Some remarks on topological orbit equivalence of Cantor minimal systems

MATUI Hiroki

August 2002

Abstract

We will study the number of discontinuities of the orbit cocycles associated with orbit equivalence between Cantor minimal systems.

1 Introduction

In [GPS] the following was stated as Theorem 2.5.

Theorem 1.1 (Original Version). *Let (X_i, φ_i) be Cantor systems $(i = 1, 2)$. The following are equivalent:*

(i) *(X_1, φ_1) is orbit equivalent to (X_2, φ_2), and there exists an orbit map $F : X_1 \to X_2$ so that the associated orbit cocycles $m, n : X_1 \to \mathbb{Z}$ each have finitely many points of discontinuity in (necessarily the same) disjoint φ_1-orbits. Assume $k + 1$ is the least possible number of discontinuity points by considering all such maps F. (If there are no discontinuity points we set $k = 0$.)*

(ii) *There exist subgroups, both isomorphic to $\mathbb{Z}^k$, of*

$$\mathrm{Inf}(K^0(X_1, \varphi_1)) \text{ and } \mathrm{Inf}(K^0(X_2, \varphi_2)),$$

respectively, so that the quotient groups

$$K^0(X_i, \varphi_i)/\mathbb{Z}^k \quad (i = 1, 2)$$

(with the induced order) are order isomorphic by a map preserving the distinguished order units (more precisely, the quotient image of the distinguished order units), where k is the least natural number with this property.

In the above statement, while the implication (i)⇒(ii) is valid, the other implication (ii)⇒(i) is not correct. We will show it by constructing a concrete counter example in this paper.

The original proof contains a gap in the final step. In page 99 of [GPS] three orbit equivalent maps G_1, G_2 and H are considered and the associated orbit cocycles each have finitely many points of discontinuity. The composition map F, however, does not have such a nice property. Thus, the orbit cocycles associated with F may have infinitely many discontinuities. What they actually proved is the following.

Theorem 1.2 (Corrected Version). *Let G_1 and G_2 be two simple (acyclic) dimension groups. The following are equivalent.*

(i) *There exist subgroups, both isomorphic to* $\mathbb{Z}^k$, *of* $\mathrm{Inf}(G_1)$ *and* $\mathrm{Inf}(G_2)$, *respectively, so that the quotient groups*

$$G_i/\mathbb{Z}^k \quad (i=1,2)$$

(with the induced order) are order isomorphic by a map preserving the distinguished order units (more precisely, the quotient image of the distinguished order units), where k *is the least natural number with this property.*

(ii) *There exist Cantor minimal systems* (X_i,φ_i), $i=1,2$, *so that* $K^0(X_1,\varphi_1)\cong G_1$ *and* $K^0(X_2,\varphi_2)\cong G_2$ *(where* $\cong$ *denotes order isomorphism by a map preserving the distinguished order units), and an orbit map* $F:X_1\to X_2$ *so that the associated orbit cocycles* $m,n:X_1\to\mathbb{Z}$ *each have* $k+1$ *discontinuity points in (necessarily the same) disjoint* φ_1*-orbits. Furthermore for* $k\geq 1$, $k+1$ *is the least possible number of discontinuity points by considering all orbit maps and all Cantor minimal systems satisfying the above condition. If* $k=0$, *the least possible number of discontinuity points is zero.*

We will collect some basic facts concerning orbit equivalence of Cantor minimal systems in Section 2. The counter example will be constructed in Section 3. In the last section we will restrict our attention to 2-strong orbit equivalence and discuss some further problems.

Acknowledgments. The author would like to express his sincerest thanks to Professor Christian Skau for the numerous insightful conversations.

2 Preliminaries

We call a compact metrizable totally disconnected and perfect space the Cantor set. The Cantor set is homeomorphic to $\{0,1\}^{\mathbb{N}}$ with the product topology. A homeomorphism $\phi\in\mathrm{Homeo}(X)$ on a topological space is said to be minimal if every ϕ-orbit is dense in X. If ϕ is a minimal homeomorphism on the Cantor set X, the pair (X,ϕ) is called a Cantor minimal system. In [GPS] it was proved that the K^0-group of (X,ϕ) is a complete invariant for the orbit equivalence class of (X,ϕ). We have to recall this fact at first.

Let (X,ϕ) and (Y,ψ) be Cantor minimal systems. When there exists a homeomorphism $F:X\to Y$ such that $F(\{\phi^n(x):n\in\mathbb{Z}\})=\{\psi^n(F(x));n\in\mathbb{Z}\}$ holds for every $x\in X$, two systems (X,ϕ) and (Y,ψ) are said to be orbit equivalent. Since ϕ and ψ have no periodic points, the orbit cocycles $n:X\to\mathbb{Z}$ and $m:Y\to\mathbb{Z}$ are uniquely determined by

$$F(\phi(x))=\psi^{n(x)}(F(x)),\ F^{-1}(\psi(y))=\phi^{m(y)}(F^{-1}(y)).$$

If each of n and m has exactly k discontinuities and these k points have distinct orbits, then we say that F gives an orbit equivalence with k discontinuities.

Definition 2.1. *Two Cantor minimal systems* (X,ϕ) *and* (Y,ψ) *are said to be* k*-strong orbit equivalent, if there exists a homeomorphism* $F:X\to Y$ *which gives an orbit equivalence with* l *discontinuities for some* $l\leq k$. 1*-strong orbit equivalence is called strong orbit equivalence simply.*

We should notice that k-strong orbit equivalence is not an equivalence relation if $k\geq 2$.

For a Cantor minimal system (X,ϕ), we set

$$K^0(X,\phi)=C(X,\mathbb{Z})/\{f-f\phi^{-1}\ ;\ f\in C(X,\mathbb{Z})\}.$$

We denote the equivalence class of $f \in C(X, \mathbb{Z})$ in $K^0(X, \phi)$ by $[f]$. The K^0-group is a unital ordered group with the positive cone

$$K^0(X, \phi)^+ = \{[f] \in K^0(X, \phi) \ ; \ f \geq 0\},$$

and the order unit $[1_X]$. Moreover $K^0(X, \phi)$ is unperforated and satisfies the Riesz interpolation property, and so it is so called a unital dimension group.

