

Torsion in coinvariants of certain Cantor minimal \mathbb{Z}^2 -systems

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Abstract

Let G be a finite abelian group. We will consider a skew product extension of a product of two Cantor minimal \mathbb{Z} -systems associated with a G -valued cocycle. When G is non-cyclic and the cocycle is non-degenerate, it will be shown that the skew product system has torsion in its coinvariants.

1 Introduction

Let X be a Cantor set and let Γ be a countable discrete abelian group which acts on X . We write the action of Γ by $\varphi : \Gamma \rightarrow \text{Homeo}(X)$. We define the group of coinvariants of (X, φ) by

$$\text{Coinv}(X, \varphi) = C(X, \mathbb{Z}) / \langle \{f - f \circ \varphi^{-\gamma} : f \in C(X, \mathbb{Z}), \gamma \in \Gamma\} \rangle,$$

where $C(X, \mathbb{Z})$ denotes the set of integer-valued continuous functions on X . When G is \mathbb{Z}^d , the group of coinvariants is known to be isomorphic to the top-dimensional dynamical cohomology $H^d(\mathbb{Z}^d, C(X, \mathbb{Z}))$ of (X, φ) . The aim of this paper is to compute the torsion part of $\text{Coinv}(X, \varphi)$ for a certain minimal action of \mathbb{Z}^2 .

Gähler [G] discovered unexpected 5-torsion in the top-dimensional cohomology of a popular two-dimensional tiling which is called the Tübingen Triangle Tiling (TTT). The TTT is one of canonical projection tilings. For these tilings, Forrest, Hunton and Kellendonk [FHK] related the cohomology of tilings to a certain group homology, and they expressed its rational rank in terms of the data defining the tiling in the cases of codimension at most three. Gähler, Hunton and Kellendonk [GHK] extended this result and computed the torsion in the cohomology for such tilings of codimension at most three. In this paper, we would like to show that torsion in the top cohomology can arise in a much elementary setting. In addition, we will see that an example from [GHK] with torsion can also be described in this context. By our method, however, we cannot ‘explain’ why the 5-torsion arises in the cohomology of the TTT. It is worth clarifying its reason in terms of dynamical systems.

Our setting is as follows. Let X and Y be Cantor sets and let $\varphi \in \text{Homeo}(X)$ and $\psi \in \text{Homeo}(Y)$ be minimal homeomorphisms. Then $\varphi \times \text{id}$ and $\text{id} \times \psi$ induce a minimal \mathbb{Z}^2 -action on $X \times Y$. This is called the product of (X, φ) and (Y, ψ) , and it is well-known that the group of coinvariants is isomorphic to the tensor product of $\text{Coinv}(X, \varphi)$ and $\text{Coinv}(Y, \psi)$. In particular, it is torsion-free. In this article, however, we will see that a skew product extension of the product system can have torsion in its coinvariants. More precisely, when G is a non-cyclic finite abelian group, a skew product extension associated with a non-degenerate G -valued cocycle always has torsion in its coinvariants. Indeed, the torsion part of coinvariants depends only on the finite abelian group G .

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2 Torsion of a certain abelian group

Let G be a finite abelian group. We use the multiplicative notation for the group operation in G . Let A and B be finite sets and let $\mu : A \cup B \rightarrow G$ be a map. Suppose that both $\mu(A)$ and $\mu(B)$ generate G . Put

$$\mathcal{M}(A, B) = \mathbb{Z}[G] \otimes \mathbb{Z}^A \otimes \mathbb{Z}^B.$$

We denote each element of $\mathcal{M}(A, B)$ by

$$r = \sum_{a \in A} \sum_{b \in B} r(a, b) \otimes a \otimes b \in \mathcal{M}(A, B),$$

where $r(a, b)$ is an element of the group ring $\mathbb{Z}[G]$.

Let $\mathcal{A} \subset \mathcal{M}(A, B)$ be the subgroup consisting of those elements

$$\sum_{a \in A} \sum_{b \in B} \alpha(a)(e - \mu(b)) \otimes a \otimes b \in \mathcal{M}(A, B)$$

with $\alpha : A \rightarrow \mathbb{Z}[G]$. In a similar fashion, let $\mathcal{B} \subset \mathcal{M}(A, B)$ be the subgroup consisting of those elements

$$\sum_{a \in A} \sum_{b \in B} \beta(b)(e - \mu(a)) \otimes a \otimes b \in \mathcal{M}(A, B)$$

with $\beta : B \rightarrow \mathbb{Z}[G]$. Put

$$\mathcal{N}(A, B) = \mathcal{M}(A, B) / (\mathcal{A} + \mathcal{B})$$

and let $\rho_{A,B} : \mathcal{M}(A, B) \rightarrow \mathcal{N}(A, B)$ be the quotient map. We denote by $T(\mathcal{N}(A, B))$ the torsion part of $\mathcal{N}(A, B)$ as a \mathbb{Z} -module, that is,

$$T(\mathcal{N}(A, B)) = \{r \in \mathcal{N}(A, B) : nr = 0 \text{ for some } n \in \mathbb{N}\}.$$

The aim of this section is to compute this group.

Let $\mathcal{M}_{\mathbb{Q}}(A, B) = \mathbb{Q}[G] \otimes \mathbb{Z}^A \otimes \mathbb{Z}^B$. We regard $\mathcal{M}(A, B)$ as a subgroup of $\mathcal{M}_{\mathbb{Q}}(A, B)$. By replacing $\mathbb{Z}[G]$ with $\mathbb{Q}[G]$, we can similarly define $\mathcal{A}_{\mathbb{Q}}$ and $\mathcal{B}_{\mathbb{Q}}$ as subgroups of $\mathcal{M}_{\mathbb{Q}}(A, B)$. It is not hard to see

$$\rho_{A,B}^{-1}(T(\mathcal{N}(A, B))) = \{s + t \in \mathcal{M}(A, B) : s \in \mathcal{A}_{\mathbb{Q}} \text{ and } t \in \mathcal{B}_{\mathbb{Q}}\}.$$

Lemma 2.1. *We have $\mathcal{A}_{\mathbb{Q}} \cap \mathcal{M}(A, B) = \mathcal{A}$ and $\mathcal{B}_{\mathbb{Q}} \cap \mathcal{M}(A, B) = \mathcal{B}$.*

Proof. It suffices to show only $\mathcal{A}_{\mathbb{Q}} \cap \mathcal{M}(A, B) = \mathcal{A}$. It is obvious that \mathcal{A} is contained in $\mathcal{A}_{\mathbb{Q}} \cap \mathcal{M}(A, B)$. Let $\alpha : A \rightarrow \mathbb{Q}[G]$ be a map and suppose that

$$r = \sum_{a \in A} \sum_{b \in B} \alpha(a)(e - \mu(b)) \otimes a \otimes b$$

belongs to $\mathcal{M}(A, B)$. We observe $\alpha(a)(e - \mu(b)) \in \mathbb{Z}[G]$ for all $a \in A$ and $b \in B$. Since $\mu(B)$ generates G , for each $a \in A$, there exists $s_a \in \mathbb{Q}$ such that

$$\alpha(a) - s_a N \in \mathbb{Z}[G],$$

where N is $\sum_{g \in G} g \in \mathbb{Z}[G]$. By replacing $\alpha(a)$ with $\alpha(a) - s_a N$, we can see that r belongs to \mathcal{A} . \square

