SQUARE ROOT CLOSED $C^*$-ALGEBRAS

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Abstract. We say that a $C^*$-algebra $A$ is approximately square root closed, if any normal element in $A$ can be approximated by a square of a normal element in $A$. We study when $A$ is approximately square root closed, and have an affirmative answer for AI-algebras, Goodearl type algebras over the torus, purely infinite simple unital $C^*$-algebras etc.

0. Introduction

D. Deckard and C. Pearcy [7, 8] proved that, for a commutative $AW^*$-algebra $M$, any algebraic equation with $M$-valued coefficients has roots in $M$. Many researchers study analogous problems for a commutative $C^*$-algebra $C(X)$, and some results are strongly related to topological properties of $X$ (e.g., covering dimension, cohomology etc.)[4, 5, 11, 17, 18].

In this paper, we consider this problem for a $C^*$-algebra which is not necessarily commutative. But we restrict our attention to a special quadratic equation, namely $x^2 = a$. We make the following definition:

Definition 0.1. Let $A$ be a $C^*$-algebra.

(1) We say that $A$ is square root closed, if for any normal element $a \in A$, there exists a normal element $b \in A$ such that $a = b^2$.

(2) We say that $A$ is approximately square root closed, if for any $\varepsilon > 0$ and any normal element $a \in A$, there exists a normal element $b \in A$ such that $\|a - b^2\| < \varepsilon$.

Needless to say, for a commutative $C^*$-algebra $A$, the square root closed property for $A$ is the same as the classical property, i.e., every element in $A$ has its square root in $A$.

Our result is as follows.

(1) Every AI-algebra is approximately square root closed. (Theorem 1.8.)

(2) If $A$ is a unital $C^*$-algebra, $A \otimes M_{\infty}$ is approximately square root closed. (Theorem 2.2.)

(3) For a Goodearl type algebra $A$ over $T$, $A$ is approximately square root closed if and only if $K_1(A)$ is 2-divisible. (Theorem 2.4.)

(4) For a purely infinite simple unital $C^*$-algebra $A$, $A$ is approximately square root closed if and only if $K_1(A)$ is 2-divisible. (Theorem 3.9.)

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1. **AI-Algebras**

It is clear that every finite dimensional $C^*$-algebra is square root closed. We say that a $C^*$-algebra $A$ has the property (FN), if any normal element in $A$ can be approximated by some normal element in $A$ with finite spectrum. If $A$ has the property (FN), then we can see that $A$ is approximately square root closed. H. Lin [14] proved that every AF-algebra has the property (FN). This implies every AF-algebra is approximately square root closed.

We give two examples of $C^*$-algebras which are approximately square root closed but not square root closed.

**Example 1.1.** There exists a unital AF-algebra $A$ such that $A$ has a maximal abelian self-adjoint subalgebra $B$ which is isomorphic to the algebra $C(T)$ of continuous functions on the torus $T$ (see [2]). Then $A$ is not square root closed.

Indeed, let $u$ be a unitary generator of $B \cong C(T)$. If $y \in A$ is normal and satisfies $y^2 = u$, then $y$ belongs to $B$ by the maximality of $B$. But $u$ does not have such an element in $B \cong C(T)$. So $A$ is not square root closed.

**Example 1.2.** Let $I = [0, 1]$ be the interval. The algebra $C(I, M_2)$ of $2 \times 2$ matrix valued continuous functions on $I$ is not square root closed but approximately square root closed.

We define a normal element $f \in C(I, M_2)$ as follows:

$$f(t) = \begin{cases} 
\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{6\pi\sqrt{-1}t} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & t \in [0, 1/3] \cup [2/3, 1] \\
\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} e^{6\pi\sqrt{-1}t} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & t \in (1/3, 2/3).
\end{cases}$$

We assume that $g$ is a normal element in $C(I, M_2)$ with $g^2 = f$. By the continuity of spectra, one of $g(1/3)$ and $g(2/3)$ must have the spectrum $\{1, -1\}$. We only consider the case $\text{Sp}(g(1/3)) = \{1, -1\}$. Since we have

$$\lim_{t \to 1/3^-} g(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \lim_{t \to 1/3^+} g(t),$$

this contradicts the assumption.

In Corollary 1.6, we will show that $C(I, M_n)$ is approximately square root closed. But, for above $f$, we construct its approximate square root here. Let $0 < \theta < 1$ and $u$ be a unitary in $C(I, M_2)$ with

$$u(\theta/3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad u(1/3) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u(1/3).$$
We define the normal element $h$ in $C(I, M_2)$ as follows:

$$h(t) = \begin{cases}
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + e^{3\pi\sqrt{-1}t/\theta} \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} & t \in [0, \theta/3] \\
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} u(t)^* & t \in [\theta/3, 1/3] \\
\begin{bmatrix}
1/2 & 1 \\
1/2 & 1
\end{bmatrix} + e^{3\pi\sqrt{-1}t} \begin{bmatrix}
1 & -1 \\
-1 & 1
\end{bmatrix} & t \in (1/3, 2/3) \\
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix} + e^{3\pi\sqrt{-1}t} \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix} & t \in [2/3, 1].
\end{cases}$$

It is easy to see that if $1 - \theta$ is sufficiently small, then so is $\|f - h\|$.

Let $f$ be a normal element of $C(I, M_n)$. For each point $t \in I$, $f(t)$ has the spectral decomposition: $f(t) = \sum_{i=1}^n \lambda_i(t)p_i(t)$, where $\lambda_1(t), \ldots, \lambda_n(t)$ are the eigenvalues of $f(t)$ and $p_i(t)$ is a one-dimensional projection corresponding to $\lambda_i$ ($1 \leq i \leq n$) and satisfying $\sum_{i=1}^n p_i(t) = 1$. By Rouché’s theorem, we may assume that $\lambda_i$ is continuous on $I$ for each $i$. But $p_i(t)$ is not necessarily continuous.

**Lemma 1.3.** Let $k \leq n$ and $\{p_i(t)\}_{i=1}^k \subset M_n$ be a family of mutually orthogonal, one-dimensional projections for each $t \in I$. If the map $I \ni t \mapsto p(t) = \sum_{i=1}^k p_i(t)$ is continuous, then there are mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $p_i(0) = q_i(0)$, $q_i(1) = p_i(1)$ and $p = \sum_{i=1}^k q_i$.