Main results of [GPS] are the following.

Theorem 2.2 ([GPS, Theorem 2.1]). *When (X, ϕ) and (Y, ψ) are Cantor minimal systems, the following are equivalent.*

(i) *(X, ϕ) and (Y, ψ) are strong orbit equivalent.*

(ii) *$K^0(X, \phi)$ and $K^0(Y, \psi)$ are order isomorphic by a map preserving the distinguished order units.*

Theorem 2.3 ([GPS, Theorem 2.2]). *When (X, ϕ) and (Y, ψ) are Cantor minimal systems, the following are equivalent.*

(i) *(X, ϕ) and (Y, ψ) are orbit equivalent.*

(ii) *$K^0(X, \phi)/\operatorname{Inf}(K^0(X, \phi))$ and $K^0(Y, \psi)/\operatorname{Inf}(K^0(Y, \psi))$ are order isomorphic by a map preserving the distinguished order units.*

Let (X, ϕ) be a Cantor minimal system and $x_0, x_1, \ldots, x_{k-1} \in X$ be k distinct points. We denote the K_0-group of the AF subalgebra $A_{\{x_0, \ldots, x_{k-1}\}}$ by $E(x_0, x_1, \ldots, x_{k-1})$ (see Section 3 of [P]). That is,

$$E(x_0, x_1, \ldots, x_{k-1}) = C(X, \mathbb{Z})/\{f - f\phi^{-1} \ ; \ f(x_i) = 0 \text{ for all } i = 0, 1, \ldots, k-1\},$$

$$E(x_0, x_1, \ldots, x_{k-1})^+ = \{[f] \in E(x_0, x_1, \ldots, x_{k-1}) \ ; \ f \geq 0\}.$$

For $i = 1, 2, \ldots, k-1$, take a clopen neighborhood U_i of x_i which does not contain the other x_j's. Define a homomorphism ι from $\mathbb{Z}^{k-1}$ to $E(x_0, x_1, \ldots, x_{k-1})$ by sending the i-th canonical basis to the representative class of $1_{U_i} - 1_{\phi(U_i)}$. Then, from [P],

$$0 \to \mathbb{Z}^{k-1} \xrightarrow{\iota} E(x_0, x_1, \ldots, x_{k-1}) \xrightarrow{q} K^0(X, \phi) \to 0$$

is exact, where q is the natural quotient map. We denote the Ext class of this exact sequence by $\zeta(x_0, x_1, \ldots, x_{k-1}) \in \operatorname{Ext}(K^0(X, \phi), \mathbb{Z}^{k-1})$.

Proposition 2.4. *In the above setting, we have the following.*

(i) $\zeta(x_0, x_1, \ldots, x_{k-1}) = \bigoplus_{i=1}^{k-1} \zeta(x_0, x_i)$

(ii) *$\zeta(x_0, x_1, \ldots, x_{k-1})$ depends only on the orbits of x_i's.*

(iii) $\zeta(x_0, \phi(x_0)) = 0$

(iv) $\zeta(x_0, x_1) + \zeta(x_1, x_2) = \zeta(x_0, x_2)$

(v) *If x_i's have distinct orbits, $E(x_0, x_1, \ldots, x_{k-1})$ is a simple dimension group and the range of the map ι is contained in the infinitesimal subgroup.*

Proof. Although every assertion is obvious from the argument in [P] or [GPS], we would like to give a proof for the reader's convenience.

(i) Let q_i be the canonical quotient map from $E(x_0, x_1, \ldots, x_{k-1})$ to $E(x_0, x_i)$. Then,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}^{k-1} & \xrightarrow{\iota} & E(x_0, x_1, \ldots, x_{k-1}) & \xrightarrow{q} & K^0(X, \phi) & \longrightarrow & 0 \\ & & \downarrow & & {\scriptstyle q_i}\downarrow & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\iota} & E(x_0, x_i) & \xrightarrow{q} & K^0(X, \phi) & \longrightarrow & 0 \end{array}$$

is commutative and q_i sends the i-th basis of $\mathbb{Z}^{k-1}$ to the basis of $\mathbb{Z}$. Hence we get the conclusion.

(ii) It suffices to show $\zeta(x_0, x_1) = \zeta(x_0, \phi(x_1))$. Let U be a clopen neighborhood of $\phi(x_1)$ which does not contain x_0. The map sending f to $f - f(\phi(x_1))(1_U - 1_{\phi(U)})$ gives rise to a homomorphism π from $E(x_0, x_1)$ to $E(x_0, \phi(x_1))$. It is easy to see that π is an isomorphism and $\zeta(x_0, x_1)$ equals $\zeta(x_0, \phi(x_1))$ via π.

(iii) The map $f \mapsto f(\phi(x_0))$ gives rise to a homomorphism from $E(x_0, \phi(x_0))$ to $\mathbb{Z}$ and this is a left inverse of ι.

(iv) Let U be a clopen neighborhood of x_2 which does not contain x_0 and x_1. Suppose $[f] \in E(x_0, x_1)$, $[g] \in E(x_1, x_2)$ and $q([f]) = q([g])$ in $K^0(X, \phi)$. There exists $h \in C(X, \mathbb{Z})$ such that $f = g + h - h\phi^{-1}$. By sending $([f], [g])$ to $[f - (h(x_1) - h(x_2))(1_U - 1_U\phi^{-1})]$, we obtain a homomorphism from

$$\{([f], [g]) \in E(x_0, x_1) \oplus E(x_1, x_2);\ q([f]) = q([g])\}$$

to $E(x_0, x_2)$. Since its kernel is

$$\{(\iota(n), -\iota(n))\ ;\ n \in \mathbb{Z}\} \cong \mathbb{Z},$$

we can conclude $\zeta(x_0, x_1) + \zeta(x_1, x_2) = \zeta(x_0, x_2)$.

