{N}$. We may assume $G = \{a \otimes 1 \mid a \in G_0\} \cup \{1 \otimes b \mid b \in G_1\}$, where $G_0$ and $G_1$ are finite subsets of $A$ and $Q$, respectively. We may further assume that $G_0$ contains $F$ and $\delta$ is less than $\varepsilon/2$. We will prove that $G_0$, $\delta$ and $4k$ meet the requirement.

Suppose that $p, q \in A$ are non-zero projections satisfying $4k[p] \leq [q]$, $\|[a,p]\| < \delta$ and $\|[a,q]\| < \delta$ for all $a \in G_0$. By Lemma 3.1, $M_2 \oplus M_3$ embeds into $A_\infty$. Hence there exist a projection $r$ and a partial isometry $s$ such that

$$r \leq q, \quad s^*s = r, \quad ss^* \leq q - r, \quad 4[r] > [q]$$

and

$$\|[a,s]\| < \delta, \quad \|[a,r]\| < \delta \quad \forall a \in G_0.$$

From $4k[p] \leq [q] < 4[r]$, we get $k[p] < [r]$. It follows that there exists a partial isometry $u \in A \otimes Q$ such that $u^*u = p \otimes 1$, $uu^* \leq r \otimes 1$ and $\|[a \otimes 1, u]\| < \varepsilon/2$ for all $a \in F$. We may assume that $u$ belongs to some $A \otimes M_m \subset A \otimes Q$. Put $u = (u_{i,j})_{1 \leq i,j \leq m}$. Define $w = (w_{i,j})_{1 \leq i,j \leq m+1} \in A \otimes M_{m+1}$ by

$$w_{i,j} = \begin{cases} u_{i,j} & \text{if } 1 \leq i,j \leq m \\ su_{i,1} & \text{if } i \neq m+1 \text{ and } j = m+1 \\ 0 & \text{if } i = m+1. \end{cases}$$

It is not so hard to see that $w^*w = p \otimes 1 \in A \otimes M_{m+1}$ and $ww^* \leq q \otimes 1 \in A \otimes M_{m+1}$. Moreover, one has $\|[a \otimes 1, w]\| < \varepsilon$ for all $a \in F$. Let $v = u \oplus w \in A \otimes (M_m \oplus M_{m+1})$. Then $v^*v = p \otimes 1$, $vv^* \leq q \otimes 1$ and $\|[a \otimes 1, v]\| < \varepsilon$ for all $a \in F$.

By [3] (see also [1, 5]), $A$ is approximately divisible. By Lemma 3.1, there exists a unital homomorphism from $M_m \oplus M_{m+1}$ to $A_\infty$, and so there exists a unital homomorphism $\pi$ from $A \otimes (M_m \oplus M_{m+1})$ to $A^\infty$ such that $\pi(a \otimes 1) = a$ for $a \in A$. It follows that $\pi(v)^*\pi(v) = p$, $\pi(v)\pi(v)^* \leq q$ and $\|[a, \pi(v)]\| < \varepsilon$ for all $a \in F$. □

**Remark 3.4.** By using the lemma above and [27, Theorem 4.5], one can show the following. Let $A$ be a unital simple AH algebra with real rank zero and slow dimension growth. Then for any $\alpha \in \mathrm{Aut}(A)$, there exists $\tilde{\alpha} \in \mathrm{Aut}(A)$ such that $\tilde{\alpha}$ has the Rohlin property in the sense of [12, Definition 4.1] and $\tilde{\alpha}$ is asymptotically unitarily equivalent to $\alpha$.

The following is a well-known fact. We have been unable to find a suitable reference in the literature, so we include a proof for completeness.

**Lemma 3.5.** *Let $A$ be a unital separable $C^*$-algebra and let $(p_n)_n$ be a central sequence of projections. For any extremal trace $\tau \in T(A)$, one has*

$$\lim_{n \to \infty} |\tau(ap_n) - \tau(a)\tau(p_n)| = 0$$

*for all $a \in A$.*

*Proof.* First, we deal with the case that there exists $\varepsilon > 0$ such that $\tau(p_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. Consider a sequence of states

$$\varphi_n(a) = \frac{\tau(ap_n)}{\tau(p_n)}$$

on $A$. Let $\psi$ be an accumulation point of $\{\varphi_n\}_n$. Since $(p_n)_n$ is a central sequence and $\tau(p_n)$ is bounded from below, one can see that $\psi$ is a trace. For any positive element

$a \in A$, it is easy to see $\varphi_n(a) \le \varepsilon^{-1}\tau(a)$, and so $\psi(a) \le \varepsilon^{-1}\tau(a)$. Hence, $\psi$ is equal to $\tau$, because $\tau$ is extremal. We have shown that any accumulation point of $\{\varphi_n\}$ is $\tau$, which implies $\varphi_n$ converges to $\tau$. Therefore, $|\tau(ap_n) - \tau(a)\tau(p_n)|$ goes to zero.

Next, we consider the general case. Fix $a \in A$. Take $\varepsilon > 0$ arbitrarily. We would like to show that $|\tau(ap_n) - \tau(a)\tau(p_n)|$ is less than $\varepsilon$ for sufficiently large $n$. We may assume $\|a\| \le 1$. Put

$$C = \{n \in \mathbb{N} \mid \tau(p_n) \ge \varepsilon/2\}.$$

If $n \notin C$, then evidently $|\tau(ap_n) - \tau(a)\tau(p_n)|$ is less than $\varepsilon$. By applying the first part of the proof to $(p_n)_{n\in C}$, we have $|\tau(ap_n) - \tau(a)\tau(p_n)| < \varepsilon$ for sufficiently large $n \in C$, thereby completing the proof. □

**Lemma 3.6.** *Let $A$ be a unital simple separable $C^*$-algebra with tracial rank zero. Suppose that $A$ has finitely many extremal traces. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exist a finite subset $G \subset A$ and $\delta > 0$ such that the following hold. If $p, q \in A$ are projections satisfying*

$$\|[a,p]\| < \delta, \quad \|[a,q]\| < \delta \quad \forall a \in G$$

*and $\tau(p) + \varepsilon < \tau(q)$ for all $\tau \in T(A)$, then there exists a partial isometry $v \in A$ such that $v^*v \le p$, $vv^* \le q$,*

$$\|[a,v]\| < \varepsilon \quad \forall a \in F$$

*and $\tau(p - v^*v) < \varepsilon$ for all $\tau \in T(A)$.*

*Proof.* The proof is by contradiction. Suppose that the assertion does not hold for a finite subset $F \subset A$ and $\varepsilon > 0$. We would have central sequences of projections $(p_n)_n$ and $(q_n)_n$ such that

$$\tau(p_n) + \varepsilon < \tau(q_n) \quad \forall \tau \in T(A),\ n \in \mathbb{N}$$

and any partial isometry does not meet the requirement.

Use tracial rank zero to find a projection $e \in A$ and a finite dimensional unital subalgebra $E \subset eAe$ such that the following are satisfied.

- For any $a \in F$, $\|[a,e]\| < \varepsilon/4$.
- For any $a \in F$ there exists $b \in E$ such that $\|eae - b\| < \varepsilon/4$.
- $\tau(1-e) < \varepsilon$ for all $\tau \in T(A)$.

Since $(p_n)_n$ and $(q_n)_n$ are central sequences of projections, we can find projections $p'_n$ and $q'_n$ in $A \cap E'$ such that $\|p_n - p'_n\| \to 0$ and $\|q_n - q'_n\| \to 0$ as $n \to \infty$. We would like to show that, for sufficiently large $n$, there exists a partial isometry $v_n \in eAe \cap E'$ such that $v_n^*v_n = ep'_n$ and $v_nv_n^* \le eq'_n$. Let $\{e_1, e_2, \dots, e_m\}$ be a family of minimal central projections in $eAe \cap E'$ such that $e_1 + e_2 + \cdots + e_m = e$. Clearly $e_i(eAe \cap E')$ is a unital simple $C^*$-algebra with tracial rank zero and the space of tracial states on $e_i(eAe \cap E')$ is naturally identified with $T(A)$. By lemma 3.5, for sufficiently large $n$, one has $\tau(e_ip'_n) < \tau(e_iq'_n)$ for all $\tau \in T(A)$ and $i = 1, 2, \dots, m$, because $A$ has finitely many extremal traces. Hence $[e_ip'_n] \le [e_iq'_n]$ in $K_0(e_i(eAe \cap E'))$. It follows that there exists a partial isometry $v_n \in eAe \cap E'$ such that $v_n^*v_n = ep'_n$ and $v_nv_n^* \le eq'_n$. Besides, $\tau(p'_n - v_n^*v_n) = \tau(p'_n(1-e)) < \varepsilon$ and $\|[a, v_n]\| < \varepsilon$ for all $a \in F$. This is a contradiction. □

By using Lemma 3.6 and 3.3, we can show the following.