**Proof.** We can choose a continuous function $I \ni t \mapsto x_1(t) \in \text{Range}(p(t))$ such that $p_1(0)x_1(0) = x_1(0)$, $p_1(1)x_1(1) = x_1(1)$ and $\|x_1(t)\| = 1$ for any $t \in I$. Define the projection $q_1 = x_1 \otimes x_1 \in C(I, M_n)$. Then $I \ni t \mapsto p(t) - q_1(t)$ is continuous. Repeating the same argument, for $l = 2, \ldots, k$, we can choose a continuous function $I \ni t \mapsto x_l(t) \in \text{Range}((p - \sum_{i=1}^{l-1} q_i(t)))$ such that $p_1(0)x_l(0) = x_l(0)$, $p_l(1)x_l(1) = x_l(1)$ and $\|x_l(t)\| = 1$ for any $t \in I$. Therefore we have $p = \sum_{i=1}^k q_i$, where $q_i = x_i \otimes x_i$ for $i = 1, \ldots, k$. \[\Box\]

**Lemma 1.4.** Let $\varepsilon > 0$, $k \leq n$ and $f = \sum_{i=1}^n \lambda_i p_i$ be a normal element of $C(I, M_n)$, where $\lambda_1, \ldots, \lambda_n \in C(I)$ and $\{p_i(t)\}_{i=1}^k \subset M_n$ is a family of mutually orthogonal projections. If $|\lambda_1(t) - \lambda_1| < \varepsilon$ and $|\lambda_i(t) - \lambda_i| < |\lambda_1(t) - \lambda_m|)$ for each $l \in \{1, \ldots, k\}$ and $m \in \{k+1, \ldots, n\}$, then $p = \sum_{i=1}^k p_i \in C(I, M_n)$.

Moreover we can choose a family of mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $q_i(0) = p_i(0)$, $q_i(1) = p_i(1)$ and

$$\left\| p \right\| < 2\varepsilon - \sum_{i=1}^k \lambda_i q_i.$$

**Proof.** We can choose a continuously differentiable function $C: I \times \mathbb{T} \to C$ such that $C(t, \cdot) = C_t$ is a simple closed curve with canonical orientation and separates $\{\lambda_j(t), \ldots, \lambda_k(t)\}$ (in its inside) and $\{\lambda_{k+1}(t), \ldots, \lambda_n(t)\}$ (in its outside) for each $t \in I$. Since we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{C_t} \frac{1}{z - f(t)} \, dz = \sum_{i=1}^k p_i(t)$$
for any $t \in I$, this implies the continuity of $p = \sum_{i=1}^{k} p_i$.

By the previous lemma, there are mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $p_1(0) = q_1(0)$, $q_i(1) = p_i(1)$ and $p = \sum_{i=1}^{k} q_i$. Then we have

$$\left\| pfp - \sum_{i=1}^{k} \lambda_i q_i \right\| \leq \left\| \sum_{i=1}^{k} \lambda_i p_i - \sum_{i=1}^{k} \lambda_i q_i \right\| \leq \varepsilon + \left\| \sum_{i=1}^{k} \lambda_i p_i - \sum_{i=1}^{k} \lambda_i q_i \right\|$$

$$= \varepsilon + \left\| \lambda_1 p - \sum_{i=1}^{k} \lambda_i q_i \right\| < 2\varepsilon.$$

\[ \Box \]

**Proposition 1.5.** Let $\varepsilon > 0$ and $f$ be a normal element of $C(I, M_n)$. Then there are $\lambda_1, \ldots, \lambda_n \in C(I)$ and mutually orthogonal projections $q_1, \ldots, q_n \in C(I, M_n)$ such that

$$\left\| f - \sum_{i=1}^{n} \lambda_i q_i \right\| < \varepsilon.$$
C([b_{l-1}, b_l], M_n) satisfying \( q_i(t)b_{l-1} = p_i(b_{l-1}) \), \( q_i(t)b_l = p_i(b_l) \) and \[
\left\| f(t) - \sum_{i=1}^n \lambda_i(t)q_i(t) \right\| < \varepsilon. \]

We define, for each \( i \), \( q_i(t) = q_i(t) \), where \( t \in [b_{l-1}, b_l] \). Then we have \( q_1, \ldots, q_n \in C(I, M_n) \) as asserted.

**Corollary 1.6.** \( C(I, M_n) \) is approximately square root closed.

**Proof.** Let \( \varepsilon > 0 \) and \( f \) be a normal element of \( C(I, M_n) \). Applying Proposition 1.5, there are \( \lambda_1, \ldots, \lambda_n \in C(I) \) and mutually orthogonal projections \( q_1, \ldots, q_n \in C(I, M_n) \) such that \( \| f - \sum_{i=1}^n \lambda_i q_i \| < \varepsilon \). For each \( i = 1, \ldots, n \), we can find \( \mu_i \in C(I) \) satisfying \( \lambda_i = \mu_i^2 \), which means that \( C(I, M_n) \) is approximately square root closed.

A \( C^* \)-algebra is called an \( AI \)-algebra if it is isomorphic to the inductive limit of a sequence \( (C(I, F_n), \varphi_n) \), where each \( F_n \) is a finite dimensional \( C^* \)-algebra and each \( \varphi_n : C(I, F_n) \rightarrow C(I, F_{n+1}) \) is an injective \( * \)-homomorphism. A \( C^* \)-algebra \( A \) is called stable rank one, if the set \( GL(A) \) of invertible elements of \( A \) is dense in \( A \).

We remark that each \( C(I, F_n) \) has stable rank one.

We need the following lemma. We have been unable to find a suitable reference in the literature, so we include a proof for completeness.

**Lemma 1.7.** Let \( A = \lim A_n \) be an inductive limit such that each \( C^* \)-algebra \( A_n \) has stable rank one. Then, for any normal element \( x \in A \) and \( \varepsilon > 0 \), there exists a normal element \( y \) in some \( A_n \) such that \( \| x - y \| < \varepsilon \).