At first, we would like to show that $T(\mathcal{N}(A, B))$ does not depend on the choice of A, B and $\mu : A \cup B \rightarrow G$. Let C and D be other finite sets and let $\nu : C \cup D \rightarrow G$ be a map. In exactly the same way, we can define $\mathcal{M}(C, D)$, \mathcal{C} , \mathcal{D} and so on. Since $\mu(A)$ generates G , we can find $\{s(a, c)\}_{a \in A, c \in C} \subset \mathbb{Z}[G]$ such that

$$\sum_{a \in A} (e - \mu(a))s(a, c) = e - \nu(c)$$

for all $c \in C$. Similarly, since $\mu(B)$ generates G , we can find $\{t(b, d)\}_{b \in B, d \in D} \subset \mathbb{Z}[G]$ such that

$$\sum_{b \in B} (e - \mu(b))t(b, d) = e - \nu(d)$$

for all $d \in D$. We can define a homomorphism π from $\mathcal{M}_{\mathbb{Q}}(A, B)$ to $\mathcal{M}_{\mathbb{Q}}(C, D)$ by

$$\pi(r \otimes a \otimes b) = \sum_{c \in C} \sum_{d \in D} rs(a, c)t(b, d) \otimes c \otimes d$$

for $r \in \mathbb{Q}[G]$ and $a \in A, b \in B$. Clearly π carries $\mathcal{M}(A, B)$ to $\mathcal{M}(C, D)$. It is also easily checked that $\pi(\mathcal{A}) \subset \mathcal{C}$, $\pi(\mathcal{B}) \subset \mathcal{D}$, $\pi(\mathcal{A}_{\mathbb{Q}}) \subset \mathcal{C}_{\mathbb{Q}}$ and $\pi(\mathcal{B}_{\mathbb{Q}}) \subset \mathcal{D}_{\mathbb{Q}}$. In particular, we obtain a homomorphism $\tilde{\pi}$ from $\mathcal{N}(A, B)$ to $\mathcal{N}(C, D)$ such that

$$\tilde{\pi}\rho_{A, B}(r) = \rho_{C, D}\pi(r)$$

for all $r \in \mathcal{M}(A, B)$.

Lemma 2.2. *The homomorphism $\tilde{\pi}$ induces an isomorphism between $T(\mathcal{N}(A, B))$ and $T(\mathcal{N}(C, D))$.*

Proof. By using $\{u(c, a)\}_{c \in C, a \in A}$ and $\{v(d, b)\}_{d \in D, b \in B}$ such that

$$\sum_{c \in C} (e - \nu(c))u(c, a) = e - \mu(a)$$

and

$$\sum_{d \in D} (e - \nu(d))v(d, b) = e - \mu(b)$$

for all $a \in A$ and $b \in B$, we can define a homomorphism $\pi' : \mathcal{M}_{\mathbb{Q}}(C, D) \rightarrow \mathcal{M}_{\mathbb{Q}}(A, B)$ in a similar fashion. It suffices to show that $\tilde{\pi}' \circ \tilde{\pi}$ induces the identity map on $T(\mathcal{N}(A, B))$.

Take $r \in \rho_{A, B}^{-1}(T(\mathcal{N}(A, B)))$ arbitrarily and let $r' = \pi'(\pi(r))$. We would like to show $r' - r \in \mathcal{A} + \mathcal{B}$. There exist $p \in \mathcal{A}_{\mathbb{Q}}$ and $q \in \mathcal{B}_{\mathbb{Q}}$ such that $r = p + q$. We can find $\alpha : A \rightarrow \mathbb{Q}[G]$ and $\beta : B \rightarrow \mathbb{Q}[G]$ such that

$$p = \sum_{a \in A} \sum_{b \in B} \alpha(a)(e - \mu(b)) \otimes a \otimes b$$

and

$$q = \sum_{a \in A} \sum_{b \in B} \beta(b)(e - \mu(a)) \otimes a \otimes b.$$

Let $p'(a', b') \in \mathbb{Q}[G]$ be the $a' \otimes b'$ -component of $\pi'(\pi(p)) \in \mathcal{A}_{\mathbb{Q}}$, that is,

$$\pi'(\pi(p)) = \sum_{a' \in A} \sum_{b' \in B} p'(a', b') \otimes a' \otimes b'.$$

Then we have

$$\begin{aligned} p'(a', b') &= \sum_{a \in A} \sum_{b \in B} \sum_{c \in C} \sum_{d \in D} \alpha(a)(e - \mu(b))s(a, c)t(b, d)u(c, a')v(d, b') \\ &= \sum_{a \in A} \sum_{c \in C} \alpha(a)(e - \mu(b'))s(a, c)u(c, a'), \end{aligned}$$

and so

$$\begin{aligned} p'(a', b') + \mathbb{Z}[G] &= \sum_{a \in A} \sum_{c \in C} -\beta(b')(e - \mu(a))s(a, c)u(c, a') + \mathbb{Z}[G] \\ &= -\beta(b')(e - \mu(a')) + \mathbb{Z}[G] \\ &= \alpha(a')(e - \mu(b')) + \mathbb{Z}[G]. \end{aligned}$$

Therefore we get

$$\pi'(\pi(p)) - p \in \mathcal{M}(A, B).$$

By Lemma 2.1, we obtain

$$\pi'(\pi(p)) - p \in \mathcal{A}.$$

In the same way,

$$\pi'(\pi(q)) - q \in \mathcal{B}$$

is obtained. Consequently,

$$\pi'(\pi(r)) - r = (\pi'(\pi(p)) - p) + (\pi'(\pi(q)) - q) \in \mathcal{A} + \mathcal{B}.$$

Hence

$$\rho_{A,B}(r) = \rho_{A,B}(\pi'(\pi(r))) = \tilde{\pi}'(\rho_{C,D}(\pi(r))) = \tilde{\pi}'(\tilde{\pi}(\rho_{A,B}(r))),$$

thereby completing the proof. \square

By the lemma above, we have noticed that $T(\mathcal{N}(A, B))$ depends only on the finite abelian group G . Let us denote this finite abelian group $T(\mathcal{N}(A, B))$ by T_G .

Let us consider the case of $G = \mathbb{Z}_m$. Let $p \in \mathbb{Z}_m$ be a generator. Thanks to Lemma 2.2, we may assume that both A and B are singletons and $\mu(A) = \mu(B) = \{p\}$. Then we have $\mathcal{A} = \mathcal{B}$. It follows from Lemma 2.1 that $\mathcal{M}(A, B)/(\mathcal{A} + \mathcal{B}) = \mathcal{M}(A, B)/\mathcal{A}$ is torsion free. Thus $T_G = T(\mathcal{N}(A, B))$ is trivial.

We now turn to the general case. Let $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_n}$ and let p_i be the generator of the i -th component. Thus,

$$G = \{p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} : k_i = 0, 1, \dots, m_i - 1\}.$$

We denote $e + p_i + p_i^2 + \cdots + p_i^{m_i-1}$ by P_i . Let

$$Q_i = \prod_{j \neq i} P_j$$

and let $N = \sum_{g \in G} g$. Notice that $N = P_1 P_2 \cdots P_n$.

Lemma 2.3. *Suppose that $\{r_i\}_{i=1}^n \subset \mathbb{Q}[G]$ satisfies $r_i P_i \in \mathbb{Z}[G]$ and*

$$r_i(e - p_j) - r_j(e - p_i) \in \mathbb{Z}[G]$$

for all $i, j = 1, 2, \dots, n$. Then we can find $x \in \mathbb{Q}[G]$ such that

$$r_i - x(e - p_i) \in \mathbb{Z}[G]$$

for all $i = 1, 2, \dots, n$.