**Lemma 3.7.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. Suppose that $A$ has finitely many extremal traces. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exist a finite subset $G \subset A$ and $\delta > 0$ such that the following hold. If $p, q \in A$ are projections satisfying*

$$\|[a, p]\| < \delta, \quad \|[a, q]\| < \delta \quad \forall a \in G$$

*and $\tau(p) + \varepsilon < \tau(q)$ for all $\tau \in T(A)$, then there exists a partial isometry $u \in A$ such that $u^*u = p$, $uu^* \leq q$ and $\|[a, u]\| < \varepsilon$ for all $a \in F$.*

*Proof.* Notice that $A$ has tracial rank zero by [15, Proposition 2.6]. Suppose that a finite subset $F \subset A$ and $\varepsilon > 0$ are given. By applying Lemma 3.3 to $F$ and $\varepsilon/2$, we obtain a finite subset $F_1 \subset A$, $\varepsilon_1 > 0$ and $k \in \mathbb{N}$. By applying Lemma 3.6 to $F \cup F_1 \cup F_1^*$ and $\min\{\varepsilon_1/4, \varepsilon/k, \varepsilon/2\}$, we obtain a finite subset $G \subset A$, $\delta > 0$. We would like to show that $G \cup F_1$ and $\min\{\delta, \varepsilon_1/2\}$ meet the requirement. Suppose that $p, q \in A$ are projections satisfying

$$\|[a, p]\| < \min\{\delta, \varepsilon_1/2\}, \quad \|[a, q]\| < \min\{\delta, \varepsilon_1/2\} \quad \forall a \in G \cup F_1$$

and $\tau(p) + \varepsilon < \tau(q)$ for all $\tau \in T(A)$. By Lemma 3.6, there exists a partial isometry $v \in A$ such that $v^*v \leq p$, $vv^* \leq q$,

$$\|[a, v]\| < \min\{\varepsilon_1/4, \varepsilon/2\} \quad \forall a \in F \cup F_1 \cup F_1^*$$

and $\tau(p - v^*v) < \varepsilon/k$ for all $\tau \in T(A)$. Let $p' = p - v^*v$ and $q' = q - vv^*$. One has

$$\tau(q') = \tau(q - vv^*) = \tau(q) - \tau(p) + \tau(p - v^*v) > \varepsilon,$$

and so $k[p'] \leq [q']$. One also has $\|[a, p']\| < \varepsilon_1$ and $\|[a, q']\| < \varepsilon_1$ for all $a \in F_1$. By Lemma 3.3, we obtain a partial isometry $w \in A$ such that $w^*w = p'$, $ww^* \leq q'$ and $\|[a, w]\| < \varepsilon/2$ for all $a \in F$. Put $u = v + w$. It is easy to see $u^*u = p$, $uu^* \leq q$ and $\|[a, u]\| < \varepsilon$ for all $a \in F$. □

Any tracial state $\tau \in T(A)$ naturally extends to a tracial state on $A^\omega$ and we write it by $\tau_\omega$.

**Proposition 3.8.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. Suppose that $A$ has finitely many extremal traces. If $p, q \in A_\omega$ are projections satisfying $\tau_\omega(p) < \tau_\omega(q)$ for all $\tau \in T(A)$, then there exists $v \in A_\omega$ such that $v^*v = p$ and $vv^* \leq q$. In particular, $A_\omega$ satisfies Blackadar's second fundamental comparability question.*

*Proof.* Let $(p_n)_n$ and $(q_n)_n$ be $\omega$-central sequences of projections such that

$$\lim_{n \to \omega} \tau(p_n) < \lim_{n \to \omega} \tau(q_n)$$

for all $\tau \in T(A)$. Since $A$ has finitely many extremal traces, there exists $\varepsilon > 0$ such that

$$C = \{n \in \mathbb{N} \mid \tau(p_n) + \varepsilon < \tau(q_n) \text{ for all } \tau \in T(A)\} \in \omega.$$

We choose an increasing sequence $\{F_m\}_{m=1}^{\infty}$ of finite subsets of $A$ whose union is dense in $A$. By applying Lemma 3.7 to $F_m$ and $\varepsilon/m$, we obtain a finite subset $G_m \subset A$ and $\delta_m > 0$. We may assume that $\{G_m\}_m$ is increasing and $\{\delta_m\}_m$ is decreasing. Put

$$C_m = \{n \in C \mid \|[a, p_n]\| < \delta_m \text{ and } \|[a, q_n]\| < \delta_m \text{ for all } a \in G_m\} \in \omega.$$

For $n \in C_m \setminus C_{m+1}$, by Lemma 3.7, there exists a partial isometry $u_n$ such that $u_n^* u_n = p_n$, $u_n u_n^* \leq q_n$ and $\|[a, u_n]\| < \varepsilon/m$ for all $a \in F_m$. For $n \in \mathbb{N} \setminus C_1$, we let $u_n = 0$. Then $(u_n)_n$ is a desired $\omega$-central sequence of partial isometries. □

The following is Lemma 4.4 of [12].

**Lemma 3.9.** *Let $A$ be a unital simple AT algebra with real rank zero. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exist a finite subset $G \subset A$ and $\delta > 0$ such that the following holds. If $u : [0,1] \to A$ is a path of unitaries satisfying $\|[a, u(t)]\| < \delta$ for all $a \in G$ and $t \in [0,1]$, then there exists a path of unitaries $v : [0,1] \to A$ satisfying*

$$v(0) = u(0), \quad v(1) = u(1), \quad \|[a, v(t)]\| < \varepsilon \quad \forall a \in F,\ t \in [0,1]$$

*and* $\mathrm{Lip}(v) < 5\pi + 1$.

We generalize the lemma above to AH algebras.

**Lemma 3.10.** *The above lemma also holds for any unital simple AH algebra with slow dimension growth and real rank zero, the Lipschitz constant being bounded by $11\pi$.*

*Proof.* Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero and let $Q$ be the UHF algebra such that $K_0(Q) \cong \mathbb{Q}$. By the classification theorem ([2, 1, 5]), $A \otimes Q$ is a unital simple AT algebra with real rank zero. Let $F \subset A$ be a finite subset and $\varepsilon > 0$. We may assume that $F$ is contained in the unit ball of $A$. Applying the lemma above to $\{a \otimes 1 \mid a \in F\} \subset A \otimes Q$ and $\varepsilon/2 > 0$, we obtain a finite subset $G \subset A \otimes Q$ and $\delta > 0$. We may assume $G = \{a \otimes 1 \mid a \in G_0\} \cup \{1 \otimes b \mid b \in G_1\}$, where $G_0$ and $G_1$ are finite subsets of $A$ and $Q$, respectively. We will prove that $G_0$ and $\delta$ meet the requirement.

Suppose that $u : [0,1] \to A$ is a path of unitaries satisfying $\|[a, u(t)]\| < \delta$ for all $a \in G_0$ and $t \in [0,1]$. Choose $N \in \mathbb{N}$ so that $\|u(\frac{k}{N}) - u(\frac{k+1}{N})\| < \varepsilon/2$ for every $k = 0, 1, \ldots, N-1$. Put $u_k = u(k/N)$. By the lemma above, we can find continuous paths $x : [0,1] \to A \otimes Q$, $y_k : [0,1] \to A \otimes Q$ and $z_k : [0,1] \to A \otimes Q$ for $k = 1, 2, \ldots, N-1$ such that

$$x(0) = u_0 \otimes 1, \quad x(1) = u_N \otimes 1, \quad \|[a \otimes 1, x(t)]\| < \varepsilon/2 \quad \forall a \in F,\ t \in [0,1],$$

$$y_k(0) = u_0 \otimes 1, \quad y_k(1) = u_k \otimes 1, \quad \|[a \otimes 1, y_k(t)]\| < \varepsilon/2 \quad \forall a \in F,\ t \in [0,1],$$

$$z_k(0) = u_k \otimes 1, \quad z_k(1) = u_N \otimes 1, \quad \|[a \otimes 1, z_k(t)]\| < \varepsilon/2 \quad \forall a \in F,\ t \in [0,1],$$

and $\mathrm{Lip}(x), \mathrm{Lip}(y_k), \mathrm{Lip}(z_k)$ are less than $5\pi + 1$. We may assume that the ranges of $x, y_k, z_k$ are contained in $A \otimes M_n$ for some $M_n \subset Q$.

Put $m = n(N-1)$. We would like to construct a path of unitaries $v : [0,1] \to A \otimes (M_m \oplus M_{m+1})$ such that $\mathrm{Lip}(v) < 11\pi$, $v(0) = u_0 \otimes 1$, $v(1) = u_N \otimes 1$ and $\|[a \otimes 1, v(t)]\| < \varepsilon$ for all $a \in F$ and $t \in [0,1]$. First, let $\tilde{x} : [0,1] \to A \otimes M_m$ be the direct sum of $N-1$

copies of $x : [0,1] \to A \otimes M_n$. Next, by using $y_1, y_2, \cdots, y_{N-1}$, we can find a path $\tilde{y} : [0,1] \to A \otimes M_{m+1}$ such that

$$\tilde{y}(0) = u_0 \otimes 1,$$

$$\tilde{y}(1) = \mathrm{diag}(u_0, \underbrace{u_1, \cdots, u_1}_{n}, \underbrace{u_2, \cdots, u_2}_{n}, \cdots, \underbrace{u_{N-1}, \cdots, u_{N-1}}_{n})$$

$$\|[a \otimes 1, \tilde{y}(t)]\| < \varepsilon/2 \quad \forall a \in F,\ t \in [0,1]$$

and $\mathrm{Lip}(\tilde{y}) < 5\pi + 1$. Likewise, by using $z_1, z_2, \ldots, z_{N-1}$, we can find a path $\tilde{z} : [0,1] \to A \otimes M_{m+1}$ such that

$$\tilde{z}(0) = \mathrm{diag}(\underbrace{u_1, \cdots, u_1}_{n}, \underbrace{u_2, \cdots, u_2}_{n}, \cdots, \underbrace{u_{N-1}, \cdots, u_{N-1}}_{n}, u_N)$$

$$\tilde{z}(1) = u_N \otimes 1,$$

$$\|[a \otimes 1, \tilde{z}(t)]\| < \varepsilon/2 \quad \forall a \in F,\ t \in [0,1]$$

and $\mathrm{Lip}(\tilde{z}) < 5\pi + 1$. Since $\|\tilde{y}(1) - \tilde{z}(0)\| < \varepsilon/2$, if $\varepsilon$ is sufficiently small, there exists a path $w : [0,1] \to M_{m+1}$ such that

$$w(0) = u_0 \otimes 1, \quad w(1) = u_N \otimes 1, \quad \|[a \otimes 1, w(t)]\| < \varepsilon \quad \forall a \in F,\ t \in [0,1],$$

and $\mathrm{Lip}(w) < 11\pi$. Then $v = \tilde{x} \oplus w$ is the desired path.