(v) This is exactly Corollary 2 of [GPS, Theorem 1.17]. □

The following proposition is clear from the proof of the implication (i)⇒(ii) of [GPS, Theorem 2.5].

Proposition 2.5 ([GPS]). *Let (X, ϕ) and (Y, ψ) be Cantor minimal systems and $F : X \to Y$ be a homeomorphism which gives an orbit equivalence with k discontinuities. Suppose $x_0, x_1, \ldots, x_{k-1} \in X$ and $y_0, y_1, \ldots, y_{k-1} \in Y$ are discontinuities of the orbit cocycles. Then $C(X, \mathbb{Z}) \ni f \mapsto fF^{-1} \in C(Y, \mathbb{Z})$ induces a unital order isomorphism from $E(x_0, x_1, \ldots, x_{k-1})$ to $E(y_0, y_1, \ldots, y_{k-1})$.*

In the next section we need the following lemma. The proof is obvious.

Lemma 2.6. *Let $\pi : (X, \phi) \to (Y, \psi)$ be a factor map between Cantor minimal systems. Suppose $x_0, x_1, \ldots, x_{k-1} \in X$ drop to distinct k points in Y. Then, $\zeta(x_0, x_1, \ldots, x_{k-1})$ is sent to $\zeta(\pi(x_0), \pi(x_1), \ldots, \pi(x_{k-1}))$ by the canonical homomorphism from* $\mathrm{Ext}(K^0(X, \phi), \mathbb{Z}^{k-1})$ *to* $\mathrm{Ext}(K^0(Y, \psi), \mathbb{Z}^{k-1})$ *induced by $\pi^* : K^0(Y, \psi) \to K^0(X, \phi)$.*

3 A counter example

At first we construct a Cantor minimal system (X_0, ϕ_0) whose K^0-group is isomorphic to $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers which satisfies

$$\alpha = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{a_n + 2}{a_n - 1} < \infty.$$

Moreover, we assume that for every $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\prod_{n=1}^{N}(a_n+2)$ and $\prod_{n=1}^{N}(a_n-1)$ are both divisible by m. We define a properly ordered simple Bratteli diagram $B=(V,E)$ as follows. Put $V_0=\{v_0\}$ and $V_n=\{u_n,v_n,w_n\}$ for all $n \in \mathbb{N}$. Connect v_0 to each vertex of V_1 by a single edge. The edge set E_{n+1} is defined so that the incidence matrix of n-th step is

$$\begin{bmatrix} a_n & 1 & 1 \\ 1 & a_n & 1 \\ 1 & 1 & a_n \end{bmatrix}.$$

It is clear that $B=(V,E)$ is a simple Bratteli diagram. Since

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} a_n+2 & & \\ & a_n-1 & \\ & & a_n-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_n & 1 & 1 \\ 1 & a_n & 1 \\ 1 & 1 & a_n \end{bmatrix},$$

by sending $xu_n+yv_n+zw_n \in \mathbb{Z}^{V_n}$ to

$$\begin{bmatrix} 3\prod_{k=1}^{n-1}(a_k+2) & & \\ & 6\prod_{k=1}^{n-1}(a_k-1) & \\ & & 2\prod_{k=1}^{n-1}(a_k-1) \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

we have

$$K_0(V,E) \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q},$$
$$K_0(V,E)^+ = \left\{ (p,q,r) \in \mathbb{Q}^3 \; ; \; \begin{array}{c} \alpha p - q + r > 0 \\ \alpha p - q - r > 0 \\ 2\alpha p + q > 0 \end{array} \right\} \cup \{0\}$$

and the order unit $1_B=(1,0,0)$. Note that $2u_1-v_1-w_1$ corresponds to $(0,1,0)$ and v_1-w_1 corresponds to $(0,0,1)$. We write this dimension group by $(G,G^+,1_G)$. Notice that the dimension group G has no non-trivial automorphisms except for the flip which changes the signal of the third coordinate.

We define a linear order on each of $r^{-1}(u_{n+1})$, $r^{-1}(v_{n+1})$ and $r^{-1}(w_{n+1})$ so that the first edge has the source vertex u_n, the second edge has the source vertex v_n and the last edge has the source vertex w_n. Then $B=(V,E)$ is obviously properly ordered. Let (X_0,ϕ_0) be the Cantor minimal system determined by $B=(V,E)$.

Let ψ be the adding machine on

$$Y = \prod_{n=1}^{\infty}\{0,1,2,\ldots,a_n+1\}.$$

Then the dimension group of (Y,ψ) is $(\mathbb{Q},\mathbb{Q}^+,1)$. It is easily seen that there exists an almost one-to-one factor map $\pi:(X_0,\phi_0)\to(Y,\psi)$ and π^* sends $p\in\mathbb{Q}$ to $(p,0,0)\in\mathbb{Q}^3$. The following lemma is also clear.

Lemma 3.1. *The factor map π is three-to-one on $\bigcup_n \psi^n(E)$, where*

$$E = \prod_{n=1}^{\infty}\{2,3,\ldots,a_n\} \subset Y,$$

and one-to-one on the other orbits.

Take $y = (y_n)_n \in E$. Suppose $x_0 \in X_0$ and $x_1 \in X_1$ are distinct preimages of y. We would like to consider $\zeta(x_0, x_1) \in \mathrm{Ext}(K^0(X_0, \phi_0), \mathbb{Z})$. We identify $\mathrm{Ext}(K^0(X_0, \phi_0), \mathbb{Z})$ with $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})^3$. Then it is not hard to see that the first summand of $\zeta(x_0, x_1)$ is zero. Let η_1 and η_2 be the second and third summands of $\zeta(x_0, x_1)$. We will compute them. Note that $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is divisible and torsion-free, thus $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ is a vector space over $\mathbb{Q}$.