Proof. The proof goes by induction on n . Let us consider the case of $n = 1$. We can write

$$r_1 = \sum_{j=1}^{m_1-1} s_j p_1^j.$$

Define

$$x = \sum_{j=1}^{m_1-1} \sum_{k=1}^j s_k p_1^j.$$

Then $r_1 P_1 \in \mathbb{Z}[G]$ ensures

$$r_1 - x(e - p_1) \in \mathbb{Z}[G].$$

Suppose that the assertion has been proved in the case of $n - 1$. Let H be the subgroup generated by p_2, p_3, \dots, p_n and let G_1 be the subgroup generated by p_1 . We regard $\mathbb{Q}[H]$ and $\mathbb{Q}[G_1]$ as subrings of $\mathbb{Q}[G]$. For each $i = 2, 3, \dots, n$, we can find $r_{i,0}, r_{i,1}, \dots, r_{i,m_1-1} \in \mathbb{Q}[H]$ such that

$$r_i = \sum_{k=0}^{m_1-1} r_{i,k} p_1^k.$$

It is easy to see $r_{i,k} P_i \in \mathbb{Z}[H]$ and

$$r_{i,k}(e - p_j) - r_{j,k}(e - p_i) \in \mathbb{Z}[H]$$

for all $i, j = 2, 3, \dots, n$ and $k = 0, 1, \dots, m_1 - 1$. For each k , by the induction hypothesis, there exists $y_k \in \mathbb{Q}[H]$ such that

$$r_{i,k} - y_k(e - p_i) \in \mathbb{Z}[H]$$

for all $i = 2, 3, \dots, n$. Put

$$y = \sum_{k=0}^{m_1-1} y_k p_1^k.$$

Then it is routine to check

$$r_i - y(e - p_i) \in \mathbb{Z}[G]$$

for all $i = 2, 3, \dots, n$. Let us consider $r_1 - y(e - p_1)$. For any $i = 2, 3, \dots, n$, we have

$$(r_1 - y(e - p_1))(e - p_i) \in \mathbb{Z}[G],$$

which means that there exists $s \in \mathbb{Q}[G_1]$ such that

$$r_1 - y(e - p_1) - s P_1 \in \mathbb{Z}[G].$$

Moreover, $r_1 P_1 \in \mathbb{Z}[G]$ implies $s P_1 \in \mathbb{Z}[G_1]$. By the induction hypothesis for $n = 1$, we can find $z \in \mathbb{Q}[G_1]$ such that

$$s - z(e - p_1) \in \mathbb{Z}[G_1].$$

We define $x \in \mathbb{Q}[G]$ by

$$x = y + zQ_1.$$

It is clear that $r_i - x(e - p_i)$ belongs to $\mathbb{Z}[G]$ for all $i = 2, 3, \dots, n$. Besides, we can see

$$r_1 - x(e - p_1) = r_1 - y(e - p_1) - z(e - p_1)Q_1 \in \mathbb{Z}[G],$$

which completes the proof. □

Lemma 2.4. *Let $r_1, r_2, \dots, r_n \in \mathbb{Q}[G]$. Suppose*

$$r_i(e - p_j) - r_j(e - p_i) \in \mathbb{Z}[G]$$

for all $i, j = 1, 2, \dots, n$. Then there exist $u_1, u_2, \dots, u_n \in \mathbb{Z}[G]$ such that

$$(r_i - u_i)(e - p_j) = (r_j - u_j)(e - p_i)$$

for all $i, j = 1, 2, \dots, n$.

Proof. Let us consider $r_i P_i$. For any $j = 1, 2, \dots, n$, we have

$$r_i P_i(e - p_j) + \mathbb{Z}[G] = r_j P_i(e - p_i) + \mathbb{Z}[G] = \mathbb{Z}[G].$$

It follows that there exists $t_i \in \mathbb{Q}$ such that

$$r_i P_i - t_i N \in \mathbb{Z}[G].$$

Define $s_i \in \mathbb{Q}[G]$ by

$$s_i = r_i - t_i Q_i.$$

It is easy to see $s_i P_i \in \mathbb{Z}[G]$ and

$$s_i(e - p_j) - s_j(e - p_i) \in \mathbb{Z}[G]$$

for all $i, j = 1, 2, \dots, n$. By Lemma 2.3, there exists $x \in \mathbb{Q}[G]$ such that

$$s_i - x(e - p_i) \in \mathbb{Z}[G].$$

It is not hard to see that $u_i = s_i - x(e - p_i)$ does the work. □

We would like to compute $T_G \cong T(\mathcal{N}(A, B))$. Thanks to Lemma 2.2, we may assume that $A = B = \{p_1, p_2, \dots, p_n\}$ and that the map μ is the inclusion to G . To simplify the notation, we omit μ .

Suppose that

$$r = \sum_{i,j=1}^n r(p_i, p_j) \otimes p_i \otimes p_j \in \mathcal{M}_{\mathbb{Q}}(A, B)$$

belongs to $\mathcal{A}_{\mathbb{Q}} + \mathcal{B}_{\mathbb{Q}}$. There exist $\alpha, \beta : \{p_1, p_2, \dots, p_n\} \rightarrow \mathbb{Q}[G]$ such that

$$r(a, b) = \alpha(a)(e - b) + \beta(b)(e - a)$$

for all $a, b \in \{p_1, p_2, \dots, p_n\}$. For every $i, j = 1, 2, \dots, n$, let us consider the element

$$s_{i,j} = (\alpha(p_i) + \beta(p_i))(e - p_j).$$

In order to show that $s_{i,j}$ does not depend on the choice of α and β , suppose that there exist $\alpha', \beta' : \{p_1, p_2, \dots, p_n\} \rightarrow \mathbb{Q}[G]$ such that

$$r(a, b) = \alpha'(a)(e - b) + \beta'(b)(e - a)$$

for all $a, b \in \{p_1, p_2, \dots, p_n\}$. As before, we put

$$s'_{i,j} = (\alpha'(p_i) + \beta'(p_i))(e - p_j).$$

For any $k = 1, 2, \dots, n$, we have

$$\begin{aligned} s_{i,j}(e - p_k) &= \alpha(p_i)(e - p_j)(e - p_k) + r(p_k, p_i)(e - p_j) - \alpha(p_k)(e - p_j)(e - p_i) \\ &= \alpha(p_i)(e - p_j)(e - p_k) + r(p_k, p_i)(e - p_j) - r(p_k, p_j)(e - p_i) + \beta(p_j)(e - p_k)(e - p_i) \\ &= r(p_k, p_i)(e - p_j) - r(p_k, p_j)(e - p_i) + r(p_i, p_j)(e - p_k), \end{aligned}$$

and so

$$s_{i,j}(e - p_k) = s'_{i,j}(e - p_k).$$

Combining this with $(s_{i,j} - s'_{i,j})P_j = 0$, we obtain $s_{i,j} = s'_{i,j}$. Thus, $s_{i,j}$ does not depend on the choice of α and β . We write $\kappa_{i,j}(r) = s_{i,j}$. Clearly $\kappa_{i,j}$ is a homomorphism from $\mathcal{A}_{\mathbb{Q}} + \mathcal{B}_{\mathbb{Q}}$ to $\mathbb{Q}[G]$.

Next, suppose that

$$r = \sum_{i,j=1}^n r(p_i, p_j) \otimes p_i \otimes p_j \in \mathcal{M}(A, B)$$

belongs to $\rho_{A,B}^{-1}(T(\mathcal{N}(A, B)))$. As mentioned before, this means that r is in $\mathcal{A}_{\mathbb{Q}} + \mathcal{B}_{\mathbb{Q}}$.