By [3] (see also [1, 5]), $A$ is approximately divisible. By Lemma 3.1, there exists a unital homomorphism from $M_m \oplus M_{m+1}$ to $A_\infty$, and so there exists a unital homomorphism $\pi$ from $A \otimes (M_m \oplus M_{m+1})$ to $A^\infty$ such that $\pi(a \otimes 1) = a$ for $a \in A$. It follows that the path $\tilde{v} : [0,1] \ni t \mapsto \pi(v(t)) \in A^\infty$ satisfies

$$\tilde{v}(0) = u_0, \quad \tilde{v}(1) = u_N, \quad \|[a, \tilde{v}(t)]\| < \varepsilon \quad \forall a \in F,\ t \in [0,1]$$

and $\mathrm{Lip}(\tilde{v}) < 11\pi$, which completes the proof. □

# 4 Automorphisms of AH algebras

In this section, we discuss the Rohlin property of automorphisms of AH algebras. For $a \in A$, we define

$$\|a\|_2 = \sup_{\tau \in T(A)} \tau(a^*a)^{1/2}.$$

If $A$ is simple and $T(A)$ is non-empty, then $\|\cdot\|_2$ is a norm.

**Proposition 4.1.** *Let $A$ be a unital simple separable $C^*$-algebra with tracial rank zero and let $\Gamma \subset \mathrm{Aut}(A)$ be a finite subset containing the identity. Suppose that there exists a sequence of projections $(e_n)_n$ in $A$ satisfying the following property.*

(1) $\|\gamma(e_n)\gamma'(e_n)\|_2 \to 0$ *for any* $\gamma, \gamma' \in \Gamma$ *such that* $\gamma \neq \gamma'$.

(2) $\|1 - \sum_{\gamma \in \Gamma} \gamma(e_n)\|_2 \to 0$.

(3) *For every* $a \in A$, *we have* $\|[a, e_n]\|_2 \to 0$.

*Then there exists a sequence of projections* $(f_n)_n$ *in* $A$ *satisfying the following.*

(1) $\|\gamma(f_n)\gamma'(f_n)\| \to 0$ *for any* $\gamma, \gamma' \in \Gamma$ *such that* $\gamma \neq \gamma'$.

(2) $\|e_n - f_n\|_2 \to 0$.

(3) *For every* $a \in A$*, we have* $\|[a, f_n]\| \to 0$.

*Proof.* This is almost the same as [19, Proposition 5.4]. In [19, Proposition 5.4], the finite set $\Gamma$ is assumed to be an orbit of a single automorphism $\gamma$ of finite order. The proof, however, does not need this. □

The following is a variant of [9, Lemma 3.1] and [26, Theorem 2.17]. See [26, Definition 2.1] for the definition of the tracial Rohlin property.

**Theorem 4.2.** *Let* $A$ *be a unital simple separable* $C^*$*-algebra with tracial rank zero. Suppose that* $A$ *has finitely many extremal traces. Let* $\alpha$ *be an automorphism of* $A$ *such that* $\alpha^m$ *is uniformly outer for any* $m \in \mathbb{N}$*. Then* $\alpha$ *has the tracial Rohlin property.*

*Proof.* Let $\{\tau_1, \dots, \tau_d\}$ be the set of extremal tracial states of $A$ and let $(\pi_i, H_i)$ be the GNS representation associated with $\tau_i$. It is well-known that $\pi_i(A)''$ is a hyperfinite $\mathrm{II}_1$-factor (see [26, Lemma 2.16]). Let $\rho = \bigoplus_{i=1}^d \pi_i$. Note that, for a bounded sequence $(a_n)_n$ in $A$, $\rho(a_n)$ converges to zero in the strong operator topology if and only if $\|a_n\|_2$ converges to zero. We regard $A$ as a subalgebra of $N = \rho(A)'' \cong \bigoplus_{i=1}^d \pi_i(A)''$ and denote the extension of the automorphism $\alpha$ to $N$ by $\bar{\alpha}$. Let $k$ be the minimum positive integer such that $\tau_i \circ \alpha^k = \tau_i$ for all $i = 1, 2, \dots, d$. In the same way as [9, Lemma 3.1], for any $l \in \mathbb{N}$, one can find a sequence $\{f_0^{(j)}, \dots, f_{kl-1}^{(j)}\}$ of orthogonal families of projections in $N$ such that $\sum_{i=0}^{kl-1} f_i^{(j)} = 1$,

$$[a, f_i^{(j)}] \to 0 \quad \forall a \in A,$$
$$\bar{\alpha}(f_i^{(j)}) - f_{i+1}^{(j)} \to 0 \quad \forall i = 0, 1, \dots, kl-1$$

in the strong operator topology as $j \to \infty$, where $f_{kl}^{(j)} = f_0^{(j)}$. By [26, Lemma 2.15], we may replace the projections $f_i^{(j)}$ with projections of $A$. From the proposition above, we can conclude that $\alpha$ has the tracial Rohlin property. □

The following is a well-known fact, but we include the proof for the reader's convenience.

**Lemma 4.3.** *Let* $A$ *be a unital separable* $C^*$*-algebra and let* $\alpha \in \overline{\mathrm{Inn}}(A)$*. For any separable subset* $C \subset A_\infty$*, there exists a unitary* $u \in A_\infty$ *such that* $uxu^* = \alpha(x)$ *for all* $x \in C$*.*

*Proof.* We choose an increasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite subsets of $A$ whose union is dense in $A$. We can find a sequence of unitaries $(v_n)_n$ in $A$ such that

$$\|v_n a v_n^* - \alpha^{-1}(a)\| < n^{-1}$$

for all $a \in F_n$, because $\alpha$ is approximately inner. We may assume that $C$ is countable. Let $C = \{x_1, x_2, \dots\}$ and let $(x_{i,j})_j$ be a representative of $x_i$. There exists an increasing sequence $(m(n))_n$ of natural numbers such that

$$\|[v_n, x_{i,j}]\| < n^{-1} \quad \forall j \geq m(n)$$

for any $i = 1, 2, \dots, n$, because $(x_{i,j})_j$ is a central sequence. Since $\alpha$ is in $\overline{\mathrm{Inn}}(A)$, one can find a sequence of unitaries $(w_n)_n$ in $A$ such that

$$\|w_n a w_n^* - \alpha(a)\| < n^{-1}$$

for all $a$ in

$$\alpha^{-1}(F_n) \cup \{x_{i,j} \mid i = 1, \dots, n,\ m(n) \le j < m(n+1)\}.$$

For $j \in \mathbb{N}$, find $n \in \mathbb{N}$ so that $m(n) \le j < m(n+1)$ and define a unitary $u_j$ by $u_j = w_n v_n$. It is easy to see

$$\|[u_j, a]\| < 2/n \quad \forall a \in F_n$$

and

$$\|u_j x_{i,j} u_j^* - \alpha(x_{i,j})\| < 2/n \quad \forall i = 1, \dots, n,$$

and so the proof is completed. □

We quote the following theorem by Lin and Osaka from [20]. See [20, Definition 2.4] for the definition of the tracial cyclic Rohlin property. Since we need to discuss an 'equivariant version' of this theorem later, we would like to include the proof briefly.

**Theorem 4.4** ([20, Theorem 3.4]). *Let $A$ be a unital simple separable $C^*$-algebra and suppose that the order on projections in $A$ is determined by traces. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial Rohlin property. If $\alpha^r$ is in $\overline{\mathrm{Inn}}(A)$ for some $r \in \mathbb{N}$, then $\alpha$ has the tracial cyclic Rohlin property.*

*Proof.* Take $m \in \mathbb{N}$ and $\varepsilon > 0$ arbitrarily. Let $l = r(mr+1)$. Since $\alpha$ has the tracial Rohlin property and the order on projections is determined by traces, there exists a central sequence of projections $(e_n)_n$ such that

$$\lim_{n\to\infty} \|e_n \alpha^i(e_n)\| = 0 \quad \forall i = 1, 2, \dots, l-1$$

and

$$\lim_{n\to\infty} \tau(1 - (e_n + \alpha(e_n) + \dots + \alpha^{l-1}(e_n)) = 0 \quad \forall \tau \in T(A).$$

Let $e \in A_\infty$ be the image of $(e_n)_n$ and define $\tilde{e}$ by

$$\tilde{e} = \sum_{i=0}^{r-1} \alpha^{i(mr+1)}(e).$$

It follows from the lemma above that there exists a partial isometry $v \in A_\infty$ such that $v^*v = \tilde{e}$ and $vv^* = \alpha(\tilde{e})$. The $C^*$-algebra $C$ generated by $v, \alpha(v), \dots, \alpha^{mr-1}(v)$ is isomorphic to $M_{mr+1}$ and its unit is $\tilde{e} + \alpha(\tilde{e}) + \dots + \alpha^{mr}(\tilde{e})$. The rest of the proof is exactly the same as that of [9, Lemma 4.3] and we omit it. □

**Remark 4.5.** The following was shown by Lin in [17, Theorem 3.4]. Let $A$ be a unital simple separable $C^*$-algebra with tracial rank zero. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rohlin property and that there exists $r \in \mathbb{N}$ such that $K_0(\alpha^r)|G = \mathrm{id}_G$ for some subgroup $G \subset K_0(A)$ for which $D_A(G)$ is dense in $D_A(K_0(A))$. Then $A \rtimes_\alpha \mathbb{Z}$ has tracial rank zero.