**Proof.** For each \( n \in \mathbb{N} \), we can find an element \( x_n \in A_n \) such that \( \| x - x_n \| \rightarrow 0 \). Then \([x_n] := (x_n) + \bigoplus A_n\) is a normal element in \( \prod A_n/ \bigoplus A_n \) and \( C^*([x_n]) \) is isomorphic to \( C(\text{Sp}([x_n])) \), where \( C^*([x_n]) \) is the \( C^* \)-algebra generated by \([x_n])]. Since \( \text{Sp}([x_n]) \) can be embedded in the closed unit disk \( \mathbb{D} \), we have a \( * \)-homomorphism from \( C(\mathbb{D}) \) onto \( C(\text{Sp}([x_n])) \). By using the argument of semi-projectivity [16, Theorem 19.2.7], there exist a natural number \( m \) and a normal element \( y_n \in A_n \) for \( n \geq m \) satisfying \([x_1, \ldots, x_{m-1}, y_m, y_{m+1}, \ldots] = [0, \ldots, 0, y_m, y_{m+1}, \ldots] \) in \( \prod A_n/ \bigoplus A_n \). If we set \( y = y_n \) for a sufficiently large \( n \), then \( y \) satisfies the desired condition.

**Theorem 1.8.** Every \( AI \)-algebra is approximately square root closed.

**Proof.** Let \( A = \lim A_n \) be an \( AI \)-algebra. Since each \( A_n \) has stable rank one, we can apply Lemma 1.7. So, for any normal element \( a \in A \) and \( \varepsilon > 0 \), there exists a normal element \( b \) in some \( A_n \) such that \( \| a - b \| < \varepsilon \). By Corollary 1.6, \( b \) can be approximated by a square of a normal element. Therefore \( A \) is approximately square root closed.

2. Two-divisibility for \( K_1 \)

**Lemma 2.1.** Let \( A \) be a \( C^* \)-algebra. If \( x \in A \) is normal, then there exists a normal element \( y \in A \otimes M_n \) such that \( x \otimes 1_n = y^n \).
We set $N = n^{\alpha}$. Let $\alpha = x \in A$ be a unital *-endomorphism defined by $\gamma(x) = 1_2 \otimes x$ ($x \in M_{2^\infty}$). For each $n$, choose a unitary $w_n \in \bigotimes_{i=1}^n M_2 \subset M_{2^\infty}$ such that
\[
\text{Ad } w_n(x_1 \otimes \cdots \otimes x_n) = w_n(x_1 \otimes \cdots \otimes x_n)w_n^* = x_n \otimes x_1 \otimes \cdots \otimes x_{n-1}
\]
for any $x_1 \otimes \cdots \otimes x_n \in \bigotimes_{i=1}^n M_2$. Then we have
\[
\lim_{n \to \infty} \| \gamma(x) - \text{Ad } w_n(x) \| = 0 \quad \text{for all } x \in M_{2^\infty}.
\]

**Theorem 2.2.** If $A$ is a unital $C^*$-algebra, then $A \otimes M_{2^\infty}$ is approximately square root closed.

**Proof.** We consider the *-endomorphism $\alpha = \text{id} \otimes \gamma$ of $A \otimes M_{2^\infty}$. It is easy to see that $\alpha(x) = \lim_{n \to \infty} \text{Ad}(1 \otimes w_n)(x)$ for all $x \in A \otimes M_{2^\infty}$.

For any normal element $x \in A \otimes M_{2^\infty}$, we can see $\alpha(x)$ like as $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. So there exists a normal element $y \in A \otimes M_{2^\infty}$ such that $y^2 = \alpha(x)$ by Lemma 2.1. It follows that
\[
\|(\text{Ad}(1 \otimes w_n^*)(y))^2 - x\| = \|y^2 - \text{Ad}(1 \otimes w_n)(x)\| \to \|y^2 - \alpha(x)\| = 0,
\]
which means that $A \otimes M_{2^\infty}$ is approximately square root closed. \hfill $\Box$

For a $C^*$-algebra $A$, we say that $K_1(A)$ is 2-divisible if any $[x] \in K_1(A)$ has an element $[y] \in K_1(A)$ with $[x] = 2[y]$.

It is known that if a unital $C^*$-algebra $A$ has stable rank one, then $M_n(A)$ has also stable rank one, and in this case the map from the unitary group of $A$ to $K_1(A)$ is surjective, see [19] for details.

Let $A$ be a unital commutative $C^*$-algebra ($A \cong C(X)$). It is well-known that $A$ has stable rank one if and only if the covering dimension of the associated compact Hausdorff space $X$ is less than one. In this case $K_1(A)$ is isomorphic to $GL(A)/GL_0(A)$, where $GL_0(A)$ is the connected component containing the identity in $GL(A)$.

**Proposition 2.3.** Let $A$ be a $C^*$-algebra with stable rank one.

1. If $A$ is approximately square root closed, then $K_1(A)$ is 2-divisible.
2. If $A$ is commutative and $K_1(A)$ is 2-divisible, then $A$ is approximately square root closed.

**Proof.** (1) Let $u$ be a unitary in $A$. There exists a normal element $a \in A$ such that $\|a^2 - 1\| < 1$. In particular, $a$ is invertible. Then we have $|u| = |a^2| = 2|a|$ in $K_1(A)$.

(2) Since $A$ has stable rank one, it suffices to show that any invertible element in $A$ can be approximated by a square of a normal element of $A$. For $a \in GL(A)$, there exists an invertible $b \in A$ such that $|a| = 2|b| = |b^2|$ in $K_1(A)$. Therefore $a$ is connected to $b^2$ in $GL(A)$. So we can choose $h_1, \ldots, h_n \in A$ such that $a = e^{h_1} \cdots e^{h_n} b^2$.\hfill $\Box$
It follows that $a = (e^{(h_1 + \cdots + h_n)/2}b)^2$.

Since $K_1(C(\mathbb{T})) = \mathbb{Z}$, we can see that $C(\mathbb{T})$ is not approximately square root closed. We define $A_n = C(\mathbb{T}) \ (n = 1, 2, \ldots)$ and a $^*$-homomorphism $\varphi_n$ from $A_n$ to $A_{n+1}$ by

$$\varphi_n(f)(z) = f(z^2) \quad (f \in C(\mathbb{T}) = A_n, z \in \mathbb{T}).$$

Then the inductive limit $A$ of this system $(A_n, \varphi_n)$ is a commutative $C^*$-algebra with stable rank one and has $K_1(A) \cong \mathbb{Z} \langle \frac{1}{2} \rangle = \{ \frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N} \}$. In fact, $A$ is approximately square root closed.