For distinct $i, j \in \{1, 2, \dots, n\}$, let us consider $\kappa_{i,j}(r)$. For any $k = 1, 2, \dots, n$, we have

$$\begin{aligned} \kappa_{i,j}(r)(e - p_k) + \mathbb{Z}[G] &= (\alpha(p_i) + \beta(p_i))(e - p_j)(e - p_k) + \mathbb{Z}[G] \\ &= \alpha(p_i)(e - p_j)(e - p_k) - \alpha(p_k)(e - p_j)(e - p_i) + \mathbb{Z}[G] \\ &= \alpha(p_i)(e - p_j)(e - p_k) + \beta(p_j)(e - p_k)(e - p_i) + \mathbb{Z}[G] \\ &= r(p_i, p_j)(e - p_k) + \mathbb{Z}[G] = \mathbb{Z}[G]. \end{aligned}$$

Therefore there exists $t_{i,j} \in \mathbb{Q}$ such that

$$\kappa_{i,j}(r) - t_{i,j}N \in \mathbb{Z}[G].$$

Because $t_{i,j}$ is uniquely determined up to \mathbb{Z} , we can define a map $\pi_{i,j}$ sending r to $e^{2\pi\sqrt{-1}t_{i,j}} \in \mathbb{T}$. It is not hard to see that $\pi_{i,j}$ is a homomorphism from $\rho_{A,B}^{-1}(T(\mathcal{N}(A, B)))$ to \mathbb{T} . We remark that $\pi_{j,i}(r) = \pi_{i,j}(r)^{-1}$.

By the definition of $\kappa_{i,j}(r)$, we have $\kappa_{i,j}(r)P_j = 0$. Then we have $t_{i,j}NP_j = t_{i,j}m_jN \in \mathbb{Z}[G]$, which implies $t_{i,j}m_j \in \mathbb{Z}$. It follows from

$$\kappa_{i,j}(r) = r(p_i, p_j) + r(p_j, p_i) - (\alpha(p_j) + \beta(p_j))(e - p_i),$$

that $\kappa_{i,j}(r)P_i$ belongs to $\mathbb{Z}[G]$. Hence $t_{i,j}NP_i = t_{i,j}m_iN \in \mathbb{Z}[G]$, which means $t_{i,j}m_i \in \mathbb{Z}$. Consequently, we obtain

$$\pi_{i,j}(r) = e^{2\pi\sqrt{-1}t_{i,j}} \in \{\omega \in \mathbb{T} : \omega^{d(i,j)} = 1\},$$

where $d(i, j)$ is the greatest common divisor of m_i and m_j . Thus, we regard $\pi_{i,j}$ as a homomorphism from $\rho_{A,B}^{-1}(T(\mathcal{N}(A, B)))$ to $\mathbb{Z}_{d(i,j)}$.

We define a homomorphism

$$\pi : \rho_{A,B}^{-1}(T(\mathcal{N}(A, B))) \rightarrow \prod_{i < j} \mathbb{Z}_{d(i,j)}$$

by $\pi(r) = (\pi_{i,j}(r))_{i,j}$.

Lemma 2.5. *The homomorphism π is surjective.*

Proof. Take distinct $k, l \in \{1, 2, \dots, n\}$ arbitrarily. Put $d = d(k, l)$. It suffices to show there exists $r \in \rho_{A,B}^{-1}(T(\mathcal{N}(A, B)))$ such that

$$\pi_{i,j}(r) = \begin{cases} e^{2\pi\sqrt{-1}/d} & \text{if } \{i, j\} = \{k, l\} \\ 1 & \text{if } \{i, j\} \neq \{k, l\}. \end{cases}$$

Let

$$r = \left(\frac{m_k}{d} Q_k - \frac{m_l}{d} Q_l \right) \otimes p_k \otimes p_l \in \mathbb{Z}[G].$$

We define $\alpha, \beta : \{p_1, p_2, \dots, p_n\} \rightarrow \mathbb{Q}[G]$ by

$$\alpha(p_i) = \begin{cases} \sum_{j=0}^{m_l-1} \frac{j}{d} p_l^j Q_l & \text{if } i = k \\ 0 & \text{if } i \neq k, \end{cases}$$

and

$$\beta(p_i) = \begin{cases} \sum_{j=0}^{m_k-1} -\frac{j}{d} p_k^j Q_k & \text{if } i = l \\ 0 & \text{if } i \neq l. \end{cases}$$

We can easily check

$$r = \sum_{i,j=1}^n (\alpha(p_i)(e - p_j) + \beta(p_j)(e - p_i)) \otimes p_i \otimes p_j.$$

In addition, we get

$$\kappa_{k,l}(r) = (\alpha(p_k) + \beta(p_k))(e - p_l) = \frac{1}{d}N - \frac{m_l}{d}Q_l.$$

It follows that

$$\kappa_{k,l}(r) - \frac{1}{d}N \in \mathbb{Z}[G].$$

Therefore we obtain $\pi_{k,l}(r) = e^{2\pi\sqrt{-1}/d}$. If $\{i, j\} \neq \{k, l\}$, then $\kappa_{i,j}(r) = 0$, and so $\pi_{i,j}(r) = 1$. \square

Lemma 2.6. *The kernel of π is equal to $\mathcal{A} + \mathcal{B}$.*

Proof. Evidently $\mathcal{A} + \mathcal{B}$ is contained in $\ker \pi$. Suppose that

$$r = \sum_{i,j=1}^n r(p_i, p_j) \otimes p_i \otimes p_j \in \mathcal{M}(A, B)$$

belongs to $\ker \pi$. We would like to show $r \in \mathcal{A} + \mathcal{B}$. There exist $\alpha, \beta : \{p_1, p_2, \dots, p_n\} \rightarrow \mathbb{Q}[G]$ such that

$$r(a, b) = \alpha(a)(e - b) + \beta(b)(e - a)$$

for all $a, b \in \{p_1, p_2, \dots, p_n\}$. By assumption, we have

$$(\alpha(p_i) + \beta(p_i))(e - p_j) \in \mathbb{Z}[G]$$

for all $i, j = 1, 2, \dots, n$. This implies that there exists $k_i \in \mathbb{Q}$ such that

$$\alpha(p_i) + \beta(p_i) \in k_i N + \mathbb{Z}[G].$$

By replacing $\alpha(p_i)$ with $\alpha(p_i) - k_i N$, we may assume that $\alpha(p_i) + \beta(p_i) \in \mathbb{Z}[G]$ for all $i = 1, 2, \dots, n$. Then we obtain

$$\alpha(p_i)(e - p_j) - \alpha(p_j)(e - p_i) = \alpha(p_i)(e - p_j) + \beta(p_i)(e - p_j) - r(p_j, p_i) \in \mathbb{Z}[G].$$

By applying Lemma 2.4 to $\alpha(p_1), \alpha(p_2), \dots, \alpha(p_n)$, we obtain $u_1, u_2, \dots, u_n \in \mathbb{Z}[G]$ such that

$$(\alpha(p_i) - u_i)(e - p_j) = (\alpha(p_j) - u_j)(e - p_i)$$

for all $i, j = 1, 2, \dots, n$. We define $\tilde{\alpha}, \tilde{\beta} : \{p_1, p_2, \dots, p_n\} \rightarrow \mathbb{Z}[G]$ by

$$\tilde{\alpha}(p_i) = u_i$$

and

$$\tilde{\beta}(p_i) = \alpha(p_i) + \beta(p_i) - u_i.$$

It is easy to check

$$r(p_i, p_j) = \tilde{\alpha}(p_i)(e - p_j) + \tilde{\beta}(p_j)(e - p_i)$$

for all $i, j = 1, 2, \dots, n$. Therefore r belongs to $\mathcal{A} + \mathcal{B}$. \square

We can summarize the results obtained above as follows.