By using Lemma 3.3, we can show the following.

**Lemma 4.6.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rohlin property and that there exists $r \in \mathbb{N}$ such that $\tau \circ \alpha^r = \tau$ for any $\tau \in T(A)$. Then, for any $m \in \mathbb{N}$, there exist projections $e, f \in A_\infty$ and a partial isometry $v \in A_\infty$ such that*

$$v^*v = f, \quad vv^* \leq e, \quad f + \sum_{i=0}^{mr} \alpha^i(e) = 1$$

*and $\alpha^{mr+1}(e) = e$.*

*Proof.* Suppose that we are given $m \in \mathbb{N}$. Let $l = r(mr+1)$. Since $\alpha$ has the tracial cyclic Rohlin property, we can find central sequences of projections $(e_n)_n$ and $(f_n)_n$ such that

$$f_n + \sum_{i=0}^{l-1} \alpha^i(e_n) \to 1, \quad e_n - \alpha^l(e_n) \to 0, \quad \sup_{\tau \in T(A)} \tau(f_n) \to 0$$

as $n \to \infty$. There exists a central sequence of projections $(\tilde{e}_n)_n$ such that

$$\lim_{n\to\infty} \tilde{e}_n - \sum_{i=0}^{r-1} \alpha^{i(mr+1)}(e_n) = 0.$$

Then

$$f_n + \sum_{i=0}^{mr} \alpha^i(\tilde{e}_n) \to 1, \quad \tilde{e}_n - \alpha^{mr+1}(\tilde{e}_n) \to 0$$

as $n \to \infty$. It is also easy to see $\tau(\alpha(\tilde{e}_n)) = \tau(\tilde{e}_n)$ for all $\tau \in T(A)$, and so $\tau(\tilde{e}_n)$ goes to $(mr+1)^{-1}$ for all $\tau \in T(A)$. Therefore, by Lemma 3.3, one can find a central sequence of partial isometries $(v_n)_n$ such that $v_n^*v_n = f_n$ and $v_nv_n^* \leq \tilde{e}_n$ for sufficiently large $n$, which completes the proof. □

By using the lemma above, we can show the following theorem.

**Theorem 4.7.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. Suppose that $\alpha \in \mathrm{Aut}(A)$ has the tracial cyclic Rohlin property. If there exists $r \in \mathbb{N}$ such that $\tau \circ \alpha^r = \tau$ for any $\tau \in T(A)$, then $\alpha$ has the Rohlin property.*

*Proof.* Suppose that we are given $M \in \mathbb{N}$. Choose a natural number $m \in \mathbb{N}$ so that $m \geq M$ and $m \equiv 1 \pmod{r}$. Let $k, l$ be sufficiently large natural numbers satisfying $k \equiv l \equiv 1 \pmod{r}$. By the lemma above, we can find projections $e, f \in A_\infty$ and a partial isometry $v \in A_\infty$ such that

$$v^*v = f, \quad vv^* \leq e, \quad f + \sum_{i=0}^{klm-1} \alpha^i(e) = 1$$

and $\alpha^{klm}(e) = e$. Define $\tilde{e}, w \in A_\infty$ by

$$\tilde{e} = \sum_{i=0}^{k-1} \alpha^{ilm}(e) \quad \text{and} \quad w = \frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} \alpha^{ilm}(v).$$

Then $\tilde{e}$ is a projection and $w$ is a partial isometry satisfying

$$f + \sum_{i=0}^{lm-1} \alpha^i(\tilde{e}) = 1, \quad \alpha^{lm}(\tilde{e}) = \tilde{e}$$

and

$$w^*w = f, \quad ww^* \leq \tilde{e}, \quad \|\alpha^{lm}(w) - w\| \leq \frac{2}{\sqrt{k}}.$$

Let $D$ be the $C^*$-algebra generated by $w, \alpha(w), \dots, \alpha^{lm-1}(w)$. Then $D$ is isomorphic to $M_{lm+1}$ and the unit $1_D$ of $D$ is equal to $f + ww^* + \cdots + \alpha^{lm-1}(ww^*)$. From the spectral property of $\alpha$ restricted to $D$, if $k$ and $l$ are sufficiently large, we can obtain projections $p_0, \dots, p_{m-1}, q_0, \dots, q_m$ of $D$ such that

$$\sum_{i=1}^{m-1} p_i + \sum_{i=1}^{m} q_i = 1_D, \quad \alpha(p_i) \approx p_{i+1}, \quad \alpha(q_i) \approx q_{i+1},$$

where $p_m = p_0$ and $q_{m+1} = q_0$. We define projections $p_i'$ in $A_\infty$ by

$$p_i' = p_i + \sum_{j=0}^{l-1} \alpha^{i+jm}(\tilde{e} - ww^*).$$

Then the projections $p_0', \dots, p_{m-1}', q_0, \dots, q_m$ meet the requirement. See [9, 10] for details. $\square$

Combining the theorems above, we obtain the following theorem which is a generalization of [11, Theorem 2.1].

**Theorem 4.8.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero and let $\alpha \in \mathrm{Aut}(A)$. Suppose that $A$ has finitely many extremal traces and that $\alpha^r$ is approximately inner for some $r \in \mathbb{N}$. Then the following are equivalent.*

(1) *$\alpha$ has the Rohlin property.*

(2) *$\alpha^m$ is uniformly outer for any $m \in \mathbb{N}$.*

We can also generalize [12, Theorem 5.1] by using Lemma 3.10 instead of [12, Lemma 4.4].

**Theorem 4.9.** *Let $A$ be a unital simple AH algebra with slow dimension growth and real rank zero. If $\alpha, \beta \in \mathrm{Aut}(A)$ have the Rohlin property and $\alpha$ is asymptotically unitarily equivalent to $\beta$, then there exist $\mu \in \overline{\mathrm{Inn}}(A)$ and a unitary $u \in A$ such that*

$$\mathrm{Ad}\, u \circ \alpha = \mu \circ \beta \circ \mu^{-1}.$$

The proof is similar to that of [12, Theorem 5.1] and we omit it.

As an application of the theorems above, we can show the following, which will be used in Section 6.

**Lemma 4.10.** *Let $A$ be a unital simple AF algebra with finitely many extremal traces and let $\alpha$ be an approximately inner automorphism of $A$ such that $\alpha^m$ is uniformly outer for all $m \in \mathbb{N}$. For any finite subset $F \subset A$ and $\varepsilon > 0$, there exist a finite subset $G \subset A$ and $\delta > 0$ satisfying the following. If $u : [0,1] \to A$ is a path of unitaries such that*

$$\|[a, u(t)]\| < \delta \quad \text{and} \quad \|u(t) - \alpha(u(t))\| < \delta \quad \forall a \in G,\ t \in [0,1],$$

*then there exists a path of unitaries $v : [0,1] \to A$ such that*

$$v(0) = u(0), \quad v(1) = u(1), \quad \|[a, v(t)]\| < \varepsilon, \quad \|v(t) - \alpha(v(t))\| < \varepsilon, \quad \forall a \in F,\ t \in [0,1]$$

*and* $\mathrm{Lip}(v) < 2\pi$.

*Proof.* Let $x_n$ be a unitary of $M_n(\mathbb{C})$ such that $\mathrm{Sp}(x_n) = \{\omega^k \mid k = 0, 1, \dots, n-1\}$, where $\omega = \exp(2\pi\sqrt{-1}/n)$. One can find an increasing sequence $\{A_n\}_{n=1}^\infty$ of unital finite dimensional subalgebras of $A$ such that $\bigcup_n A_n$ is dense in $A$ and there exists a unital embedding $\pi_n : M_n \oplus M_{n+1} \to A_{n+1} \cap A_n'$. Let $y_n = \pi_n(x_n \oplus x_{n+1})$. Define an automorphism $\sigma$ of $A$ by $\sigma = \lim_{n\to\infty} \mathrm{Ad}(y_1 y_2 \dots y_n)$. Then $\sigma$ is approximately inner and $\sigma^m$ is uniformly outer for all $m \in \mathbb{N}$.