We take a sequence $\{x_n\}$ of a compact Hausdorff space $X$ and an increasing sequence $\{k_n\}$ of positive integers such that $k_n$ divides $k_{n+1}$ for each $n$. For each $n$, we define a $^*$-homomorphism $\varphi_n$ from $C(X, M_{k_n})$ to $C(X, M_{k_{n+1}})$ by

$$\varphi_n(f)(x) = \text{diag}(f(x), \ldots, f(x), f(x_n), \ldots, f(x_n))$$

for $f \in C(X, M_{k_n})$ and $x \in X$. Then we call the inductive limit $A$ of the inductive system $(C(X, M_{k_n}), \varphi_n)$ a Goodearl type algebra over $X$. We note that if $\{x_n\}$ is dense in $X$, then $A$ becomes simple and is called a Goodearl algebra [10]. But, in our setting $\{x_n\}$ is not necessarily dense in $X$.

**Theorem 2.4.** Let $A$ be a Goodearl type algebra over $\mathbb{T}$. Then the following are equivalent.

1. $A$ is approximately square root closed.
2. For any $n \in \mathbb{N}$, there exists $m \geq n$ such that $s(m)$ is even.
3. $K_1(A)$ is 2-divisible.

**Proof.** (1) $\Rightarrow$ (3). It follows from Proposition 2.3.

(3) $\Rightarrow$ (2). We remark that $K_1(A_n) \cong \mathbb{Z}$ for each $n \in \mathbb{N}$ and denote by $1_n$ the unit of $K_1(A_n)$. Then we have $(\varphi_n)_*(1_n) = s(n)1_{n+1} \in K_1(A_{n+1})$. By the assumption we can choose a positive integer $N(> n)$ such that

$$s(N)s(N-1) \cdots s(n)1_{N+1} \in 2K_1(A_{N+1}).$$

This means that $s(m)$ is even for some $m \in \{n, \ldots, N\}$.

(2) $\Rightarrow$ (1). Let $f$ be a normal element in $A$ and $\varepsilon > 0$. Since each $A_n$ has stable rank one, by the same argument in Theorem 1.8, we can choose a number $n$ and a normal element $g \in A_n$ such that $\|f - g\| < \varepsilon$. Then we may assume that $s(n)$ is even. By Lemma 2.1 we can show that

$$\varphi_n(g) = g \otimes 1_{s(n)} \oplus g(x_n) \oplus \cdots \oplus g(x_n)$$

has a square root in $A_{n+1}$.

\[ \square \]

3. **Purely infinite simple unital $C^*$-algebras**

Let $A$ be a unital simple $C^*$-algebra. We say that $A$ is purely infinite, if every non-zero hereditary $C^*$-subalgebra of $A$ contains an infinite projection. The simplicity and the pure infiniteness of $A$ ([20]) implies that $A$ has real rank zero, i.e., the invertible self-adjoint elements are dense in the set of the self-adjoint elements of $A$. It is also known that the following are equivalent:

i. $A$ has real rank zero.

ii. $A$ has the property (HP), i.e., every non-zero hereditary $C^*$-subalgebra $B$ of $A$ has an approximate identity of projections in $B$ ([3]).
We denote the unit circle in $\mathbb{C}$ by $\mathbb{T}$. If $\text{Sp}(u)$ is not the whole of $\mathbb{T}$, then $u$ has a square root. Therefore we may assume $\text{Sp}(u) = \mathbb{T}$. Let $F \subseteq \mathbb{T}$ be an $\varepsilon$-dense finite subset of $\mathbb{T}$, that is, for any $\xi \in \mathbb{T}$ there exists $\eta \in F$ such that $|\xi - \eta| < \varepsilon$. Since $A$ has real rank zero, applying $[13, \text{Lemma } 2]$, there exist a unitary $u_0 \in A$ and a family of mutually orthogonal nonzero projections $\{e_\eta\}_{\eta \in F}$ such that $\|u - u_0\| < \varepsilon$ and

$$e_\eta u_0 = u_0 e_\eta = \eta e_\eta$$

for all $\eta \in F$. Let $e = 1 - \sum_{\eta \in F} e_\eta$ and $B = e A e$. Then $u_1 = u_0 e$ is a unitary of $B$. Note that $[u_1 + 1 - e]$ is equal to $[u]$ in $K_1(A)$. Hence there exists a unitary $v \in B$ such that $[u_2] = -2 [v]$ in $K_1(B) \cong K_1(A)$. Since $M_2(B)$ has the property weak (FU), there exist projections $q_1, q_2, \ldots, q_n \in M_2(B)$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{T}$ such that

$$\sum_{i=1}^n q_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\left\| \begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix} - \sum_{i=1}^n \xi_i q_i \right\| < \varepsilon.$$  

Because $F$ is $\varepsilon$-dense in $\mathbb{T}$, for each $i = 1, 2, \ldots, n$, there exists $\eta_i \in F$ such that $|\xi_i - \eta_i| < \varepsilon$. It follows that

$$\left\| \begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix} - \sum_{i=1}^n \eta_i q_i \right\| < 2\varepsilon.$$  

Since $A$ is simple and purely infinite, there exists a family of mutually orthogonal projections $r_i$ in $A$ such that $r_i \leq e_{\eta_i}$ and $[r_i] = [q_i]$ in $K_0(A)$. Put $r = \sum_{i=1}^n r_i$. Then we have

$$u_0 r = \sum_{i=1}^n \eta_i r_i,$$

and so we can find a unitary $u_2 \in r A r$ which is a copy of the unitary

$$\begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix},$$

and $\|u_2 - u_0 r\|$ is less than $2\varepsilon$. It follows that $u_3 = u_1 + u_2 + u_0 (1 - e - r)$ is a unitary of $A$ and $\|u_3 - u\|$ is less than $3\varepsilon$. Moreover $u_1 + u_2$ looks like

$$\begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_1 & 0 \\ 0 & 0 & v^2 \end{bmatrix},$$
which is a square of
\[
\begin{bmatrix}
0 & u_1 & 0 \\
1 & 0 & 0 \\
0 & 0 & v \\
\end{bmatrix}.
\]
Because \( u_0(1 - e - r) \) has finite spectrum, the proof is completed. □