Let $F(V)$ be the free abelian group over V and $\partial : F(V) \to F(V)$ be the homomorphism defined by $\partial(v) = v - \sum_{s(e)=v} r(e)$. Then

$$0 \to F(V) \xrightarrow{\partial} F(V) \longrightarrow K_0(V, E) \to 0$$

gives a projective resolution of $K_0(V, E)$, and $\mathrm{Ext}(K^0(X_0, \phi_0), \mathbb{Z})$ is the quotient of $\mathrm{Hom}(F(V), \mathbb{Z})$ by the image of ∂^*. By the same computation as in [GPS], we see that $\zeta(x_0, x_1)$ has a representative $\rho : F(V) \to \mathbb{Z}$ given by

$$\rho(v) = \#\{e \in E_{n+1}; s(e) = v \text{ and } x_1(n+1) < e\} - \#\{e \in E_{n+1}; s(e) = v \text{ and } x_0(n+1) < e\}$$

for $v \in V_n = \{u_n, v_n, w_n\}$. Suppose x_0 goes through u_n's and x_1 goes through v_n's. Then we get

$$(\rho(u_n), \rho(v_n), \rho(w_n)) = (y_n - a_n, a_n - y_n, 0).$$

We denote the basis of the free abelian group $F(\mathbb{N})$ by $\{e_n\}_{n \in \mathbb{N}}$. Put $\partial(e_n) = e_n - (a_n - 1)e_{n+1}$. Then

$$0 \to F(\mathbb{N}) \xrightarrow{\partial} F(\mathbb{N}) \longrightarrow \mathbb{Q} \to 0$$

gives a projective resolution of $\mathbb{Q}$. Let $\iota_1 : F(\mathbb{N}) \to F(V)$ be the homomorphism defined by $\iota_1(e_n) = 2u_n - v_n - w_n$. Then we have $\partial\iota_1 = \iota_1\partial$, and so a representative of η_1 is given by $e_n \mapsto 3(y_n - a_n)$. Similarly, by considering $\iota_2(e_n) = v_n - w_n$, we know that a representative of η_2 is given by $e_n \mapsto a_n - y_n$. Hence we obtain $-3\eta_2 = \eta_1$. Similar computation can be done, when x_0 or x_1 goes through w_n's.

These observations give us the following.

Lemma 3.2. *Suppose that $x_0, x_1 \in X_0$ have distinct orbits. If $\zeta(x_0, x_1) = (0, \eta_1, \eta_2) \in \mathrm{Ext}(\mathbb{Q}, \mathbb{Z})^3$, then η_1 and η_2 are linearly dependent over $\mathbb{Q}$.*

By [GPS2], we get a Cantor minimal system (X_1, ϕ_1) and a factor map $\pi_1 : (X_1, \phi_1) \to (X_0, \phi_0)$ which satisfy the following:

- The dimension group $K^0(X_1, \phi_1)$ is isomorphic to $G \oplus \mathbb{Z}$ equipped with the positive cone

$$\{(g, n) \in G \oplus \mathbb{Z} \ ; \ g \in G^+ \setminus \{0\}\} \cup \{(0, 0)\}.$$

- The factor map π_1 induces the embedding $G \ni p \mapsto (p, 0) \in G \oplus \mathbb{Z}$.

- The factor map π_1 is at most two-to-one and the factor map $\pi\pi_1$ is at most three-to-one.

Since π^* induces the canonical isomorphism from $\mathrm{Ext}(K^0(X_1, \phi_1), \mathbb{Z})$ to $\mathrm{Ext}(K^0(X_0, \phi_0), \mathbb{Z})$, we get the exactly same statement as Lemma 3.2 for (X_1, ϕ_1).

Take two elements ξ_1 and ξ_2 in $\mathrm{Ext}(\mathbb{Q}, \mathbb{Z})$ which are linearly independent over $\mathbb{Q}$. Let

$$0 \to \mathbb{Z} \to D \to G \to 0$$

be an exact sequence corresponding to $(0, \xi_1, \xi_2) \in \mathrm{Ext}(G, \mathbb{Z})$. Let D^+ be the union of zero and the inverse image of $G^+ \setminus \{0\}$, and let 1_D be a preimage of 1_G. Then $(D, D^+, 1_D)$ is a unital simple dimension group, and so there exists a Cantor minimal system (X_2, ϕ_2) whose dimension group is isomorphic to $(D, D^+, 1_D)$.

Clearly, $\operatorname{Inf}(K^0(X_1, \phi_1)) \cong \operatorname{Inf}(K^0(X_2, \phi_2)) \cong \mathbb{Z}$, and $K^0(X_1, \phi_1)/\operatorname{Inf}(K^0(X_1, \phi_1))$ is isomorphic to $K^0(X_2, \phi_2)/\operatorname{Inf}(K^0(X_2, \phi_2))$ as unital simple dimension group. However, we have the following.

Theorem 3.3. *In the above setting, (X_1, ϕ_1) and (X_2, ϕ_2) are not 2-strong orbit equivalent.*

Proof. Because $K^0(X_1, \phi_1)$ and $K^0(X_2, \phi_2)$ are not isomorphic, these two systems are not strong orbit equivalent. Suppose there exists a homeomorphism $F : X_1 \to X_2$ which gives an orbit equivalence with two discontinuities. Let $x_0, x_1 \in X_1$ and $y_0, y_1 \in X_2$ be the discontinuities of the orbit cocycles. From Proposition 2.5, the unital dimension groups $E(x_0, x_1)$ and $E(y_0, y_1)$ are isomorphic. Let θ be the isomorphism.

Both of $E(x_0, x_1)$ and $E(y_0, y_1)$ have the infinitesimal subgroup isomorphic to $\mathbb{Z}^2$. Hence θ on $\mathbb{Z}^2$ gives a matrix $S \in GL(2, \mathbb{Z})$, and the following diagram is obtained:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & E(x_0, x_1) & \longrightarrow & G & \longrightarrow & 0 \\
 & & \cong \downarrow S & & \cong \downarrow \theta & & \cong \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & E(y_0, y_1) & \longrightarrow & G & \longrightarrow & 0
\end{array}$$

Since G has no non-trivial automorphisms except for the flip on the third coordinate, we may assume that θ induces the identity map on G.