We would like to show that the assertion holds for $\sigma$. Suppose that we are given $F \subset A$ and $\varepsilon > 0$. Without loss of generality, we may assume that there exists $n \in \mathbb{N}$ such that $F$ is contained in the unit ball of $A_n$. Applying [8, Lemma 4.2] to $\varepsilon/2$, we obtain a positive real number $\delta_1 > 0$. We may assume $\delta_1$ is less than $\min\{2, \varepsilon\}$. Choose a finite subset $G \subset A$ and $\delta_2 > 0$ so that if $z : [0,1] \to A$ is a path of unitaries such that $\|[a, z(t)]\| < \delta_2$ for all $a \in G$ and $t \in [0,1]$, then there exists a path of unitaries $\tilde{z} : [0,1] \to A \cap A_n'$ such that $\|z(t) - \tilde{z}(t)\| < \delta_1/6$. Let $\delta = \min\{\delta_1/6, \delta_2\}$. Suppose that $u : [0,1] \to A$ is a path of unitaries such that

$$\|[a, u(t)]\| < \delta \quad \text{and} \quad \|u(t) - \sigma(u(t))\| < \delta \quad \forall a \in G,\ t \in [0,1].$$

By the choice of $\delta$, we can find $\tilde{u} : [0,1] \to A \cap A_n'$ such that $\|u(t) - \tilde{u}(t)\| < \delta_1/6$. We may assume that there exists $m > n$ such that the range of $\tilde{u}$ is contained in $A_m$. Put $y = y_n y_{n+1} \dots y_{m-1} \in A_m \cap A_n'$. Then

$$\|[y, \tilde{u}(t)]\| = \|\tilde{u}(t) - \sigma(\tilde{u}(t))\| < \|u(t) - \sigma(u(t))\| + \delta_1/3 < \delta + \delta_1/3 \le \delta_1/2$$

for every $t \in [0,1]$. Hence $\|[y, \tilde{u}(t)\tilde{u}(0)^*]\|$ is less than $\delta_1$. It follows from [8, Lemma 4.2] that one can find a path of unitaries $w : [0,1] \to A_m \cap A_n'$ such that

$$w(0) = 1, \quad w(1) = \tilde{u}(1)\tilde{u}(0)^*, \quad \mathrm{Lip}(w) \le \pi + \varepsilon$$

and $\|[y, w(t)]\| < \varepsilon/2$ for every $t \in [0,1]$. Note that $yw(t)y^*$ is equal to $\sigma(w(t))$. By perturbing $w(t)u(0)$ a little bit, the required $v : [0,1] \to A$ is obtained.

Suppose that $\alpha$ is an approximately inner automorphism of $A$ such that $\alpha^m$ is uniformly outer for all $m \in \mathbb{N}$. By Theorem 4.8 and Theorem 4.9, there exist $\mu \in \mathrm{Aut}(A)$ and a unitary $u \in A$ such that $\mathrm{Ad}\, u \circ \alpha = \mu \circ \sigma \circ \mu^{-1}$. Moreover, one can choose $u$ arbitrarily close to 1, because $A$ is AF (see [6, 12]). Therefore the assertion also holds for $\alpha$. $\square$

**Remark 4.11.** In the proof of the lemma above, it is easily seen that (the $\mathbb{Z}$-action generated by) $\sigma$ is asymptotically representable ([7, Definition 2.2]). Hence the automorphism $\alpha$ stated in the lemma above is also asymptotically representable. Besides, it is not so hard to see that the crossed product $C^*$-algebra $A \rtimes_\sigma \mathbb{Z}$ is a unital simple AT algebra with real rank zero. Therefore, $A \rtimes_\alpha \mathbb{Z}$ is a unital simple AT algebra with real rank zero, too.

## 5 The Rohlin property of $\mathbb{Z}^2$-actions on AF algebras

In this section, we would like to show that certain $\mathbb{Z}^2$-actions on an AF algebra have the Rohlin property. This is a generalization of Nakamura's theorem [23, Theorem 3].

Throughout this section, we keep the following setting. Let $A$ be a unital simple separable $C^*$-algebra with tracial rank zero and suppose that $A$ has a unique tracial state $\tau$. Suppose that automorphisms $\alpha, \beta \in \mathrm{Aut}(A)$ and a unitary $w \in A$ satisfy

$$\beta \circ \alpha = \mathrm{Ad}\, w \circ \alpha \circ \beta$$

and $\alpha^m \circ \beta^n$ is uniformly outer for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$. We remark that $\alpha$ and $\beta$ induce a $\mathbb{Z}^2$-action on $A_\infty$.

**Lemma 5.1.** *For any $m_1, m_2 \in \mathbb{N}$, there exists a central sequence of projections $(e_n)_n$ in $A$ such that*

$$\lim_{n\to\infty} \tau(e_n) = \frac{1}{m_1 m_2}$$

*and*

$$\lim_{n\to\infty} \|\beta^j(\alpha^i(e_n))\beta^l(\alpha^k(e_n))\| = 0$$

*for all $(i, j) \neq (k, l)$ in $\{(i, j) \mid 0 \leq i \leq m_1{-}1,\ 0 \leq j \leq m_2{-}1\}$.*

*Proof.* Set $I = \{(i, j) \mid 0 \leq i \leq m_1{-}1,\ 0 \leq j \leq m_2{-}1\}$. Let $(\pi_\tau, H_\tau)$ be the GNS representation associated with $\tau$. It is well-known that $\pi_\tau(A)''$ is a hyperfinite $\mathrm{II}_1$-factor (see [26, Lemma 2.16]). For $(i, j) \in \mathbb{Z}^2$, we put $\varphi_{(i,j)} = \alpha^i \circ \beta^j$. Then $\varphi : \mathbb{Z}^2 \to \mathrm{Aut}(A)$ is a cocycle action of $\mathbb{Z}^2$. We denote its extension to $\pi_\tau(A)''$ by $\bar{\varphi}$. Since $\alpha^m \circ \beta^n$ is uniformly outer for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, $\bar{\varphi}$ is an outer cocycle action of $\mathbb{Z}^2$ on $\pi_\tau(A)''$. It follows from [25] that there exists a sequence of projections $(e_n)_n$ in $\pi_\tau(A)''$ such that

$$\sum_{(i,j)\in I} \beta^j(\alpha^i(e_n)) \to 1, \quad [x, e_n] \to 0 \quad \forall x \in \pi_\tau(A)''$$

and

$$\beta^j(\alpha^i(e_n))\beta^l(\alpha^k(e_n)) \to 0 \quad \forall (i, j), (k, l) \in I \text{ with } (i, j) \neq (k, l)$$

in the strong operator topology as $n \to \infty$. By [26, Lemma 2.15], we may replace $e_n$ with projections in $A$. Applying Proposition 4.1 to $\Gamma = \{\beta^j \circ \alpha^i \mid (i, j) \in I\}$, we obtain the conclusion. □

Let $(e_n)_n$ be the projections as in the lemma above. If $\alpha^r$ is in $\overline{\mathrm{Inn}}(A)$ for some $r \in \mathbb{N}$ and $m_1$ is large enough, then we can construct a central sequence of projections $(e'_n)_n$ in $A$ such that

$$\lim_{n\to\infty} \|e'_n(e_n + \alpha(e_n) + \cdots + \alpha^{m_1-1}(e_n)) - e'_n\| = 0,$$

$$\alpha(e'_n) \approx e'_n \quad \text{and} \quad \tau(e'_n) \approx m_1\tau(e_n),$$

by using the arguments in [9, Lemma 3.1] (see also [23, Lemma 6]). Consequently, we get the following.

**Lemma 5.2.** *If $\alpha^r$ is in $\overline{\mathrm{Inn}}(A)$ for some $r \in \mathbb{N}$, then for any $m \in \mathbb{N}$, there exists a central sequence of projections $(e_n)_n$ in $A$ such that*

$$\lim_{n\to\infty} \tau(e_n) = \frac{1}{m}, \qquad \lim_{n\to\infty} \|e_n - \alpha(e_n)\| = 0$$

*and*

$$\lim_{n\to\infty} \|e_n \beta^j(e_n)\| = 0$$

*for all $j = 1, 2, \ldots, m{-}1$.*

Our next task is to achieve the cyclicity condition $\beta^m(e_n) \approx e_n$.

**Lemma 5.3.** *Suppose that $A$ is AF. Suppose that a projection $e \in A_\infty$ and a partial isometry $u \in A_\infty$ satisfy $e = \alpha(e)$ and $e = u^*u = uu^*$. Then there exists a partial isometry $w \in A_\infty$ such that $w^*w = ww^* = e$ and $u = w^*\alpha(w)$.*

*Proof.* Theorem 4.8 tells us that $\alpha$ possesses the Rohlin property. We can modify the standard argument deducing stability from the Rohlin property (see [6, 4]) and apply it to the unitary $u + (1 - e)$. We leave the details to the readers. $\square$

**Lemma 5.4.** *Suppose that either of the following holds.*

1. *$A$ is AF, $\alpha^r$ is approximately inner for some $r \in \mathbb{N}$ and $\beta^s$ is approximately inner for some $s \in \mathbb{N}$.*
2. *$\alpha^r$ is approximately inner for some $r \in \mathbb{N}$ and there exist a natural number $s \in \mathbb{N}$ and a sequence of unitaries $(u_n)_n$ in $A$ such that*

$$\lim_{n\to\infty} \|u_n - \alpha(u_n)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n a u_n^* - \beta^s(a)\| = 0 \quad \forall a \in A.$$

*Then for any $m \in \mathbb{N}$, there exists a central sequence of projections $(e_n)_n$ such that*

$$\lim_{n\to\infty} \tau(e_n) = \frac{1}{m}, \quad \lim_{n\to\infty} \|e_n - \alpha(e_n)\| = 0, \quad \lim_{n\to\infty} \|e_n \beta^j(e_n)\| = 0$$