**Corollary 3.2.** Let \( A \) be a unital simple purely infinite \( C^* \)-algebra. Suppose that \( K_1(A) \) is 2-divisible. If \( x \in A \) is a normal element and \( \text{Sp}(x) \) is homeomorphic to the circle, then for any \( \varepsilon > 0 \) there exists a normal element \( y \in A \) such that
\[
\|x - y^2\| < \varepsilon.
\]

**Proof.** Since the circle is one-dimensional, by perturbing \( x \) a little bit, we may assume that \( x \) is invertible. Let \( f : \mathbb{T} \to \text{Sp}(x) \) be a homeomorphism. Because \( f \) is a homeomorphism onto \( \text{Sp}(x) \), the rotation number of \( f \) is \(-1\) or \( 0 \) or \( 1 \). If the rotation number of \( f \) is zero, then \( x \) has a square root. Hence, without loss of generality, we may assume that the rotation number of \( f \) is one. We denote the inverse of \( f \) by \( f^{-1} : \text{Sp}(x) \to \mathbb{T} \).

There exists \( \delta > 0 \) such that if \( u, v \in A \) are unitaries with \( \|u - v\| < \delta \), then \( \|f(u) - f(v)\| < \varepsilon \). Applying Proposition 3.1 to the unitary \( f^{-1}(x) \), we get a unitary \( v \in A \) such that
\[
\|f^{-1}(x) - v^2\| < \delta,
\]
which means that
\[
\|x - f(v^2)\| < \varepsilon.
\]
Since the rotation number of the function
\[
\mathbb{T} \ni \xi \to f(\xi^2) \in \mathbb{C}
\]
is two, we can find a continuous function \( g : \mathbb{T} \to \mathbb{C} \) such that
\[
g^2(\xi) = f(\xi^2)
\]
for all \( \xi \in \mathbb{T} \). Put \( y = g(v) \). Then \( y \) is a normal element and \( y^2 = g^2(v) = f(v^2) \), which completes the proof. □

Let \( a \) and \( b \) be two elements of a \( C^* \)-algebra and \( \varepsilon > 0 \). We write \( a \sim b \), if \( \|a - b\| < \varepsilon \).

**Lemma 3.3.** Let \( A \) be a unital \( C^* \)-algebra and \( x \in A \) be a normal element. Suppose that there exist \( \zeta \in \text{Sp}(x) \) and closed subsets \( G_0, G_1 \subset \text{Sp}(x) \) such that \( \text{Sp}(x) = G_0 \cup G_1 \) and \( G_0 \cap G_1 = \{\zeta\} \). Then, for any \( \varepsilon > 0 \), there exist normal elements \( x_0, x_1 \in A \) and a unitary \( u \in M_2(A) \) such that \( \text{Sp}(x_i) = G_i \) and
\[
\left\|u \begin{bmatrix} x & \zeta \\ \zeta & x \end{bmatrix} u^* - \begin{bmatrix} x_0 & x_1 \\ x_1 & x_0 \end{bmatrix}\right\| < \varepsilon.
\]

**Proof.** We can identify \( C(\text{Sp}(x)) \) with the abelian \( C^* \)-subalgebra of \( A \) which is generated by \( x \) and \( 1 \). Put \( O = \{\xi \in \mathbb{C} : |\xi - \zeta| < \varepsilon/2\} \).

Since \( G_0 \setminus O \) and \( G_1 \setminus O \) are disjoint, there exists a unitary \( u \in M_2(C(\text{Sp}(x))) \cong C(\text{Sp}(x), M_2) \) such that
\[
u(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{for } \xi \in G_0 \setminus O \quad \text{and} \quad u(\xi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{for } \xi \in G_1 \setminus O.
\]
Define \( x_i \in C(\text{Sp}(x)) \) by
\[
x_i(\xi) = \begin{cases} \xi & \xi \in G_i \\ \zeta & \xi \in G_{1-i} \end{cases}.
\]

If \( \xi \notin O \), then we can check
\[
u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix} \nu(\xi)^* = \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.
\]

If \( \xi \in G_0 \cap O \), then
\[
u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix} \nu(\xi)^* \approx \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} \approx \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.
\]

When \( \xi \in G_1 \cap O \), we can obtain the same estimate. \( \square \)

We put\[
H_+ = \{ a + b\sqrt{-1} \in \mathbb{C} : b \geq 0 \}
\]
and
\[
H_- = \{ a + b\sqrt{-1} \in \mathbb{C} : b \leq 0 \}.
\]

We identify the real line \( \mathbb{R} \) with \( H_+ \cap H_- \).

**Lemma 3.4.** Let \( A \) be a unital \( C^* \)-algebra and \( x \in A \) be a normal element. Suppose that there exists a homeomorphism \( f : \mathbb{C} \to \mathbb{C} \) such that \( f(\mathbb{R}) \cap \text{Sp}(x) = f(\mathbb{R}) \cap \text{Sp}(x) \). Then, for any \( \varepsilon > 0 \), there exist normal elements \( x_0, x_1, a \in A \) and a unitary \( u \in M_2(A) \) such that
\[
\text{Sp}(x_0) = f(H_+ \cap \text{Sp}(x)), \quad \text{Sp}(x_1) = f(H_- \cap \text{Sp}(x)), \quad \text{Sp}(a) = f([-1, 1])
\]
and
\[
\left\| u \begin{bmatrix} x & 0 \\ 0 & a \end{bmatrix} u^* - \begin{bmatrix} x_0 & 0 \\ 0 & x_1 \end{bmatrix} \right\| < \varepsilon.
\]

**Proof.** We identify \( C(\text{Sp}(x)) \) with the abelian \( C^* \)-subalgebra of \( A \) which is generated by \( x \) and \( 1 \in A \). We first deal with the case that \( f : \mathbb{C} \to \mathbb{C} \) is the identity map. Let \( h_0 : H_+ \to [-1, 1] \) and \( h_1 : H_- \to [-1, 1] \) be continuous functions such that \( h_i(\xi) = \xi \) for \( \xi \in [-1, 1] \). Define \( a, x_0, x_1 \in C(\text{Sp}(x)) \) by
\[
a(\xi) = \begin{cases} h_0(\xi) & \xi \in H_+ \\ h_1(\xi) & \xi \in H_- \end{cases},
\]
\[
x_0(\xi) = \begin{cases} \xi & \xi \in H_+ \\ h_1(\xi) & \xi \in H_- \end{cases}
\]
and
\[
x_1(\xi) = \begin{cases} h_0(\xi) & \xi \in H_+ \\ \xi & \xi \in H_- \end{cases}.
\]