Theorem 2.7. *Let $G = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \dots \times \mathbb{Z}_{m_n}$ and let $d(i, j)$ be the greatest common divisor of m_i and m_j . Then T_G is isomorphic to the direct product of all $\mathbb{Z}_{d(i, j)}$'s for $1 \leq i < j \leq n$. In other words T_G is isomorphic to the wedge product $G \wedge G$.*

3 Skew product extensions of product systems

Let (X, φ) be a Cantor minimal \mathbb{Z} -system. The group of coinvariants of (X, φ)

$$\text{Coinv}(X, \varphi) = C(X, \mathbb{Z}) / \{f - f \circ \varphi^{-1} : f \in C(X, \mathbb{Z})\}.$$

was denoted by $K^0(X, \varphi)$ in [HPS]. At first, we must recall that $\text{Coinv}(X, \varphi)$ can be written as an inductive limit of free abelian groups of finite rank. The reader may refer to [HPS] for the details.

A family of non-empty clopen subsets

$$\mathcal{P} = \{X(v, k) : v \in V, 1 \leq k \leq h(v)\}$$

indexed by a finite set V and natural numbers $k = 1, 2, \dots, h(v)$ is called a Kakutani-Rohlin partition of (X, φ) , if the following conditions are satisfied:

- \mathcal{P} is a partition of X .

- For all $v \in V$ and $k = 1, 2, \dots, h(v) - 1$, we have $\varphi(X(v, k)) = X(v, k + 1)$.

We define the roof set $R(\mathcal{P})$ and the base set $B(\mathcal{P})$ by

$$R(\mathcal{P}) = \bigcup_{v \in V} X(v, h(v))$$

and

$$B(\mathcal{P}) = \bigcup_{v \in V} X(v, 1).$$

Notice that $\varphi(R(\mathcal{P})) = B(\mathcal{P})$. For each $v \in V$, the family of clopen sets $X(v, 1), X(v, 2), \dots, X(v, h(v))$ is called a tower, and $h(v)$ is called the height of the tower. We may identify the label v with the corresponding tower.

Let $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ be a sequence of Kakutani-Rohlin partitions. We denote the set of towers in \mathcal{P}_n by V_n and clopen sets belonging to \mathcal{P}_n by $X(n, v, k)$ ($v \in V_n, k = 1, 2, \dots, h(v)$). We say that $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ gives a Bratteli-Vershik model for (X, φ) , if the following are satisfied:

- The roof sets $R(\mathcal{P}_n) = \bigcup_{v \in V_n} X(n, v, h(v))$ form a decreasing sequence of clopen sets, which shrinks to a single point.
- \mathcal{P}_{n+1} is finer than \mathcal{P}_n for all $n \in \mathbb{N}$ as partitions, and $\bigcup_n \mathcal{P}_n$ generates the topology of X .

Let \mathbb{Z}^{V_n} be the free abelian group which has basis $\{v : v \in V_n\}$. For any $v \in V_n$ and $v' \in V_{n+1}$, we put

$$A_n(v, v') = \#\{1 \leq k \leq h(v') : X(n+1, v', k) \subset X(n, v, 1)\}.$$

We can define a homomorphism $\alpha_n : \mathbb{Z}^{V_n} \rightarrow \mathbb{Z}^{V_{n+1}}$ by

$$\alpha_n(v) = \sum_{v' \in V_{n+1}} A_n(v, v') v' \in \mathbb{Z}^{V_{n+1}}$$

for all $v \in V_n$.

Theorem 3.1. *The group of coinvariants $\text{Coinv}(X, \varphi)$ is isomorphic to the inductive limit of \mathbb{Z}^{V_n} via α_n , that is, $\text{Coinv}(X, \varphi) \cong \lim_{n \rightarrow \infty} (\mathbb{Z}^{V_n}, \alpha_n)$.*

We would like to consider a skew product extension of (X, φ) associated with a finite group valued cocycle. Let G be a finite group and let $\xi : X \rightarrow G$ be a continuous map. We call ξ a cocycle taking its values in G . Define a homeomorphism $\varphi \times \xi$ on $X \times G$ by

$$\varphi \times \xi(x, g) = (\varphi(x), g\xi(x))$$

for all $x \in X$ and $g \in G$. The cocycle ξ is said to be non-degenerate, if $\varphi \times \xi$ is a minimal homeomorphism.

Given a cocycle ξ , we may assume that ξ is constant on every clopen set $U \in \mathcal{P}_1$. Since \mathcal{P}_n is finer than \mathcal{P}_1 , this means that ξ is constant on every clopen set $U \in \mathcal{P}_n$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $v \in V_n$, we define

$$\tilde{\xi}(v) = \xi(x)\xi(\varphi(x))\xi(\varphi^2(x)) \dots \xi(\varphi^{h(v)-1}(x)),$$

where x is an arbitrary point in $X(n, v, 1)$. It is easy to see that $\tilde{\xi}(v)$ does not depend on the choice of x , because ξ is constant on each $X(n, v, k)$. The next lemma is quoted from [M].

Lemma 3.2 ([M, Lemma 2.3]). *The cocycle ξ is non-degenerate if and only if $\{\tilde{\xi}(v) : v \in V_n\}$ generates G for every $n \in \mathbb{N}$.*

From now on, we assume that G is a finite abelian group. We still use the multiplicative notation. Let (X, φ) and (Y, ψ) be two Cantor minimal \mathbb{Z} -systems and let $\xi : X \rightarrow G$ and $\eta : Y \rightarrow G$ be two non-degenerate cocycles.

Put $Z = X \times Y \times G$. We define $\tilde{\varphi}, \tilde{\psi} \in \text{Homeo}(Z)$ by

$$\tilde{\varphi}(x, y, g) = (\varphi(x), y, g\xi(x))$$

and

$$\tilde{\psi}(x, y, g) = (x, \psi(y), g\eta(y)).$$

Evidently $\tilde{\varphi}$ commutes with $\tilde{\psi}$, and so they form a \mathbb{Z}^2 -action ω on Z . It is easy to see (Z, ω) is a minimal dynamical system. The following is the main theorem of this paper.

Theorem 3.3. *The torsion part of the group of coinvariants $\text{Coinv}(Z, \omega)$ is isomorphic to T_G .*

For example, if G is the direct product of n copies of \mathbb{Z}_m , then the torsion subgroup of $\text{Coinv}(Z, \omega)$ is isomorphic to the direct product of $n(n-1)/2$ copies of \mathbb{Z}_m .

We need a series of lemmas to prove this theorem.

The group of coinvariants $\text{Coinv}(Z, \omega)$ is the quotient of $C(Z, \mathbb{Z})$ by

$$D = \{f_1 - f_1 \circ \tilde{\varphi}^{-1} + f_2 - f_2 \circ \tilde{\psi}^{-1} : f_1, f_2 \in C(Z, \mathbb{Z})\}.$$

Fix $x_0 \in X$ and $y_0 \in Y$. Let $D' \subset D$ be the subgroup consisting of all elements $f_1 - f_1 \circ \tilde{\varphi}^{-1} + f_2 - f_2 \circ \tilde{\psi}^{-1}$ with $f_1(x_0, y, g) = f_2(x, y_0, g) = 0$ for all $(x, y, g) \in Z$. At first, we would like to consider the quotient of $C(Z, \mathbb{Z})$ by D' .