*for all $j = 1, 2, \ldots, m{-}1$ and*

$$\lim_{n\to\infty} \|e_n - \beta^m(e_n)\| = 0.$$

*Proof.* Choose a large natural number $l$ such that $l \equiv 1 \pmod{s}$. By using Lemma 5.2 and the assumption that $\beta^s$ is in $\overline{\mathrm{Inn}}(A)$ for some $s \in \mathbb{N}$, one can find a projection $e \in A_\infty$ and a partial isometry $v \in A_\infty$ such that

$$e = \alpha(e), \quad v^*v = e, \quad vv^* = \beta(e) \quad \text{and} \quad e\beta^j(e) = 0 \quad \forall j = 1, 2, \ldots, l{-}1$$

in the same way as the proof of Theorem 4.4. Moreover, we have

$$\lim_{n\to\infty} \tau(e_n) = l^{-1},$$

where $(e_n)_n$ is a representative sequence of $e$ consisting of projections. Note that $\beta^j(e)$ is fixed by $\alpha$, because $e$ is a central sequence. In the case (2), clearly we may further assume $v = \alpha(v)$. In the case (1), the lemma above applies to $v^*\alpha(v)$ and yields $w \in A_\infty$ satisfying $w^*w = e$, $ww^* = e$ and $v^*\alpha(v) = w^*\alpha(w)$. By replacing $v$ with $vw^*$, we get $v = \alpha(v)$, too. Then the conclusion follows from exactly the same argument as [9, Lemma 4.3]. $\square$

**Theorem 5.5.** *Suppose that the conclusion of Lemma 5.4 holds. Then for any $m \in \mathbb{N}$, there exist projections $e$ and $f$ in $A_\infty$ such that*

$$\alpha(e) = e, \quad \alpha(f) = f, \quad \beta^m(e) = e, \quad \beta^{m+1}(f) = f$$

*and*

$$\sum_{i=0}^{m-1} \beta^i(e) + \sum_{j=0}^{m} \beta^j(f) = 1.$$

*Proof.* Let $(e_n)_n$ be the central sequence of projections obtained in Lemma 5.4. Define

$$f_n = 1 - \sum_{j=0}^{m-1} \beta^j(e_n).$$

There exists a sequence of unitaries $(u_n)_n$ in $A$ such that $u_n \to 1$ as $n \to \infty$ and $u_n\alpha(e_n)u_n^* = e_n$ for sufficiently large $n$. The $\mathbb{Z}$-action on $e_nAe_n$ generated by $\mathrm{Ad}\, u_n \circ \alpha$ is uniformly outer, and so it has the tracial Rohlin property by Theorem 4.2 (or [26, Theorem 2.17]). It follows that, for any $k \in \mathbb{N}$, there exists a central sequence of projections $(\tilde{e}_n)_n$ such that

$$\tilde{e}_n \leq e_n, \quad \lim_{n\to\infty} \tau(\tilde{e}_n) = 1/mk, \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{e}_n\alpha^i(\tilde{e}_n)\| = 0 \quad \forall i = 1, 2, \dots, k-1.$$

Let $e, f, \tilde{e} \in A_\infty$ be the images of $(e_n)_n, (f_n)_n, (\tilde{e}_n)_n$, respectively. By Lemma 3.3, there exists a partial isometry $v$ such that $v^*v = f$ and $vv^* \leq \tilde{e}$. We define a partial isometry $\tilde{v} \in A_\infty$ by

$$\tilde{v} = \frac{1}{\sqrt{k}} \sum_{i=0}^{k-1} \alpha^i(v).$$

Then one has

$$\tilde{v}^*\tilde{v} = f, \quad \tilde{v}\tilde{v}^* \leq e \quad \text{and} \quad \|\tilde{v} - \alpha(\tilde{v})\| < 2/\sqrt{k}.$$

By a standard trick on central sequences, we may assume $\alpha(\tilde{v}) = \tilde{v}$. Thus, we have obtained the $\alpha$-invariant version of the conclusion of Lemma 4.6. We can complete the proof by the same argument as in Theorem 4.7. □

The following is a generalization of [23, Theorem 3].

**Corollary 5.6.** *Let $\varphi : \mathbb{Z}^2 \curvearrowright A$ be a $\mathbb{Z}^2$-action on a unital simple AF algebra $A$ with unique trace. When $\varphi_{(r,0)}$ and $\varphi_{(0,s)}$ are approximately inner for some $r, s \in \mathbb{N}$, the following are equivalent.*

(1) *$\varphi$ has the Rohlin property.*

(2) *$\varphi$ is uniformly outer.*

*Proof.* This immediately follows from Theorem 5.5 and [23, Remark 2] (see also [22, Remark 2.2]). □

The next corollary also follows from Theorem 5.5 immediately, because condition (2) of Lemma 5.4 is satisfied in this case. See [7, Definition 2.2] for the definition of approximate representability.

**Corollary 5.7.** *Let $\varphi : \mathbb{Z}^2 \curvearrowright A$ be an approximately representable $\mathbb{Z}^2$-action on a unital simple AH algebra $A$ with real rank zero and slow dimension growth. Suppose that $A$ has a unique trace. Then the following are equivalent.*

(1) *$\varphi$ has the Rohlin property.*

(2) *$\varphi$ is uniformly outer.*

# 6 Classification of certain $\mathbb{Z}^2$-actions on AF algebras

In this section, we will show a classification result of a certain class of $\mathbb{Z}^2$-actions on unital simple AF algebras. We freely use the terminology and notation introduced in [7, Definition 2.1]. For an automorphism $\alpha$ of a $C^*$-algebra $A$, we write the crossed product $C^*$-algebra $A \rtimes_\alpha \mathbb{Z}$ by $C^*(A, \alpha)$ and the implementing unitary by $\lambda_\alpha$. The mapping torus $M(A, \alpha)$ is defined by

$$M(A, \alpha) = \{f \in C([0, 1], A) \mid \alpha(f(0)) = f(1)\}.$$

When $A$ is an AF algebra, '$KK$-triviality' of $\alpha \in \mathrm{Aut}(A)$ is equivalent to $K_0(\alpha) = \mathrm{id}$, and also equivalent to $\alpha$ being approximately inner.

The following theorem is a $\mathbb{Z}$-equivariant version of Theorem 4.9. Let $A$ be a unital simple AF algebra with unique trace and let $\alpha \in \overline{\mathrm{Inn}}(A)$. Let $\mathrm{Aut}_{\mathbb{T}}(C^*(A, \alpha))$ denote the set of all automorphisms of $C^*(A, \alpha)$ commuting with the dual action $\hat{\alpha}$. For $i = 1, 2$, we suppose that an automorphism $\beta_i \in \mathrm{Aut}(A)$ and a unitary $w_i \in A$ are given and satisfy

$$\beta_i \circ \alpha = \mathrm{Ad}\, w_i \circ \alpha \circ \beta_i.$$

Then $\beta_i$ extends to $\tilde{\beta}_i \in \mathrm{Aut}_{\mathbb{T}}(C^*(A, \alpha))$ by setting $\tilde{\beta}_i(\lambda_\alpha) = w_i \lambda_\alpha$. Suppose further that $\alpha^m \circ \beta_i^n$ is uniformly outer for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and that $\beta_i^{s_i}$ is approximately inner for some $s_i \in \mathbb{N}$.

**Theorem 6.1.** *In the setting above, if $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are asymptotically unitarily equivalent, then there exist an approximately inner automorphism $\mu \in \mathrm{Aut}_{\mathbb{T}}(C^*(A, \alpha))$ and a unitary $v \in A$ such that $\mu|A$ is also approximately inner and*

$$\mu \circ \tilde{\beta}_1 \circ \mu^{-1} = \mathrm{Ad}\, v \circ \tilde{\beta}_2.$$

*Proof.* We can apply the argument of [24, Theorem 5] to $\tilde{\beta}_1$ and $\tilde{\beta}_2$ in a similar fashion to [7, Theorem 4.11]. By Remark 4.11, (the $\mathbb{Z}$-action generated by) $\alpha$ is asymptotically representable. Then [7, Theorem 4.8] implies that $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are $\mathbb{T}$-asymptotically unitarily equivalent. Moreover, by Theorem 5.5, we can find Rohlin projections for $\tilde{\beta}_i$ in the fixed point algebra $(A_\infty)^\alpha$. Hence, by using Lemma 4.10 instead of [24, Theorem 7], the usual intertwining argument shows the statement. □

Let us recall the OrderExt invariant introduced in [13]. Let $G_0, G_1, F$ be abelian groups and let $D : G_0 \to F$ be a homomorphism. When

$$\xi : \quad 0 \longrightarrow G_0 \xrightarrow{\ \iota\ } E_\xi \xrightarrow{\ q\ } G_1 \longrightarrow 0$$

is exact, $R$ is in $\mathrm{Hom}(E_\xi, F)$ and $R \circ \iota = D$, the pair $(\xi, R)$ is called an order-extension. Two order-extensions $(\xi, R)$ and $(\xi', R')$ are equivalent if there exists an isomorphism $\theta : E_\xi \to E_{\xi'}$ such that $R = R' \circ \theta$ and

$$\begin{array}{cccccccccc} \xi: & 0 & \longrightarrow & G_0 & \longrightarrow & E_\xi & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\theta} & & \| & & \\ \xi': & 0 & \longrightarrow & G_0 & \longrightarrow & E_{\xi'} & \longrightarrow & G_1 & \longrightarrow & 0 \end{array}$$

is commutative. Then $\mathrm{OrderExt}(G_1, G_0, D)$ consists of equivalence classes of all order-extensions. As shown in [13], $\mathrm{OrderExt}(G_1, G_0, D)$ is equipped with an abelian group structure. The map sending $(\xi, R)$ to $\xi$ induces a homomorphism from $\mathrm{OrderExt}(G_1, G_0, D)$ onto $\mathrm{Ext}(G_1, G_0)$.