Since \( \text{Sp}(x) \cap \mathbb{R} = [-1, 1] \), there exists \( \delta > 0 \) such that if \( \xi = s + t\sqrt{-1} \in \text{Sp}(x) \) with \( |t| < \delta \), then \( |h_i(\xi) - \xi| < \varepsilon/2 \) for each \( i = 0, 1 \). We can find a unitary \( u \in M_2(C(\text{Sp}(x))) \cong C(\text{Sp}(x), M_2) \) such that
\[
u(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
for \( \xi = s + t\sqrt{-1} \in \text{Sp}(x) \) with \( t \geq \delta \) and
\[
u(\xi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
for \( \xi = s + t\sqrt{-1} \in \text{Sp}(x) \) with \( t \leq -\delta \). If \( |t| \geq \delta \), then for \( \xi = s + t\sqrt{-1} \in \text{Sp}(x) \) we can check
\[
u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & a(\xi) \end{bmatrix} \nu(\xi)^* = \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}
\]
If \( |t| < \delta \), then for \( \xi = s + t\sqrt{-1} \in \text{Sp}(x) \) we can also check
\[
u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & a(\xi) \end{bmatrix} \nu(\xi)^* \cdot \nu(\xi) \begin{bmatrix} a(\xi) & 0 \\ 0 & a(\xi) \end{bmatrix} \nu(\xi)^* = \begin{bmatrix} a(\xi) & 0 \\ 0 & a(\xi) \end{bmatrix} \cdot \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}
\]
Now let us turn to the general case. Because \( K = f^{-1}(\text{Sp}(x)) = \text{Sp}(f^{-1}(x)) \) is compact, there exists \( \delta > 0 \) such that if \( y_0 \) and \( y_1 \) are normal elements in some \( C^* \)-algebra \( B \) with \( \text{Sp}(y_1) \subset K \) and \( \|y_0 - y_1\| < \delta \), then \( \|f(y_0) - f(y_1)\| < \varepsilon \). Applying the first part of this proof to \( f^{-1}(x) \) and \( \delta \), we get
\[
\left\| u \begin{bmatrix} f^{-1}(x) & 0 \\ 0 & a \end{bmatrix} u^* - \begin{bmatrix} x_0 & 0 \\ 0 & x_1 \end{bmatrix} \right\| < \delta.
\]
By the choice of \( \delta \), we obtain
\[
\left\| u \begin{bmatrix} z & f(a) \\ f(\xi) & \end{bmatrix} u^* - \begin{bmatrix} f(x_0) & \end{bmatrix} \right\| < \varepsilon,
\]
thereby completing the proof.

We define \( I_0 \) and \( I_1 \) by
\[
I_0 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq a \leq 1, b = 0\}
\]
and
\[
I_1 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq b \leq 1, a = 0\}.
\]
Let \( G \) be a compact subset of \( \mathbb{C} \). We say that \( G \) is a lattice graph, if there exist finite subsets \( F_0 \) and \( F_1 \) of \( Z + Z\sqrt{-1} \) such that
\[
G = \bigcup_{i=0,1} \bigcup_{\zeta \in F_i} I_i + \zeta.
\]
We call each point in \( G \cap (Z + Z\sqrt{-1}) \) a vertex of \( G \) and each \( I_i + \zeta \) contained in \( G \) an edge of \( G \). We denote by \( |G| \) the number of edges of \( G \).

**Proposition 3.5.** For any nonempty connected lattice graph \( G \), there exists a natural number \( N(G) \in \mathbb{N} \) such that the following holds: Let \( A \) be a unital \( C^* \)-algebra and \( x \in A \) be a normal element with \( \text{Sp}(x) = G \). For any \( \varepsilon > 0 \), there exist a natural number \( N \leq N(G) \), normal elements \( a_1, a_2, \ldots, a_N, x_0, x_1, \ldots, x_N \in A \), and a unitary \( u \in M_{N+1}(A) \) such that the following are satisfied.

1. \( \|u \text{diag}(x, a_1, a_2, \ldots, a_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon \).
2. \( \text{Sp}(x_i) \) is contained in \( G \).
3. \( \text{Sp}(x_i) \) is homeomorphic to the closed interval \([-1, 1]\) or the circle.
4. \( \text{Sp}(a_i) \) is contained in \( G \).
5. \( \text{Sp}(a_i) \) is a single point or homeomorphic to the closed interval \([-1, 1] \).
Proof. The proof goes by induction concerning $|G|$. If $|G| = 1$, then $G$ is homeomorphic to the closed interval, and so we have nothing to do.

We may assume that the assertion has been proved for all $G$ with $|G| < L$. Let us consider a connected lattice graph $G$ with $|G| = L$. We would like to show that

$$N(G) = 2 \max \{N(G_0) : G_0 \text{ is a connected lattice graph with } G_0 \subseteq G \} + 1$$

does the work. Suppose that $A$ is a unital $C^*$-algebra and $x \in A$ is a normal element with $G = \text{Sp}(x)$. Take $\varepsilon > 0$.

Suppose that there exists a vertex $\zeta \in G$ such that $G \setminus \{\zeta\}$ is not connected. We can find nonempty connected lattice graphs $G_0$ and $G_1$ such that $G = G_0 \cup G_1$ and $G_0 \cap G_1 = \{\zeta\}$. Applying Lemma 3.3 to $G_0, G_1, \zeta$ and $\varepsilon/2$, we obtain normal elements $x_0, x_1 \in A$ and a unitary $u \in M_2(A)$ such that $\text{Sp}(x_i) = G_i$ and

$$\| u \begin{bmatrix} x & \zeta \\ \zeta^* & u^* \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \| < \frac{\varepsilon}{2}.$$ 

By the induction hypothesis, there exists $N_i \leq N(G_i)$ such that the assertion holds for $x_i$ and $\varepsilon/2$. Hence $N = N_0 + N_1 + 1 \leq N(G)$ works for $x$ and $\varepsilon$.