We take Bratteli-Vershik models $\{\mathcal{P}_n\}_{n \in \mathbb{N}}$ and $\{\mathcal{Q}_n\}_{n \in \mathbb{N}}$ for (X, φ) and (Y, ψ) , respectively. Let V_n and W_n denote the towers in \mathcal{P}_n and \mathcal{Q}_n , respectively. We write

$$\mathcal{P}_n = \{X(n, v, k) : v \in V_n, 1 \leq k \leq h(v)\}$$

and

$$\mathcal{Q}_n = \{Y(n, w, l) : w \in W_n, 1 \leq l \leq h(w)\}$$

for every $n \in \mathbb{N}$. We may assume $\bigcap_n R(\mathcal{P}_n) = \{x_0\}$ and $\bigcap_n R(\mathcal{Q}_n) = \{y_0\}$. Moreover, we may also assume that ξ (resp. η) is constant on every clopen set in \mathcal{P}_1 (resp. \mathcal{Q}_1). Then $\tilde{\xi}(v)$ and $\tilde{\eta}(w)$ are well-defined for every $v \in V_n$ and $w \in W_n$. Besides, we define $\tilde{\xi}(v, k)$ by $\tilde{\xi}(v, 1) = e$ and by

$$\tilde{\xi}(v, k) = \xi(x)\xi(\varphi(x))\xi(\varphi^2(x)) \dots \xi(\varphi^{k-2}(x))$$

for $k = 2, 3, \dots, h(v)$, where x is a point in $X(n, v, 1)$. For every $w \in W_n$ and $l = 1, 2, \dots, h(w)$, $\tilde{\eta}(w, l)$ can be defined in a similar fashion.

Let $\chi(v, w, k, l, g) \in C(Z, \mathbb{Z})$ be the characteristic function on $X(n, v, k) \times Y(n, w, l) \times \{g\}$ and let $C_n \subset C(Z, \mathbb{Z})$ be the subgroup generated by

$$\{\chi(v, w, k, l, g) : v \in V_n, w \in W_n, 1 \leq k \leq h(v), 1 \leq l \leq h(w), g \in G\}.$$

Evidently C_n is a finitely generated free abelian group and $C(Z, \mathbb{Z})$ is equal to $\bigcup_n C_n$. Let $D'_n \subset D'$ be the subgroup consisting of all elements $f_1 - f_1 \circ \tilde{\varphi}^{-1} + f_2 - f_2 \circ \tilde{\psi}^{-1}$ with $f_1, f_2 \in C_n$ and

$$f_1|_{R(\mathcal{P}_n) \times Y \times G} = f_2|_{X \times R(\mathcal{Q}_n) \times G} = 0.$$

Since ξ (resp. η) is constant on each clopen set belonging to \mathcal{P}_n (resp. \mathcal{Q}_n), we can see that D'_n is contained in C_n . Clearly we have $D'_n \subset D'_{n+1}$ and $D' = \bigcup_n D'_n$. It follows that $C(Z, \mathbb{Z})/D'$ is isomorphic to the inductive limit of C_n/D'_n 's.

It is easily verified that

$$\chi(v, w, 1, 1, g) - \chi(v, w, k, l, g\tilde{\xi}(v, k)\tilde{\eta}(w, l))$$

belongs to D'_n and D'_n is spanned by these elements. We put $\mathcal{M}(V_n, W_n) = \mathbb{Z}[G] \otimes \mathbb{Z}^{V_n} \otimes \mathbb{Z}^{W_n}$ and denote each element r of $\mathcal{M}(V_n, W_n)$ by

$$r = \sum_{v \in V_n} \sum_{w \in W_n} r(v, w) \otimes v \otimes w,$$

where $r(v, w)$ is an element of the group ring $\mathbb{Z}[G]$. Define $\tau_n : C_n \rightarrow \mathcal{M}(V_n, W_n)$ by

$$\tau_n(\chi(v, w, k, l, g)) = g\tilde{\xi}(v, k)^{-1}\tilde{\eta}(w, l)^{-1} \otimes v \otimes w.$$

Then the kernel of τ_n agrees with D'_n . Let $\pi_n : \mathcal{M}(V_n, W_n) \rightarrow \mathcal{M}(V_{n+1}, W_{n+1})$ be the connecting homomorphism induced by the inclusion $C_n \subset C_{n+1}$. Thus, we get the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & D'_n & \longrightarrow & C_n & \xrightarrow{\tau_n} & \mathcal{M}(V_n, W_n) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow \pi_n & & \\ 0 & \longrightarrow & D'_{n+1} & \longrightarrow & C_{n+1} & \xrightarrow{\tau_{n+1}} & \mathcal{M}(V_{n+1}, W_{n+1}) & \longrightarrow & 0, \end{array}$$

where the vertical arrows from D'_n to D'_{n+1} and from C_n to C_{n+1} are inclusions. As a direct consequence, we have the following.

Lemma 3.4. *In this setting, $C(Z, \mathbb{Z})/D'$ is isomorphic to the inductive limit of $\mathcal{M}(V_n, W_n) = \mathbb{Z}[G] \otimes \mathbb{Z}^{V_n} \otimes \mathbb{Z}^{W_n}$. In particular, $C(Z, \mathbb{Z})/D'$ is torsion free.*

We would like to see the connecting homomorphism π_n from $\mathcal{M}(V_n, W_n)$ to $\mathcal{M}(V_{n+1}, W_{n+1})$. For $v \in V_n$ and $v' \in V_{n+1}$, we set

$$E_n(v, v') = \{1 \leq k \leq h(v') : X(n+1, v', k) \subset X(n, v, 1)\}$$

and

$$s(v, v') = \sum_{k \in E_n(v, v')} \tilde{\xi}(v', k)^{-1} \in \mathbb{Z}[G].$$

In the same way, $t(w, w') \in \mathbb{Z}[G]$ can be defined for $w \in W_n$ and $w' \in W_{n+1}$. Since

$$\chi(v', w', 1, 1, g\tilde{\xi}(v', k)^{-1}\tilde{\eta}(w', l)^{-1}) - \chi(v', w', k, l, g)$$

belongs to D'_{n+1} , we obtain

$$\pi_n(r \otimes v \otimes w) = \sum_{v' \in V_{n+1}} \sum_{w' \in W_{n+1}} rs(v, v')t(w, w') \otimes v' \otimes w'$$

for all $r \in \mathbb{Z}[G]$.