Let $B$ be a unital $C^*$-algebra with $T(B)$ non-empty. We denote by $\mathrm{Aut}_0(B)$ the set of all automorphisms $\gamma$ of $B$ such that $K_0(\gamma) = K_1(\gamma) = \mathrm{id}$ and $\tau \circ \gamma = \tau$ for all $\tau \in T(B)$. When $B$ is a unital simple AT algebra with real rank zero, $\mathrm{Aut}_0(B)$ equals $\overline{\mathrm{Inn}}(B)$. Let $D_B : K_0(B) \to \mathrm{Aff}(T(B))$ denote the dimension map defined by $D_B([p])(\tau) = \tau(p)$. As described in [13], there exist natural homomorphisms

$$\tilde{\eta}_0 : \mathrm{Aut}_0(B) \to \mathrm{OrderExt}(K_1(B), K_0(B), D_B)$$

and

$$\eta_1 : \mathrm{Aut}_0(B) \to \mathrm{Ext}(K_0(B), K_1(B)).$$

The following is the main result of [13]. See [21, 18] for further developments.

**Theorem 6.2** ([13, Theorem 4.4]). *Suppose that $B$ is a unital simple AT algebra with real rank zero. Then the homomorphism*

$$\tilde{\eta}_0 \oplus \eta_1 : \overline{\mathrm{Inn}}(B) \to \mathrm{OrderExt}(K_1(B), K_0(B), D_B) \oplus \mathrm{Ext}(K_0(B), K_1(B))$$

*is surjective and its kernel equals the set of all asymptotically inner automorphisms of $B$.*

By using this OrderExt invariant, we introduce an invariant of certain $\mathbb{Z}^2$-actions as follows. Let $A$ be a unital simple AF algebra and let $\varphi : \mathbb{Z}^2 \curvearrowright A$ be an action of $\mathbb{Z}^2$ on $A$. Suppose that $\varphi$ is uniformly outer and locally $KK$-trivial (i.e. locally approximately inner). We write $B = C^*(A, \varphi_{(1,0)})$. Then $\varphi_{(0,1)}$ extends to $\tilde{\varphi}_{(0,1)} \in \mathrm{Aut}(B)$ by setting $\tilde{\varphi}_{(0,1)}(\lambda_{\varphi_{(1,0)}}) = \lambda_{\varphi_{(1,0)}}$. Let $\iota : A \to B = C^*(A, \varphi_{(1,0)})$ be the canonical inclusion. One can check the following immediately.

- $K_0(\iota)$ is an isomorphism from $K_0(A)$ to $K_0(B)$.
- The connecting map $\partial : K_1(B) \to K_0(A)$ in the Pimsner-Voiculescu exact sequence is an isomorphism and $\partial^{-1}([p]) = [\lambda_{\varphi_{(1,0)}} \iota(p) + \iota(v(1-p))]$ for any projection $p \in A$, where $v$ is a unitary of $A$ satisfying $vpv^* = \varphi_{(1,0)}(p)$.
- The map $\iota^* : T(B) \to T(A)$ sending $\tau$ to $\tau \circ \iota$ is an isomorphism and satisfies $D_B(K_0(\iota)(x))(\tau) = D_A(x)(\iota^*(\tau))$ for $x \in K_0(A)$ and $\tau \in T(B)$.

From these properties, we can obtain a natural isomorphism

$$\zeta_{\varphi_{(1,0)}} : \mathrm{OrderExt}(K_1(B), K_0(B), D_B) \to \mathrm{OrderExt}(K_0(A), K_0(A), D_A).$$

In addition, it is easy to see $K_0(\tilde{\varphi}_{(0,1)}) = K_1(\tilde{\varphi}_{(0,1)}) = \mathrm{id}$ and $\tau \circ \tilde{\varphi}_{(0,1)} = \tau$ for all $\tau \in T(B)$, that is, $\tilde{\varphi}_{(0,1)}$ belongs to $\mathrm{Aut}_0(B)$.

**Lemma 6.3.** *In the setting above, $\eta_1(\tilde{\varphi}_{(0,1)}) \in \mathrm{Ext}(K_0(B), K_1(B))$ is zero.*

*Proof.* There exists a natural commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0((0,1), B) & \longrightarrow & M(B, \tilde{\varphi}_{(0,1)}) & \longrightarrow & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & C_0((0,1), A) & \longrightarrow & M(A, \varphi_{(0,1)}) & \longrightarrow & A & \longrightarrow & 0,
\end{array}$$

where the horizontal sequences are exact. From the naturality of the six-term exact sequence, we obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(B) & \longrightarrow & K_0(M(B, \tilde{\varphi}_{(0,1)})) & \longrightarrow & K_0(B) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow{\scriptstyle K_0(\iota)} & & \\
0 & \longrightarrow & K_1(A) & \longrightarrow & K_0(M(A, \varphi_{(0,1)})) & \longrightarrow & K_0(A) & \longrightarrow & 0,
\end{array}$$

where the horizontal sequences are exact. Since $K_1(A)$ is zero and $K_0(\iota)$ is an isomorphism, we can conclude $\eta_1(\tilde{\varphi}_{(0,1)}) = 0$. □

**Definition 6.4.** In the setting above, we define our invariant $[\varphi]$ by

$$[\varphi] = \zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\tilde{\varphi}_{(0,1)})) \in \mathrm{OrderExt}(K_0(A), K_0(A), D_A).$$

**Proposition 6.5.** *Let $\varphi, \psi : \mathbb{Z}^2 \curvearrowright A$ be uniformly outer, locally $KK$-trivial $\mathbb{Z}^2$-actions on a unital simple AF algebra $A$. If $\varphi$ and $\psi$ are $KK$-trivially cocycle conjugate, then $[\varphi] = [\psi]$.*

*Proof.* For $\mu \in \overline{\mathrm{Inn}}(A)$, it is straightforward to see that the $\mathbb{Z}^2$-action $\mu \circ \varphi \circ \mu^{-1}$ has the same invariant as $\varphi$. Hence, it suffices to show $[\varphi] = [\varphi^u]$ for any $\varphi$-cocycle $\{u_n\}_{n \in \mathbb{Z}^2}$. Define an isomorphism $\pi$ from $C^*(A, \varphi_{(1,0)})$ to $C^*(A, \varphi^u_{(1,0)})$ by

$$\pi(\lambda_{\varphi_{(1,0)}}) = u^*_{(1,0)} \lambda_{\varphi^u_{(1,0)}} \quad \text{and} \quad \pi(a) = a \quad \forall a \in A,$$

where $A$ is identified with subalgebras of the crossed products. For $\gamma \in \mathrm{Aut}(C^*(A, \varphi^u_{(1,0)}))$, one can check

$$\zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\pi^{-1} \circ \gamma \circ \pi)) = \zeta_{\varphi^u_{(1,0)}}(\tilde{\eta}_0(\gamma)) \in \mathrm{OrderExt}(K_0(A), K_0(A), D_A),$$

where $\tilde{\eta}_0$ in the left hand side is defined for $C^*(A, \varphi_{(1,0)})$ and $\tilde{\eta}_0$ in the right hand side is defined for $C^*(A, \varphi^u_{(1,0)})$. We also have

$$(\pi^{-1} \circ \tilde{\varphi}^u_{(0,1)} \circ \pi)(a) = \pi^{-1}(\tilde{\varphi}^u_{(0,1)}(a)) = \tilde{\varphi}^u_{(0,1)}(a) = (\mathrm{Ad}\, u_{(0,1)} \circ \tilde{\varphi}_{(0,1)})(a) \quad \forall a \in A$$

and

$$\begin{aligned}(\pi^{-1}\circ\tilde{\varphi}^u_{(0,1)}\circ\pi)(\lambda_{\varphi_{(1,0)}}) &= (\pi^{-1}\circ\tilde{\varphi}^u_{(0,1)})(u^*_{(1,0)}\lambda_{\varphi^u_{(1,0)}})\\ &= \pi^{-1}(\tilde{\varphi}^u_{(0,1)}(u^*_{(1,0)})\lambda_{\varphi^u_{(1,0)}})\\ &= \varphi^u_{(0,1)}(u^*_{(1,0)})u_{(1,0)}\lambda_{\varphi_{(1,0)}}\\ &= u_{(0,1)}\varphi_{(0,1)}(u^*_{(1,0)})u^*_{(0,1)}u_{(1,0)}\lambda_{\varphi_{(1,0)}}\\ &= u_{(0,1)}\varphi_{(1,0)}(u^*_{(0,1)})u^*_{(1,0)}u_{(1,0)}\lambda_{\varphi_{(1,0)}}\\ &= u_{(0,1)}\varphi_{(1,0)}(u^*_{(0,1)})\lambda_{\varphi_{(1,0)}}\\ &= u_{(0,1)}\lambda_{\varphi_{(1,0)}}u^*_{(0,1)}\\ &= (\operatorname{Ad}u_{(0,1)}\circ\tilde{\varphi}_{(0,1)})(\lambda_{\varphi_{(1,0)}}).\end{aligned}$$

Thus $\pi^{-1}\circ\tilde{\varphi}^u_{(0,1)}\circ\pi=\operatorname{Ad}u_{(0,1)}\circ\tilde{\varphi}_{(0,1)}$. Since inner automorphisms are contained in the kernel of $\tilde{\eta}_0$, we obtain

$$\begin{aligned}\zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\tilde{\varphi}_{(0,1)})) &= \zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\operatorname{Ad}u_{(0,1)}\circ\tilde{\varphi}_{(0,1)}))\\ &= \zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\pi^{-1}\circ\tilde{\varphi}^u_{(0,1)}\circ\pi))\\ &= \zeta_{\varphi^u_{(1,0)}}(\tilde{\eta}_0(\tilde{\varphi}^u_{(0,1)})),\end{aligned}$$

which completes the proof. □

**Theorem 6.6.** *Let $\varphi,\psi:\mathbb{Z}^2\curvearrowright A$ be uniformly outer, locally $KK$-trivial $\mathbb{Z}^2$-actions on a unital simple AF algebra $A$ with unique trace. The following are equivalent.*

(1) $[\varphi]=[\psi]$.