Therefore we may assume that $G \setminus \{\zeta\}$ is connected for all vertices $\zeta$ in $G$. Let $O$ be the unbounded connected component of $\mathbb{C} \setminus G$ and $\partial O$ be the boundary of $O$ in $\mathbb{C}$. Then $\partial O \subseteq G$ is homeomorphic to the circle. If $G = \partial O$, then we have nothing to do. Let us assume that $G \neq \partial O$. We can find an edge $e \subset G$ such that $e$ is not contained in $\partial O$ and an endpoint $\zeta_0$ of $e$ belongs to $\partial O$. Let $\zeta_1$ be the other endpoint of $e$. Since $G \setminus \{\zeta_0\}$ is connected, we can find a path in $G$ from $\zeta_1$ to a vertex $\zeta_2 \in \partial O$ which is distinct from $\zeta_0$. Let $P$ be the union of this path and $e$. Then $P \subseteq G$ is homeomorphic to the closed interval $[-1, 1]$ and its endpoints are $\zeta_0$ and $\zeta_2$. There exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{R}) \cap G = P$ and $f([-1, 1]) = P$. Applying Lemma 3.4 to $f$ and $\varepsilon/2$, we obtain normal elements $x_0, x_1, a \in A$ and a unitary $u \in M_2(A)$ such that

$$\text{Sp}(x_0) = f(H_+) \cap \text{Sp}(x), \quad \text{Sp}(x_1) = f(H_-) \cap \text{Sp}(x), \quad \text{Sp}(a) = f([-1, 1])$$

and

$$\| u \begin{bmatrix} x & a \\ a^* & u^* \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \| < \frac{\varepsilon}{2}.$$ 

Put $G_i = \text{Sp}(x_i)$ for $i = 0, 1$. Note that $G_i$ is a connected lattice graph. By the induction hypothesis, there exists a natural number $N_i \leq N(G_i)$ such that the assertion holds for $x_i$ and $\varepsilon/2$. Hence $N = N_0 + N_1 + 1 \leq N(G)$ works for $x$ and $\varepsilon$.

Lemma 3.6. Let $A$ be a unital $C^*$-algebra and $a \in A$ be a normal element. Suppose that $\text{Sp}(a)$ is homeomorphic to the closed interval $[-1, 1]$. For any $\varepsilon > 0$, there exist complex numbers $\xi_1, \xi_2, \ldots, \xi_N, \eta_0, \eta_1, \ldots, \eta_N \in \text{Sp}(a)$ and a unitary $u \in M_{N+1}(A)$ such that

$$\| u \text{diag}(a, \xi_1, \xi_2, \ldots, \xi_N)u^* - \text{diag}(\eta_0, \eta_1, \ldots, \eta_N) \| < \varepsilon.$$ 

Proof. By using Lemma 3.3 repeatedly, we can find $\xi_1, \xi_2, \ldots, \xi_N \in \text{Sp}(a)$ and normal elements $x_0, x_1, \ldots, x_N \in A$ and a unitary $u \in M_{N+1}(A)$ such that

$$\| u \text{diag}(a, \xi_1, \xi_2, \ldots, \xi_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N) \| < \frac{\varepsilon}{2}$$

and $\text{Sp}(x_i)$ has diameter less than $\varepsilon/2$. Replacing $x_i$ with some $\eta_i \in \text{Sp}(x_i)$, we get the conclusion. □
This lemma together with Proposition 3.5 directly implies the following.

**Proposition 3.7.** Let $A$ be a unital $C^*$-algebra and $x \in A$ be a normal element. Suppose that $G = \text{Sp}(x)$ is a lattice graph. For any $\varepsilon > 0$, there exist $N \in \mathbb{N}$, $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{C}$, normal elements $x_0, x_1, \ldots, x_N \in A$ and a unitary $u \in M_{N+1}(A)$ such that the following are satisfied.

1. $\|u \text{diag}(x, \xi_1, \xi_2, \ldots, \xi_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon$.
2. $\text{Sp}(x_i)$ is contained in $G$.
3. $\text{Sp}(x_i)$ is a single point or homeomorphic to the closed interval $[-1, 1]$ or the circle.
4. $\xi_i$ is contained in $G$.

Combining this with Corollary 3.2, we get the following.

**Lemma 3.8.** Let $A$ be a unital simple purely infinite $C^*$-algebra. Suppose that $K_1(A)$ is $2$-divisible. If $x \in A$ is a normal element and $\text{Sp}(\varepsilon^{-1}x)$ is a connected lattice graph for some $\varepsilon > 0$, then there exists a normal element $y \in A$ such that $\|x - y^2\| < 2\varepsilon$.

**Proof.** Put $G = \text{Sp}(\varepsilon^{-1}x)$ and $F = \text{Sp}(x) \cap (\varepsilon \mathbb{Z} + \varepsilon \mathbb{Z} \sqrt{-1})$.

Thus, $\varepsilon^{-1}F$ is the set of vertices of the lattice graph $G$. Clearly $F$ is an $\varepsilon/2$-dense finite subset of $\text{Sp}(x)$. As before, we put

$$I_0 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq a \leq 1, b = 0\}$$

and

$$I_1 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq b \leq 1, a = 0\}.$$

We define a continuous function $f : \text{Sp}(x) \to \text{Sp}(x)$ as follows: If $\xi = a + b\sqrt{-1} \in \text{Sp}(x)$ belongs to $\varepsilon I_0 + \zeta$ with $\zeta = t + b\sqrt{-1} \in F$, then we set

$$f(\xi) = \begin{cases} 
\zeta & t \leq a \leq t + \frac{\varepsilon}{3} \\
\zeta + 3(a - t - \frac{\varepsilon}{3}) & t + \frac{\varepsilon}{3} \leq a \leq t + \frac{2\varepsilon}{3} \\
\zeta + \varepsilon & t + \frac{2\varepsilon}{3} \leq a \leq t + \varepsilon.
\end{cases}$$