Suppose the above two exact sequences are given by $(0, \eta), (\xi, \zeta) \in \operatorname{Ext}(G, \mathbb{Z}) \oplus \operatorname{Ext}(G, \mathbb{Z})$ respectively, where $\xi = (0, \xi_1, \xi_2) \in \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})^3$. Then we get

$$\begin{bmatrix} \xi \\ \zeta \end{bmatrix} = S \begin{bmatrix} 0 \\ \eta \end{bmatrix}.$$

Therefore

$$S_{12}\eta = S_{12}(\eta_0, \eta_1, \eta_2) = (0, \xi_1, \xi_2),$$

which shows $\eta_0 = 0$. From Lemma 3.2, however, η_1 and η_2 are linearly dependent over $\mathbb{Q}$ and this contradicts the linear independence of ξ_1 and ξ_2. □

By starting from linearly independent three elements $\xi_1, \xi_2, \xi_3 \in \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})$ and the corresponding exact sequence, we get a counter example for 3-strong orbit equivalence in a similar fashion. For $k \geq 4$, we do not need the linear independence and we can get a contradiction much easily, because the factor map π is at most three-to-one.

In Theorem 3.3 we have shown that (X_1, ϕ_1) and (X_2, ϕ_2) are not 2-strong orbit equivalent. But they may be k-strong orbit equivalent for some $k \geq 3$. Therefore we may have a chance to show the following statement.

Conjecture 3.4. *When (X_1, ϕ_1) and (X_2, ϕ_2) are Cantor minimal systems, the following are equivalent.*

(i) *(X_1, ϕ_1) is k-strong orbit equivalent to (X_2, ϕ_2) for some $k \geq 0$.*

(ii) *For some $l \geq 0$, there exist subgroups, both isomorphic to $\mathbb{Z}^l$, of*

$$\operatorname{Inf}(K^0(X_1, \phi_1)) \text{ and } \operatorname{Inf}(K^0(X_2, \phi_2)),$$

respectively, so that the quotient groups

$$K^0(X_i, \phi_i)/\mathbb{Z}^l \quad (i = 1, 2)$$

(with the induced order) are order isomorphic by a map preserving the distinguished order units (more precisely, the quotient image of the distinguished order units).

It seems rather hard to find a counter example for the above conjecture.

4 More on 2-strong orbit equivalence

The 2-strong orbit equivalence closely relates to the surjectivity of the map $X^2 \ni (x_0, x_1) \mapsto \zeta(x_0, x_1) \in \mathrm{Ext}(K^0(X,\phi), \mathbb{Z})$. Of course, this map is not surjective in general, which was shown in Lemma 3.2. But we can show the surjectivity in some cases. Let us denote the real analogue of K^0-groups by $K^0_{\mathbb{R}}(X,\phi)$ (see [O]). That is,

$$K^0_{\mathbb{R}}(X,\phi) = C(X,\mathbb{R})/\{f - f\phi^{-1} \ ; \ f \in C(X,\mathbb{R})\}.$$

Notice that $K^0_{\mathbb{R}}(X,\phi)$ is a real vector space.

Theorem 4.1. *When (X,ϕ) is a Cantor minimal system and $K^0(X,\phi)$ is isomorphic to $\mathbb{Q}$ or $\mathbb{Q}^2$, the map $(x_0,x_1) \mapsto \zeta(x_0,x_1)$ is surjective. Moreover we can make x_0 and x_1 have distinct orbits, unless (X,ϕ) is an odometer system.*

Proof. Suppose $K^0(X,\phi)$ is isomorphic to $\mathbb{Q}$. There exists a factor map π from (X,ϕ) to the odometer system (Y,ψ) whose K^0-group is $\mathbb{Q}$. For every $\xi \in \mathrm{Ext}(\mathbb{Q},\mathbb{Z})$, there exist distinct points $y_0, y_1 \in Y$ such that $\zeta(y_0,y_1) = \xi$. Then, Lemma 4 leads us to the conclusion. When $\xi = 0$ and (X,ϕ) is not an odometer system, take distinct points x_0 and x_1 with $\pi(x_0) = \pi(x_1)$. Then $\zeta(x_0,x_1) = 0$.

Next, let (X,ϕ) be a Cantor minimal system whose K^0-group is $\mathbb{Q}\oplus\mathbb{Q}$. Take a clopen set U so that $[1_X]$ and $[1_U]$ are linearly independent over $\mathbb{Q}$ in $K^0(X,\phi)$. If they are linearly independent over $\mathbb{R}$ in $K^0_{\mathbb{R}}(X,\phi)$, then $f = 1_U$ satisfies the condition (i) of Lemma 4.3. If $[1_U] = s[1_X]$ in $K^0_{\mathbb{R}}(X,\phi)$ for some irrational number $s \in (0,1)$, there exist $f \in C(X,\mathbb{Z})$ and $h \in C(X,\mathbb{R})$ such that $0 \le h \le 4/3$, $[f] = [1_U]$ and $f = s1_X + h - h\phi^{-1}$. That is, the condition (ii) of Lemma 4.3 is satisfied. We put

$$G = \{s \in K^0(X,\phi) \ ; \ ns = m[1_X] \text{ for some } n \in \mathbb{N}, \ m \in \mathbb{Z}\}$$

and

$$H = \{t \in K^0(X,\phi) \ ; \ nt = m[f] \text{ for some } n \in \mathbb{N}, \ m \in \mathbb{Z}\}.$$

Then they are both isomorphic to $\mathbb{Q}$ and $G\oplus H \ni (s,t) \mapsto s+t \in K^0(X,\phi)$ is an isomorphism. Let π_G and π_H be the natural homomorphism from $\mathrm{Ext}(K^0(X,\phi),\mathbb{Z})$ to $\mathrm{Ext}(G,\mathbb{Z})$ and $\mathrm{Ext}(H,\mathbb{Z})$. Of course $\pi_G \oplus \pi_H$ is an isomorphism.