(2) *$\varphi$ and $\psi$ are $KK$-trivially cocycle conjugate.*

*Proof.* (2)⇒(1) was shown in the proposition above without assuming that $A$ has a unique trace. Let us consider the other implication (1)⇒(2). By Theorem 4.8 and Theorem 4.9, we may assume that there exists a unitary $u\in A$ such that $\psi_{(1,0)}=\operatorname{Ad}u\circ\varphi_{(1,0)}$. By Theorem 4.8 and Remark 4.11 (or Remark 4.5), the crossed product $C^*$-algebra $C^*(A,\varphi_{(1,0)})$ is a unital simple AT algebra with real rank zero.

Clearly $\varphi_{(0,1)}$ extends to $\tilde{\varphi}_{(0,1)}\in\operatorname{Aut}(C^*(A,\varphi_{(1,0)}))$ by

$$\tilde{\varphi}_{(0,1)}(a)=a\quad\forall a\in A\quad\text{and}\quad\tilde{\varphi}_{(0,1)}(\lambda_{\varphi_{(1,0)}})=\lambda_{\varphi_{(1,0)}}.$$

Since

$$\psi_{(0,1)}\circ\varphi_{(1,0)}=\operatorname{Ad}(\psi_{(0,1)}(u^*)u)\circ\varphi_{(1,0)}\circ\psi_{(0,1)},$$

we can extend $\psi_{(0,1)}$ to $\omega\in\operatorname{Aut}(C^*(A,\varphi_{(1,0)}))$ by

$$\omega(a)=a\quad\forall a\in A\quad\text{and}\quad\omega(\lambda_{\varphi_{(1,0)}})=\psi_{(0,1)}(u^*)u\lambda_{\varphi_{(1,0)}}.$$

In order to apply Theorem 6.1 to $\tilde{\varphi}_{(0,1)}$ and $\omega$, we would like to check that these automorphisms are asymptotically unitarily equivalent. There exists an isomorphism $\pi: C^*(A,\varphi_{(1,0)})\to C^*(A,\psi_{(1,0)})$ defined by

$$\pi(a)=a\quad\forall a\in A\quad\text{and}\quad\pi(\lambda_{\varphi_{(1,0)}})=u^*\lambda_{\psi_{(1,0)}}.$$

As mentioned in the proof of Proposition 6.5, for any $\gamma \in \mathrm{Aut}(C^*(A, \varphi_{(1,0)}))$, one has

$$\zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\gamma)) = \zeta_{\psi_{(1,0)}}(\tilde{\eta}_0(\pi \circ \gamma \circ \pi^{-1})).$$

Moreover it is easy to see that $\pi \circ \omega \circ \pi^{-1}$ is equal to $\tilde{\psi}_{(0,1)}$, which is defined by

$$\tilde{\psi}_{(0,1)}(a) = a \quad \forall a \in A \quad \text{and} \quad \tilde{\psi}_{(0,1)}(\lambda_{\psi_{(1,0)}}) = \lambda_{\psi_{(1,0)}}.$$

It follows that

$$\begin{aligned} \zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\omega)) &= \zeta_{\psi_{(1,0)}}(\tilde{\eta}_0(\pi \circ \omega \circ \pi^{-1})) \\ &= \zeta_{\psi_{(1,0)}}(\tilde{\eta}_0(\tilde{\psi}_{(0,1)})) = [\psi] = [\varphi] = \zeta_{\varphi_{(1,0)}}(\tilde{\eta}_0(\tilde{\varphi}_{(0,1)})), \end{aligned}$$

and so $\tilde{\eta}_0(\omega) = \tilde{\eta}_0(\tilde{\varphi}_{(0,1)})$. By Lemma 6.3, $\eta_1(\omega) = \eta_1(\tilde{\varphi}_{(0,1)}) = 0$. Therefore, by Theorem 6.2, $\tilde{\varphi}_{(0,1)}$ and $\omega$ are asymptotically unitarily equivalent.

Then, Theorem 6.1 applies and yields an approximately inner automorphism $\mu \in \mathrm{Aut}_{\mathbb{T}}(C^*(A, \varphi_{(1,0)}))$ and a unitary $v \in A$ such that $\mu|A$ is in $\overline{\mathrm{Inn}}(A)$ and

$$\mu \circ \omega \circ \mu^{-1} = \mathrm{Ad}\, v \circ \tilde{\varphi}_{(0,1)}. \tag{6.1}$$

By restricting this equality to $A$, we get

$$(\mu|A) \circ \psi_{(0,1)} \circ (\mu|A)^{-1} = \mathrm{Ad}\, v \circ \varphi_{(0,1)}. \tag{6.2}$$

Let $z \in A$ be the unitary satisfying $\mu(\lambda_{\varphi_{(1,0)}}) = z\lambda_{\varphi_{(1,0)}}$. Then

$$(\mu|A) \circ \psi_{(1,0)} \circ (\mu|A)^{-1} = (\mu|A) \circ \mathrm{Ad}\, u \circ \varphi_{(1,0)} \circ (\mu|A)^{-1} = \mathrm{Ad}\, \mu(u)z \circ \varphi_{(1,0)}. \tag{6.3}$$

From (6.1), one can see that

$$\begin{aligned} (\mu \circ \omega \circ \mu^{-1})(\lambda_{\varphi_{(1,0)}}) &= (\mu \circ \omega)(\mu^{-1}(z^*)\lambda_{\varphi_{(1,0)}}) \\ &= \mu(\psi_{(0,1)}(\mu^{-1}(z^*))\psi_{(0,1)}(u^*)u\lambda_{\varphi_{(1,0)}}) \\ &= (\mathrm{Ad}\, v \circ \varphi_{(0,1)})(z^*\mu(u^*))\mu(u)z\lambda_{\varphi_{(1,0)}} \\ &= v\varphi_{(0,1)}(z^*\mu(u^*))v^*\mu(u)z\lambda_{\varphi_{(1,0)}} \end{aligned}$$

is equal to

$$\begin{aligned} (\mathrm{Ad}\, v \circ \tilde{\varphi}_{(0,1)})(\lambda_{\varphi_{(1,0)}}) &= v\lambda_{\varphi_{(1,0)}}v^* \\ &= v\varphi_{(1,0)}(v^*)\lambda_{\varphi_{(1,0)}}. \end{aligned}$$

Hence one obtains

$$v\varphi_{(0,1)}(\mu(u)z) = \mu(u)z\varphi_{(1,0)}(v). \tag{6.4}$$

It follows from (6.2), (6.3), (6.4) that $\psi$ and $\varphi$ are $KK$-trivially cocycle conjugate. $\square$

**Remark 6.7.** We do not know the precise range of our invariant which takes its values in OrderExt. At least, the following observation shows that the range does not exhaust OrderExt. Let $\varphi : \mathbb{Z}^2 \curvearrowright A$ be a locally $KK$-trivial and uniformly outer $\mathbb{Z}^2$-action on a unital simple AF algebra. Suppose that $(\xi, R)$ is a representative of $[\varphi] \in \mathrm{OrderExt}(K_0(A), K_0(A), D_A)$. Since

$$\xi: \quad 0 \longrightarrow K_0(A) \xrightarrow{\iota} E_\xi \xrightarrow{q} K_0(A) \longrightarrow 0$$

is exact and $R : E_\xi \to \mathrm{Aff}(T(A))$ satisfies $R \circ \iota = D_A$, there exists a homomorphism $R_0 : K_0(A) \to \mathrm{Aff}(K_0(A))/\operatorname{Im} D_A$ such that $R_0(q(x)) = R(x) + D_A(K_0(A))$ for any $x \in E_\xi$. It is easy to see $R_0([1_A]) = 0$, because the implementing unitary $\lambda_{\varphi_{(1,0)}}$ is fixed by $\tilde{\varphi}_{(0,1)}$. Thus, $[\varphi]$ belongs to the subgroup

$$\{[(\xi, R)] \in \mathrm{OrderExt}(K_0(A), K_0(A), D_A) \mid R_0([1_A]) = 0\}.$$

When $A$ is a UHF algebra, one can see that this subgroup coincides with the range of the invariant introduced in [8]. Therefore, Theorem 6.6 yields a new proof of [8, Theorem 6.5].

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