Lemma 3.5. (1) *For every $v' \in V_{n+1}$, we have*

$$\sum_{v \in V_n} (e - \tilde{\xi}(v)^{-1})s(v, v') = e - \tilde{\xi}(v')^{-1}.$$

(2) For every $w' \in W_{n+1}$, we have

$$\sum_{w \in W_n} (e - \tilde{\eta}(w)^{-1})t(w, w') = e - \tilde{\eta}(w')^{-1}.$$

Proof. It suffices to show (1). Put

$$E = \bigcup_{v \in V_n} E_n(v, v').$$

It is clear that

$$\sum_{v \in V_n} s(v, v') = \sum_{k \in E} \tilde{\xi}(v', k)^{-1}.$$

Note that E contains one, because

$$X(n+1, v', 1) \subset B(\mathcal{P}_{n+1}) \subset B(\mathcal{P}_n) = \bigcup_{v \in V_n} X(n, v, 1).$$

Let $1 = k_1 < k_2 < \dots < k_L$ be the arranged list of elements in E . If k_i belongs to $E_n(v, v')$ and $i \neq L$, then we can see that

$$\tilde{\xi}(v)^{-1} \tilde{\xi}(v', k_i)^{-1} = \tilde{\xi}(v', k_{i+1})^{-1}.$$

If k_L belongs to $E_n(v, v')$, then we get

$$\tilde{\xi}(v)^{-1} \tilde{\xi}(v', k_L)^{-1} = \tilde{\xi}(v')^{-1}.$$

It follows that

$$\sum_{v \in V_n} \tilde{\xi}(v)^{-1} s(v, v') = \sum_{i=2}^L \tilde{\xi}(v', k_i)^{-1} + \tilde{\xi}(v')^{-1}.$$

Hence we obtain

$$\sum_{v \in V_n} (e - \tilde{\xi}(v)^{-1})s(v, v') = \tilde{\xi}(v', 1) - \tilde{\xi}(v')^{-1} = e - \tilde{\xi}(v')^{-1}.$$

□

For $v \in V_n$ and $g \in G$, we define $\partial_Y(v, g) \in C_n$ by

$$\partial_Y(v, g) = \sum_{w \in W_n} \chi(v, w, 1, 1, g) - \sum_{w \in W_n} \chi(v, w, 1, 1, g) \circ \tilde{\psi}.$$

Similarly, for $w \in W_n$ and $g \in G$, we define $\partial_X(w, g) \in C_n$ by

$$\partial_X(w, g) = \sum_{v \in V_n} \chi(v, w, 1, 1, g) - \sum_{v \in V_n} \chi(v, w, 1, 1, g) \circ \tilde{\varphi}.$$

Let D_n be the subgroup of C_n generated by D'_n and

$$\{\partial_X(w, g) : w \in W_n, g \in G\} \cup \{\partial_Y(v, g) : v \in V_n, g \in G\}.$$

Lemma 3.6. (1) For any $g \in G$, $v \in V_n$ and $k = 1, 2, \dots, h(v)$,

$$\sum_{w \in W_n} \chi(v, w, k, 1, g) - \sum_{w \in W_n} \chi(v, w, k, 1, g) \circ \tilde{\psi}$$

is contained in D_n .

(2) For any $g \in G$, $w \in W_n$ and $l = 1, 2, \dots, h(w)$,

$$\sum_{v \in V_n} \chi(v, w, 1, l, g) - \sum_{v \in V_n} \chi(v, w, 1, l, g) \circ \tilde{\psi}$$

is contained in D_n .

Proof. It suffices to show (1). At first, we have

$$\chi(v, w, k, 1, g) - \chi(v, w, 1, 1, g\tilde{\xi}(v, k)^{-1}) \in D'_n$$

for every $w \in W_n$. It follows that

$$\sum_{w \in W_n} \chi(v, w, k, 1, g) - \sum_{w \in W_n} \chi(v, w, 1, 1, g\tilde{\xi}(v, k)^{-1}) \in D'_n.$$

Put $\eta_w = \tilde{\eta}(w)^{-1}\tilde{\eta}(w, h(w))$ for every $w \in W_n$. It is not hard to see

$$\sum_{w \in W_n} \chi(v, w, k, 1, g) \circ \tilde{\psi} = \sum_{w \in W_n} \chi(v, w, k, h(w), g\eta_w)$$

for every $g \in G$. By combining this with

$$\chi(v, w, k, h(w), g\eta_w) - \chi(v, w, 1, h(w), g\eta_w\tilde{\xi}(v, k)^{-1}) \in D'_n,$$

we obtain

$$\sum_{w \in W_n} \chi(v, w, k, 1, g) \circ \tilde{\psi} - \sum_{w \in W_n} \chi(v, w, 1, 1, g\tilde{\xi}(v, k)^{-1}) \circ \tilde{\psi} \in D'_n.$$

Since

$$\sum_{w \in W_n} \chi(v, w, 1, 1, g\tilde{\xi}(v, k)^{-1}) - \sum_{w \in W_n} \chi(v, w, 1, 1, g\tilde{\xi}(v, k)^{-1}) \circ \tilde{\psi} = \partial_Y(v, g\tilde{\xi}(v, k)^{-1}),$$

we can conclude

$$\sum_{w \in W_n} \chi(v, w, k, 1, g) - \sum_{w \in W_n} \chi(v, w, k, 1, g) \circ \tilde{\psi} \in D_n.$$

□

Lemma 3.7. *In the setting above, we have $D_n \subset D_{n+1}$ and $D = \bigcup_n D_n$.*

Proof. For any $v \in V_n$, $X(n, v, 1)$ is a union of some clopen sets in \mathcal{P}_{n+1} , because \mathcal{P}_{n+1} is finer than \mathcal{P}_n . It follows from (1) of the lemma above that $\partial_Y(v, g)$ is contained in D_{n+1} for every $g \in G$. In the same way, $\partial_X(w, g)$ is contained in D_{n+1} for every $w \in W_n$ and $g \in G$. Therefore we get $D_n \subset D_{n+1}$.

Let us prove $D = \bigcup_n D_n$. It suffices to show that $1_O - 1_O \circ \tilde{\varphi}$ and $1_O - 1_O \circ \tilde{\psi}$ belong to $\bigcup_n D_n$ for any clopen set $O \subset Z$. We may assume that there exist $g \in G$ and clopen sets $U \subset X$, $V \subset Y$ such that $O = U \times V \times \{g\}$.

If V does not contain $\psi(y_0)$, it is clear that $1_O - 1_O \circ \tilde{\psi}$ belongs to $D' = \bigcup_n D'_n$. Then we have $1_O - 1_O \circ \tilde{\psi} \in \bigcup_n D_n$, because $D'_n \subset D_n$. Suppose that V contains $\psi(y_0)$. Since $\bigcup_n \mathcal{P}_n$ generates the topology of X and $\bigcap_n B(\mathcal{Q}_n) = \psi(y_0)$, there exists $n \in \mathbb{N}$ such that $1_U \in C_n$ and $B(\mathcal{Q}_n) \subset V$. Put $O' = U \times B(\mathcal{Q}_n) \times \{g\}$ and $O'' = O \setminus O'$. We have already seen that $1_{O''} - 1_{O''} \circ \tilde{\psi}$ is contained in $\bigcup_n D_n$, and so it suffices to show $1_{O'} - 1_{O'} \circ \tilde{\psi} \in \bigcup_n D_n$. By $1_U \in C_n$, the clopen set U is a union of some clopen sets in \mathcal{P}_n . For any $X(n, v, k) \in \mathcal{P}_n$, the characteristic function on $X(n, v, k) \times B(\mathcal{Q}_n) \times \{g\}$ equals $\sum_{w \in W_n} \chi(v, w, k, 1, g)$. Hence, from (1) of the lemma above, we can conclude $1_{O'} - 1_{O'} \circ \tilde{\psi} \in \bigcup_n D_n$.