If $\xi = a + b\sqrt{-1} \in \text{Sp}(x)$ belongs to $\varepsilon I_1 + \zeta$ with $\zeta = a + t\sqrt{-1} \in F$, then we set

$$f(\xi) = \begin{cases} 
\zeta & t \leq b \leq t + \frac{\varepsilon}{3} \\
\zeta + 3(b - t - \frac{\varepsilon}{3})\sqrt{-1} & t + \frac{\varepsilon}{3} \leq b \leq t + \frac{2\varepsilon}{3} \\
\zeta + \varepsilon \sqrt{-1} & t + \frac{2\varepsilon}{3} \leq b \leq t + \varepsilon.
\end{cases}$$

Define $z = f(x)$. Evidently we have $\|x - z\| \leq \varepsilon/3$ and $\text{Sp}(z) = \text{Sp}(x) = \varepsilon G$. For each $\eta \in F$, let $g_\eta : F \to [0, 1]$ be a continuous function such that $g_\eta(\xi) = 1$ and $g_\eta(\xi) = 0$ if $|\xi - \eta| \geq \varepsilon/3$. Since $A$ has real rank zero, there exists a nonzero projection $e_\eta \in g_\eta(x)Ag_\eta(x)$. It is not hard to see that $e_\eta z = ze_\eta = \eta e_\eta$. Note that $(e_\eta)_{\eta \in F}$ is a family of mutually orthogonal projections. Put $e = 1 - \sum_{\eta \in F} e_\eta$, $B = eAe$ and $z_0 = ze$.

Then we have

$$z = z_0 + \sum_{\eta \in F} \eta e_\eta,$$

and so the spectrum of $z_0$ in $B$ is equal to $\varepsilon G = \text{Sp}(z)$. 

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*SQUARE ROOT CLOSED C*-ALGEBRAS*
By applying Proposition 3.7 to $e^{-1}z_0 \in B$ and $1$, we obtain complex numbers $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{C}$, normal elements $x_0, x_1, \ldots, x_N \in B$ and a unitary $u \in M_{N+1}(B)$ such that

1. $\|u \text{diag}(e^{-1}z_0, \xi_1, \xi_2, \ldots, \xi_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N)\| < 1$.
2. $\text{Sp}(x_i)$ is a single point or homeomorphic to the closed interval $[-1, 1]$ or the circle.
3. $\xi_i$ is contained in $G$.

By replacing $\xi_i$ and $x_i$ with $e^{-1}\xi_i$ and $e^{-1}x_i$, we get

1. $\|u \text{diag}(z_0, \xi_1, \xi_2, \ldots, \xi_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon$.
2. $\text{Sp}(x_i)$ is a single point or homeomorphic to the closed interval $[-1, 1]$ or the circle.
3. $\xi_i$ is contained in $\text{Sp}(x)$.

Because $F$ is $\varepsilon/2$-dense in $\text{Sp}(x)$, for each $i = 1, 2, \ldots, N$ we can find $\eta_i \in F$ such that $|\xi_i - \eta_i| \leq \varepsilon/2$. It follows that

$$\|u \text{diag}(z_0, \eta_1, \eta_2, \ldots, \eta_N)u^* - \text{diag}(x_0, x_1, \ldots, x_N)\| < \frac{3\varepsilon}{2}.$$ 

Since $A$ is purely infinite, there exists a family of mutually orthogonal projections $q_i$ such that $q_i \leq \varepsilon_i$ and $[q_i] = [e]$ in $K_0(A)$. Put $q = \sum q_i$. Then we have

$$(e + q)z = z_0 + \sum_{i=1}^{N} \eta_i q_i,$$

and so there exists a normal element $w \in (e + q)A(e + q)$ which is a unitary conjugation of $\text{diag}(x_0, x_1, \ldots, x_N)$ and

$$\|(e + q)z - w\| < \frac{3\varepsilon}{2}.$$ 

Thanks to Corollary 3.2, we can find a normal element $y_0 \in (e + q)A(e + q)$ such that

$$\|w - y_0^2\| < \frac{\varepsilon}{6}.$$ 

Since $(1 - e - q)z$ has finite spectrum, it has a square root $y_1$. Put $y = y_0 + y_1$. Then we have

$$\|z - y^2\| = \|(e + q)z - y_0^2\| < \|w - y_0^2\| + \frac{3\varepsilon}{2} < \frac{3\varepsilon}{2} + \frac{\varepsilon}{6}.$$ 

This estimate together with $\|x - z\| \leq \varepsilon/3$ implies

$$\|x - y^2\| < 2\varepsilon.$$ 

Now we are ready to prove the main result of this section.

**Theorem 3.9.** For a unital simple purely infinite $C^*$-algebra $A$, the following are equivalent.

1. $A$ is approximately square root closed.
2. $K_1(A)$ is 2-divisible.

**Proof.** (1)⇒(2). Since $K_1(A) \cong U(A)/U_0(A)$, it suffices to show that every unitary in $A$ is divided by 2 in $K_1(A)$. Let $u$ be a unitary in $A$. Then there exists a unitary $v \in A$ such that $\|u - v^2\| < 1$. Therefore $[u] = 2[e]$ in $K_1(A)$. 

(2)⇒(1). Take a normal element \( x \in A \) and a small real number \( \varepsilon > 0 \). By [9, Lemma 3.2], there exists a normal element \( z \in A \) such that \( \|x - z\| < \varepsilon \) and \( \text{Sp}(z) \) is contained in

\[
\{ a + b\sqrt{-1} \in \mathbb{C} : a \in \varepsilon \mathbb{Z} \text{ or } b \in \varepsilon \mathbb{Z} \}.
\]

By perturbing \( z \) a little bit more, we can find a normal element \( w \in A \) such that \( \|z - w\| < \varepsilon \) and \( G = \varepsilon^{-1} \text{Sp}(w) \) is a lattice graph. Let \( G_1, G_2, \ldots, G_n \) be connected components of \( G \). Each \( G_i \) is a connected lattice graph. Let \( h_i \) be the characteristic function on \( \varepsilon G_i \) and put \( v_i = h_i(w) \). Then \( w \) is the direct sum of \( v_1, v_2, \ldots, v_n \) and \( \text{Sp}(w) = \varepsilon G_i \). By using the lemma above, we get mutually orthogonal normal elements \( y_1, y_2, \ldots, y_n \) such that \( \|w_i - y_i^2\| < 2\varepsilon \). Put \( y = y_1 + y_2 + \cdots + y_n \). We can easily see that \( \|x - y^2\| < 4\varepsilon \).

\[\square\]

References