To prove the surjectivity, assume $\xi \in \mathrm{Ext}(K^0(X,\phi),\mathbb{Z})$ is given and $\xi \neq 0$. Let $\eta_G \in \mathrm{Hom}(G,\mathbb{R}/\mathbb{Z})$ and $\eta_H \in \mathrm{Hom}(H,\mathbb{R}/\mathbb{Z})$ be representatives of $\pi_G(\xi)$ and $\pi_H(\xi)$. We may assume that $\eta_G([1_X]) = \eta_H([f]) = 0$. We use the notation of Lemma 4.2. For every natural number n we set

$$F_n = \left\{(x,y) \in X^2 \ ; \ \eta_G(\frac{1}{n!}[1_X]) = \rho^{x,y}_{1_X}(\frac{1}{n!}), \ \eta_H(\frac{1}{n!}[f]) = \rho^{x,y}_f(\frac{1}{n!})\right\}.$$

From the definition of $\rho^{x,y}_f$ it is easy to see that F_n is a closed set and $F_{n+1} \subset F_n$. Thanks to Lemma 4.3 each F_n is not empty. Hence we obtain $(x_0,x_1) \in \bigcap F_n$. It is clear that $\zeta(x_0,x_1)$ is equal to ξ.

In the case of $\xi = 0$, we get the conclusion by Lemma 4.8. □

Lemma 4.2. *Let (X,ϕ) be a Cantor minimal system and $x,y \in X$ be distinct points. Suppose $\tau : \mathbb{Q} \to K^0(X,\phi)$ is an injection and $\tau(1) = [f]$. If a sequence of natural numbers $\{a_n\}_n$ satisfies $\lim_{n\to\infty} \phi^{a_n}(y) = x$, then*

$$\mathbb{Q} \ni r \mapsto \left(\lim_{n\to\infty} r \sum_{k=1}^{a_n} f(\phi^k(y))\right) + \mathbb{Z}$$

gives a well-defined homomorphism $\rho^{x,y}_f$ from $\mathbb{Q}$ to $\mathbb{R}/\mathbb{Z}$ and it is a representative of $\tau^*(\zeta(x,y))$.

Proof. For any natural number N, there exists $g, h \in C(X, \mathbb{Z})$ with $f - Ng = h - h\phi^{-1}$. Then

$$\sum_{k=a_l+1}^{a_m} f(\phi^k(y)) = N \sum_{k=a_l+1}^{a_m} g(\phi^k(y)) + h(\phi^{a_m}(y)) - h(\phi^{a_l}(y))$$

is zero modulo N for sufficiently large $l < m$, since the last two terms are canceled. Hence $\rho_f^{x,y} : \mathbb{Q} \to \mathbb{R}/\mathbb{Z}$ is a well-defined homomorphism.

Let us show that $\rho_f^{x,y}$ is a representative of $\tau^*(\zeta(x, y))$. Put

$$F = \{s \in E(x, y) \ ; \ q(s) \in \tau(\mathbb{Q})\}.$$

Suppose $[g] \in F$ and $q([g]) = \tau(r)$. There exists a locally constant function $h : X \to \mathbb{Q}$ such that $rf - g = h - h\phi^{-1}$. By sending $[g] \in F$ to

$$\lim_{n\to\infty} \sum_{k=1}^{a_n} (rf(\phi^k(y)) - g(\phi^k(y))) = \lim_{n\to\infty} h(\phi^{a_n}(y)) - h(y) = h(x) - h(y),$$

we get a well-defined homomorphism $\rho : F \to \mathbb{Q}$. If U is a clopen neighborhood of y which does not contain x, we see $\rho([1_U - 1_{\phi(U)}]) = 1$. The proof is completed because ρ induces $\rho_f^{x,y}$. □

Lemma 4.3. *Let (X, ϕ) be a Cantor minimal system and suppose a function $f \in C(X, \mathbb{Z})$ satisfies either of the following:*

- (i) *$[f]$ and $[1_X]$ are linearly independent over $\mathbb{R}$ in $K^0_{\mathbb{R}}(X, \phi)$ and f is a characteristic function on a clopen set.*
- (ii) *There exist an irrational number $s \in (0, 1)$ and $h \in C(X, \mathbb{R})$ such that $0 \le h \le 4/3$ and $f = s1_X + h - h\phi^{-1}$.*

Then, for any natural numbers N, m and l, there exist $x \in X$ and $k \in \mathbb{N}$ such that

$$k \equiv m, \ f(\phi(x)) + f(\phi^2(x)) + \cdots + f(\phi^k(x)) \equiv l \pmod N.$$

Proof. Put

$$g_n = f\phi + f\phi^2 + \cdots + f\phi^{nN+m}$$

for every $n \in \mathbb{N}$.

(i) Take an invariant measure μ. Since $f - \mu(f)1_X$ is not a coboundary, we get $\sup_n ||g_n - (nN + m)\mu(f)1_X|| = \infty$. Besides, we have

$$\max\{g_n(x) - (nN + m)\mu(f) \ ; \ x \in X\} \ge 0$$

and

$$\min\{g_n(x) - (nN + m)\mu(f) \ ; \ x \in X\} \le 0$$

because $\mu(g_n - (nN + m)\mu(f)1_X) = 0$. Hence we obtain

$$\max_{x\in X} g_n(x) - \min_{x\in X} g_n(x) \ge N$$

for some $n \in \mathbb{N}$. As f is a characteristic function, $g_n(x) - g_n(\phi(x)) \in \{-1, 0, 1\}$ for every $x \in X$. Therefore we can find $x \in X$ with $g_n(x) \equiv l \pmod N$.