In the same way we can prove $1_O - 1_O \circ \tilde{\varphi} \in \bigcup_n D_n$, and so we obtain $D = \bigcup_n D_n$. □

For any $v \in V_n$ and $g \in G$, it is easy to see

$$\tau_n(\partial_Y(v, g)) = \sum_{w \in W_n} g(e - \tilde{\eta}(w)^{-1}) \otimes v \otimes w.$$

Similarly, for any $w \in W_n$ and $g \in G$,

$$\tau_n(\partial_X(w, g)) = \sum_{v \in V_n} g(e - \tilde{\xi}(v)^{-1}) \otimes v \otimes w.$$

Let $\mathcal{V}_n \subset \mathbb{Z}[G] \otimes \mathbb{Z}^{V_n} \otimes \mathbb{Z}^{W_n}$ be the subgroup consisting of

$$\sum_{v \in V_n} \sum_{w \in W_n} \alpha(v)(e - \tilde{\eta}(w)^{-1}) \otimes v \otimes w$$

with some map $\alpha : V_n \rightarrow \mathbb{Z}[G]$. In the same way, let $\mathcal{W}_n \subset \mathbb{Z}[G] \otimes \mathbb{Z}^{V_n} \otimes \mathbb{Z}^{W_n}$ be the subgroup consisting of

$$\sum_{v \in V_n} \sum_{w \in W_n} \beta(w)(e - \tilde{\xi}(v)^{-1}) \otimes v \otimes w$$

with some map $\beta : W_n \rightarrow \mathbb{Z}[G]$. Then it is evident that $\tau_n(D_n)$ coincides with $\mathcal{V}_n + \mathcal{W}_n$, that is, C_n/D_n is isomorphic to $\mathcal{M}(V_n, W_n)/(\mathcal{V}_n + \mathcal{W}_n)$. By defining the map $\mu : V_n \cup W_n \rightarrow G$ by $\mu(v) = \tilde{\xi}(v)^{-1}$ for $v \in V_n$ and $\mu(w) = \tilde{\eta}(w)^{-1}$ for $w \in W_n$, we can regard $\mathcal{M}(V_n, W_n)/(\mathcal{V}_n + \mathcal{W}_n)$ as the abelian group $\mathcal{N}(V_n, W_n)$ which was defined in Section 2. Let $\tilde{\pi}_n$ denote the connecting homomorphism from

$$\mathcal{N}(V_n, W_n) = \mathcal{M}(V_n, W_n)/(\mathcal{V}_n + \mathcal{W}_n)$$

to

$$\mathcal{N}(V_{n+1}, W_{n+1}) = \mathcal{M}(V_{n+1}, W_{n+1})/(\mathcal{V}_{n+1} + \mathcal{W}_{n+1})$$

induced by $\pi_n : \mathcal{M}(V_n, W_n) \rightarrow \mathcal{M}(V_{n+1}, W_{n+1})$.

From the lemma above, the group of coinvariants $\text{Coinv}(Z, \omega)$ is isomorphic to the inductive limit of C_n/D_n . Consequently, we get the following.

Lemma 3.8. *The group of coinvariants $\text{Coinv}(Z, \omega)$ is isomorphic to the inductive limit of $\mathcal{N}(V_n, W_n) = \mathcal{M}(V_n, W_n)/(\mathcal{V}_n + \mathcal{W}_n)$.*

Now we are ready to prove our main theorem.

Proof of Theorem 3.3. Theorem 2.7 tells us that the torsion subgroup $T(\mathcal{N}(V_n, W_n))$ is isomorphic to $T_G \cong G \wedge G$. By Lemma 3.5, the homomorphism π_n from $\mathcal{M}(V_n, W_n)$ to $\mathcal{M}(V_{n+1}, W_{n+1})$ is exactly the same as the homomorphism considered in Lemma 2.2. It follows that $\tilde{\pi}_n$ induces an isomorphism between the torsion subgroups of $\mathcal{N}(V_n, W_n)$ and $\mathcal{N}(V_{n+1}, W_{n+1})$. Hence, by the lemma above, we can conclude that the torsion subgroup of $\text{Coinv}(Z, \omega)$ is isomorphic to $G \wedge G$. \square

4 An example

In [GHK, Section 6.3], it was observed that the octagonal tiling described by the data $(V, \Gamma_8, \mathcal{W}_8^b)$ has a torsion component \mathbb{Z}_2 in the top-dimensional cohomology. We would like to see that this torsion can be explained in our context.

Instead of Γ_8 , it is convenient to use the lattice Γ which is rotated by $\pi/8$ with respect to Γ_8 , that is, Γ is the \mathbb{Z} -span of the vectors

$$e_k = \left(\cos \frac{(2k-1)\pi}{8}, \sin \frac{(2k-1)\pi}{8} \right)$$

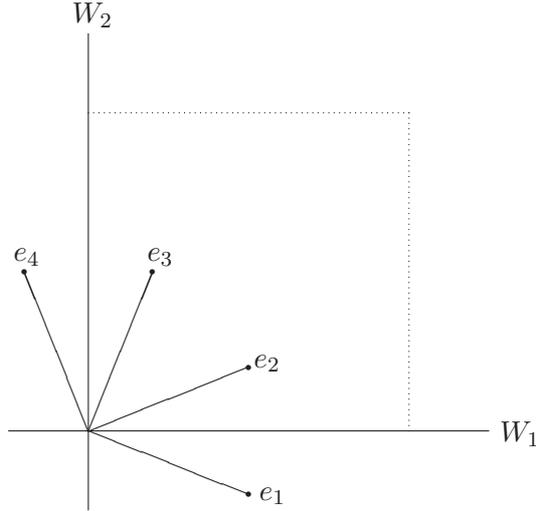


Figure 1: The lattice Γ and the square Z

in \mathbb{R}^2 for $k = 0, 1, 2, 3$. The set of representative singular lines \mathcal{W}_8^b consisted of four lines. But, we ‘cut’ the plane along only two directions, horizontal lines and vertical lines, because they are enough to obtain the torsion component. More precisely, letting W_1 (resp. W_2) be the horizontal (resp. vertical) line which passes through the origin, we define a totally disconnected (noncompact) space Π by ‘cutting’ the plane along the lines $W_i + \gamma$ for $i = 1, 2$ and $\gamma \in \Gamma$. Then Γ acts on Π naturally by translation.

Let $\Gamma_0 \subset \Gamma$ be the subgroup generated by $e_0 + e_1$ and $e_2 + e_3$ and let Z be the quotient of Π by Γ_0 . We can identify Z with the square surrounded by W_i and $W_i + e_0 + e_1 + e_2 + e_3$ for $i = 1, 2$. Clearly Z is a Cantor set and the translations by e_1 and e_2 induce a \mathbb{Z}^2 -action ω on Z . It is well-known that (Π, Γ) has the same dynamical cohomology as (Z, ω) .

Let us look at (Z, ω) . It is not hard to see that the translations by $(e_0 + e_1)/2$ and $(e_2 + e_3)/2$ are well-defined on Π . It follows that they induce a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action on Z . We can easily see that the square surrounded by W_i and $W_i + (e_0 + e_1 + e_2 + e_3)/2$ for $i = 1, 2$ is a fundamental domain for this $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action. Moreover, the translations by e_1 and e_2 can be regarded as the translations by $e_1 - (e_0 + e_1)/2$ and $e_2 - (e_2 + e_3)/2$ on this small square. We can observe that $e_1 - (e_0 + e_1)/2$ is vertical and $e_2 - (e_2 + e_3)/2$ is horizontal. Therefore the quotient of Z by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -action is the product of two Cantor minimal \mathbb{Z} -systems and each of them is conjugate to the same Cantor minimal \mathbb{Z} -system (X, φ) . Here, the Cantor set X is obtained by ‘cutting’ the torus \mathbb{R}/\mathbb{Z} at $\mathbb{Z} + \mathbb{Z}\sqrt{2}$, and the minimal homeomorphism φ is the translation by $\sqrt{2}$ on X . Since (Z, ω) is conjugate to a skew product extension of two copies of (X, φ) associated with $\mathbb{Z}_2 \times \mathbb{Z}_2$ -valued cocycles, by Theorem 3.3, the torsion subgroup of $\text{Coinv}(Z, \omega)$ is isomorphic to \mathbb{Z}_2 .

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