(ii) Since s is irrational, we can find $n \in \mathbb{N}$ and $t \in (1/3, 2/3)$ with $(nN + m)s - t \in N\mathbb{Z} + l$. Then we have

$$g_n = (nN + m)s1_X + h\phi^{nN+m} - h$$

and $h\phi^{nN+m}(x) - h(x)$ equals $-t$ or $1 - t$ because of $0 \le h \le 4/3$. As a coboundary cannot be positive, there exists $x \in X$ with $h\phi^{nN+m}(x) - h(x) = -t$. Consequently we get $g_n(x) \equiv l \pmod N$. □

In [GPS] the following was stated as a corollary.

Conjecture 4.4. *Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. If there exists an order isomorphism from $K^0(X, \phi)$ to $K^0(Y, \psi)$ preserving the order units modulo infinitesimal subgroups, then (X, ϕ) and (Y, ψ) are 2-strong orbit equivalent.*

If the discontinuities of the orbit cocycle are allowed to lie in the same orbit, the above conjecture is solved positively. This is because we can prove that $E(x_0, \phi(x_0))$ is unital order isomorphic to $E(y_0, \psi(y_0))$ for every $x_0 \in X$ and $y_0 \in Y$. But, 2-strong orbit equivalence requires the discontinuities to lie in distinct orbits. Thus, we can show the above conjecture if the following is true.

Conjecture 4.5. *Let (X, ϕ) be a Cantor minimal system which is not conjugate to an odometer system. Then there exist $x_0, x_1 \in X$ lying in distinct orbits such that $\zeta(x_0, x_1)$ is equal to zero.*

We do not know whether the above conjectures are correct or not.

Definition 4.6. *Let (X, ϕ) be a Cantor minimal system. Two distinct points $x_0, x_1 \in X$ are said to be positively asymptotic, if $d(\phi^n(x_0), \phi^n(x_1))$ converges to zero as $n \to \infty$.*

Proposition 4.7. *When (X, ϕ) is a Cantor minimal system and two distinct points $x_0, x_1 \in X$ are positively asymptotic, $\zeta(x_0, x_1)$ is zero in* $\mathrm{Ext}(K^0(X, \phi), \mathbb{Z})$.

Proof. For $f \in C(X, \mathbb{Z})$, put

$$\rho(f) = \sum_{n=1}^{\infty} f(\phi^n(x_0)) - f(\phi^n(x_1)),$$

which is well-defined because $f(\phi^n(x_0)) = f(\phi^n(x_1))$ for sufficiently large n. When $f(x_0) = f(x_1) = 0$, we can see $\rho(f - f\phi^{-1}) = 0$. Hence ρ gives rise to a homomorphism from $E(x_0, x_1)$ to $\mathbb{Z}$. If a clopen neighborhood U of x_1 does not contain x_0, then $\rho(1_U - 1_{\phi(U)}) = 1$. Therefore $\zeta(x_0, x_1)$ is equal to zero. □

Thus, if (X, ϕ) and (Y, ψ) have positively asymptotic pairs, Conjecture 4.4 is true. It is well known that every minimal subshift has positively asymptotic pairs. In a recent paper [BHR], it was proved that systems of positive entropy also have positively asymptotic pairs. Hence Conjecture 4.4 is true for these kinds of Cantor minimal systems. In general, however, it is not known whether or not a Cantor minimal system always has positively asymptotic pairs unless it is an odometer system.

We would like to conclude this section with the following lemma, which implies Conjecture 4.4 is also true when K^0-groups are of finite rank.

Lemma 4.8. *When (X, ϕ) is a Cantor minimal system except for odometer systems and $K^0(X, \phi)$ is of finite rank, there exist $x_0, x_1 \in X$ lying in distinct orbits and $\zeta(x_0, x_1) = 0$.*

Proof. Let $p_1, p_2, \dots, p_m \in C(X, \mathbb{Z})$ be a maximal family of independent basis of $K^0(X, \phi)$. We can define a map π from X to $(\mathbb{Z}^m)^{\mathbb{Z}}$ by

$$\pi(x)_k = (p_1(x), p_2(x), \dots, p_m(x))$$

for $k \in \mathbb{Z}$. The infinite sequence $\pi(x)$ actually consists of finite alphabets, and so π is regarded as a factor map to a subshift. Thus, we can find a factor map $\pi : (X, \phi) \to (Y, \psi)$ so that (Y, ψ) is a minimal subshift and $\pi^*(K^0(Y, \psi))$ contains $[p_1], [p_2], \dots, [p_m]$.

Let $\Gamma = K^0(X,\phi)/\pi^*(K^0(Y,\psi))$. Note that Γ is a countable torsion group. Suppose $[f] \in K^0(X,\phi)$ is of order n in Γ. Then there exist $g \in C(Y,\mathbb{Z})$ and $h \in C(X,\mathbb{Z})$ such that $nf + h - h\phi^{-1} = g\pi$. Let $\tilde{\psi}$ be a homeomorphism on $Y \times \mathbb{Z}/n\mathbb{Z}$ determined by

$$\tilde{\psi}(y,k) = (\psi(y), k + g(\psi(x))),$$

where the addition is understood modulo n. The dynamical system $(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$ is called the skew product extension of (Y,ψ) associated with the $\mathbb{Z}/n\mathbb{Z}$-valued cocycle g. Lemma 3.6 of [M] tells us that $(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$ is a Cantor minimal system. When we define a map π_1 from X to $Y \times \mathbb{Z}/n\mathbb{Z}$ by $\pi_1(x) = (\pi(x), h(x))$, it is easy to see that π_1 is a factor map from (X,ϕ) to $(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$ and $\pi = \pi_0\pi_1$ where π_0 is the canonical projection from $Y \times \mathbb{Z}/n\mathbb{Z}$ to Y. Let $\gamma \in \mathrm{Homeo}(Y \times \mathbb{Z}/n\mathbb{Z})$ be the canonical centralizer determined by $\gamma(y,k) = (y, k+1)$. Take $s \in K^0(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$. We have $ms \in \pi_0^*(K^0(Y,\psi))$ for some $m \in \mathbb{N}$ because Γ is a torsion group. Hence $ms = \mathrm{mod}(\gamma)(ms) = m\,\mathrm{mod}(\gamma)(s)$, which implies $s = \mathrm{mod}(\gamma)(s)$. That is, $\mathrm{mod}(\gamma)$ is the identity map on $K^0(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$. Thanks to Theorem 3.7 of [M2], we can conclude that $K^0(Y,\psi)/nK^0(Y,\psi)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and $[g]$ is a generator. We also remark that $[g\pi_0]$ is n-divisible in $K^0(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi})$, and so $[f] \in \pi_1^*(K^0(Y \times \mathbb{Z}/n\mathbb{Z}, \tilde{\psi}))$.

If $[f], [f'] \in K^0(X,\phi)$ are both of order n, the above argument implies that both of $n[f] = [g\pi]$ and $n[f'] = [g'\pi]$ are generators of $K^0(Y,\psi)/nK^0(Y,\psi) \cong \mathbb{Z}/n\mathbb{Z}$. Therefore we get $[g'] - k[g] \in nK^0(Y,\psi)$ for some $k \in \mathbb{N}$, and so $[f'] - k[f] \in \pi^*(K^0(Y,\psi))$. Consequently Γ has only one n-cyclic component, which implies that Γ is a subgroup of $\mathbb{Q}/\mathbb{Z}$.

We can find a sequence of natural numbers $\{a_n\}_n$ such that $a_1|a_2|a_3|\dots$ and

$$\Gamma \cong \bigcup_{n=1}^{\infty} \frac{1}{a_n}\mathbb{Z} + \mathbb{Z}.$$

It is well-known that the dual group of the discrete abelian group Γ is

$$\hat{\Gamma} = \mathrm{proj}\lim \mathbb{Z}/a_n\mathbb{Z}$$

which is a compact zero-dimensional abelian group. Suppose $f_1 \in C(X,\mathbb{Z})$ is of order a_1 in Γ and $\{f_n\}_{n=1}^{\infty} \subset C(X,\mathbb{Z})$ satisfies $f_1 - a_1^{-1}a_nf_n \in \pi^*(K^0(Y,\psi))$. Let $h_n \in C(X,\mathbb{Z})$ and $g_n \in C(Y,\mathbb{Z})$ be functions satisfying

$$a_nf_n + h_n - h_n\phi^{-1} = g_1\pi + \sum_{k=1}^{n-1} a_kg_{k+1}\pi.$$

For some fixed $x_0 \in X$ we may assume that $h_n(x_0) = 0$ for all $n \in \mathbb{N}$. Then we have $h_{n+1}(x) - h_n(x)$ is a_n-divisible for all $n \in \mathbb{N}$ and $x \in X$. Hence

$$H : X \ni x \mapsto (h_1(x), h_2(x), h_3(x), \dots)$$

is a well-defined continuous map from X to $\hat{\Gamma}$. Similarly

$$G : Y \ni y \mapsto (g_1(y), g_1(y) + a_1g_2(y), g_1(y) + a_1g_2(y) + a_2g_3(y), \dots)$$

is well-defined as a continuous map from Y to $\hat{\Gamma}$. In the same way as the case of the cyclic group valued cocycle, we can define a homeomorphism $\tilde{\psi}$ on $Y \times \hat{\Gamma}$ by

$$\tilde{\psi}(y,k) = (\psi(y), k + G(\psi(y))).$$

Moreover $\pi_1 : X \ni x \mapsto (\pi(x), H(x))$ gives a factor map from (X,ϕ) to $(Y \times \hat{\Gamma}, \tilde{\psi})$ and $\pi = \pi_0\pi_1$ where π_0 is the canonical projection from $Y \times \hat{\Gamma}$ to Y. For every $f \in C(X,\mathbb{Z})$, there exists a_n

such that $a_n[f] \in \pi^*(K^0(Y,\psi))$. Since every element of $K^0(Y,\psi)$ is a_n-divisible in $K^0(Y\times\hat{\Gamma},\tilde{\psi})$, we can conclude that π_1^* is an isomorphism.

There exists a positively asymptotic pair (y_0,y_1) in Y, as (Y,ψ) is a minimal subshift. Then

$$k=\lim_{n\to\infty}\sum_{i=1}^{n}(G(\psi^i(y_0))-G(\psi^i(y_1)))$$

exists in $\hat{\Gamma}$. Therefore $((y_0,0),(y_1,k))$ is a positively asymptotic pair in $(Y\times\hat{\Gamma},\tilde{\psi})$ and, by virtue of Proposition 4.7, $\zeta((y_0,0),(y_1,k))$ is zero. Because π_1^* is an isomorphism, preimages of $(y_0,0)$ and (y_1,k) by π_1 do the work. □

References

[BHR] Blanchard, F.; Host, B.; Ruette, S.; *Asymptotic pairs in positive-entropy systems*, Ergodic Theory Dynam. Systems 22 (2002), 671–686.

[GPS] Giordano, T.; Putnam, I. F.; Skau, C. F.; *Topological orbit equivalence and C^*-crossed products*, J. reine angew. Math. 469 (1995), 51–111.

[GPS2] Giordano, T.; Putnam, I. F.; Skau, C. F.; *K-theory and asymptotic index for certain almost one-to-one factors*, Math. Scand. 89 (2001), 297–319.

[M] Matui, H.; *Ext and OrderExt classes of certain automorphisms of C^*-algebras arising from Cantor minimal systems*, Canad. J. Math. 53 (2001), 325–354.

[M2] Matui, H.; *Finite order automorphisms and dimension groups of Cantor minimal systems*, J. Math. Soc. Japan 54 (2002), 135–160.

[O] Ormes, N.; *Real coboundaries for minimal Cantor systems*, Pacific J. Math. 195 (2000), 453–476.

[P] Putnam, I. F.; *The C^*-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math. 136 (1989), 329–353.

e-mail; matui@math.s.chiba-u.ac.jp
Department of Mathematics and Informatics,
Faculty of Science, Chiba University,
Yayoityô 1-33, Inageku,
Chiba, 263-8522,
Japan.