SQUARE ROOT CLOSED C^*-ALGEBRAS

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Abstract. We say that a C^*-algebra A is approximately square root closed, if any normal element in A can be approximated by a square of a normal element in A. We study when A is approximately square root closed, and have an affirmative answer for AI-algebras, Goodearl type algebras over the torus, purely infinite simple unital C^*-algebras etc.

0. Introduction

D. Deckard and C. Pearcy [7, 8] proved that, for a commutative AW^*-algebra M, any algebraic equation with M-valued coefficients has roots in M. Many researchers study analogous problems for a commutative C^*-algebra $C(X)$, and some results are strongly related to topological properties of X (e.g., covering dimension, cohomology etc.)[4, 5, 11, 17, 18].

In this paper, we consider this problem for a C^*-algebra which is not necessarily commutative. But we restrict our attention to a special quadratic equation, namely $x^2 = a$. We make the following definition:

Definition 0.1. Let A be a C^*-algebra.

(1) We say that A is *square root closed*, if for any normal element $a \in A$, there exists a normal element $b \in A$ such that $a = b^2$.
(2) We say that A is *approximately square root closed*, if for any $\varepsilon > 0$ and any normal element $a \in A$, there exists a normal element $b \in A$ such that $\|a - b^2\| < \varepsilon$.

Needless to say, for a commutative C^*-algebra A, the square root closed property for A is the same as the classical property, i.e., every element in A has its square root in A.

Our result is as follows.

(1) Every AI-algebra is approximately square root closed. (Theorem 1.8.)
(2) If A is a unital C^*-algebra, $A \otimes M_{2^\infty}$ is approximately square root closed. (Theorem 2.2.)
(3) For a Goodearl type algebra A over $\mathbb{T}$, A is approximately square root closed if and only if $K_1(A)$ is 2-divisible. (Theorem 2.4.)
(4) For a purely infinite simple unital C^*-algebra A, A is approximately square root closed if and only if $K_1(A)$ is 2-divisible. (Theorem 3.9.)

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1. AI-algebras

It is clear that every finite dimensional C^*-algebra is square root closed. We say that a C^*-algebra A has the property (FN), if any normal element in A can be approximated by some normal element in A with finite spectrum. If A has the property (FN), then we can see that A is approximately square root closed. H. Lin [14] proved that every AF-algebra has the property (FN). This implies every AF-algebra is approximately square root closed.

We give two examples of C^*-algebras which are approximately square root closed but not square root closed.

Example 1.1. There exists a unital AF-algebra A such that A has a maximal abelian self-adjoint subalgebra B which is isomorphic to the algebra $C(\mathbb{T})$ of continuous functions on the torus $\mathbb{T}$ (see [2]). Then A is not square root closed.

Indeed, let u be a unitary generator of $B \cong C(\mathbb{T})$. If $y \in A$ is normal and satisfies $y^2 = u$, then y belongs to B by the maximality of B. But u does not have such an element in $B \cong C(\mathbb{T})$. So A is not square root closed.

Example 1.2. Let $I = [0,1]$ be the interval. The algebra $C(I, M_2)$ of 2×2 matrix valued continuous functions on I is not square root closed but approximately square root closed.

We define a normal element $f \in C(I, M_2)$ as follows:

$$f(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{6\pi\sqrt{-1}t} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & t \in [0, 1/3] \cup [2/3, 1] \\ \dfrac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \dfrac{1}{2} e^{6\pi\sqrt{-1}t} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & t \in (1/3, 2/3). \end{cases}$$

We assume that g is a normal element in $C(I, M_2)$ with $g^2 = f$. By the continuity of spectra, one of $g(1/3)$ and $g(2/3)$ must have the spectrum $\{1, -1\}$. We only consider the case $\mathrm{Sp}(g(1/3)) = \{1, -1\}$. Since we have

$$\lim_{t \to 1/3-0} g(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \lim_{t \to 1/3+0} g(t),$$

this contradicts the assumption.

In Corollary 1.6, we will show that $C(I, M_n)$ is approximately square root closed. But, for above f, we construct its approximate square root here. Let $0 < \theta < 1$ and u be a unitary in $C(I, M_2)$ with

$$u(\theta/3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad u(1/3) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} u(1/3).$$

We define the normal element h in $C(I, M_2)$ as follows:

$$h(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{3\pi\sqrt{-1}t/\theta} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & t \in [0, \theta/3] \\ u(t) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} u(t)^* & t \in [\theta/3, 1/3] \\ \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{1}{2} e^{3\pi\sqrt{-1}t} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & t \in (1/3, 2/3) \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{3\pi\sqrt{-1}t} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & t \in [2/3, 1]. \end{cases}$$

It is easy to see that if $1 - \theta$ is sufficiently small, then so is $\|f - h^2\|$.

Let f be a normal element of $C(I, M_n)$. For each point $t \in I$, $f(t)$ has the spectral decomposition: $f(t) = \sum_{i=1}^n \lambda_i(t) p_i(t)$, where $\lambda_1(t), \ldots, \lambda_n(t)$ are the eigenvalues of $f(t)$ and $p_i(t)$ is a one-dimensional projection corresponding to λ_i ($1 \leq i \leq n$) and satisfying $\sum_{i=1}^n p_i(t) = 1$. By Rouché's theorem, we may assume that λ_i is continuous on I for each i. But $p_i(t)$ is not necessarily continuous.

Lemma 1.3. *Let $k \leq n$ and $\{p_i(t)\}_{i=1}^k \subset M_n$ be a family of mutually orthogonal, one-dimensional projections for each $t \in I$. If the map $I \ni t \mapsto p(t) = \sum_{i=1}^k p_i(t)$ is continuous, then there are mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $p_i(0) = q_i(0)$, $q_i(1) = p_i(1)$ and $p = \sum_{i=1}^k q_i$.*

Proof. We can choose a continuous function $I \ni t \mapsto x_1(t) \in \mathrm{Range}(p(t))$ such that $p_1(0)x_1(0) = x_1(0)$, $p_1(1)x_1(1) = x_1(1)$ and $\|x_1(t)\| = 1$ for any $t \in I$. We define the projection $q_1 = x_1 \otimes x_1 \in C(I, M_n)$. Then $I \ni t \mapsto p(t) - q_1(t)$ is continuous. Repeating the same argument, for $l = 2, \ldots, k$, we can choose a continuous function $I \ni t \mapsto x_l(t) \in \mathrm{Range}((p - \sum_{i=1}^{l-1} q_i(t)))$ such that $p_l(0)x_l(0) = x_l(0)$, $p_l(1)x_l(1) = x_l(1)$ and $\|x_l(t)\| = 1$ for any $t \in I$. Therefore we have $p = \sum_{i=1}^k q_i$, where $q_i = x_i \otimes x_i$ for $i = 1, \ldots, k$. □

Lemma 1.4. *Let $\varepsilon > 0$, $k \leq n$ and $f = \sum_{i=1}^n \lambda_i p_i$ be a normal element of $C(I, M_n)$, where $\lambda_1, \ldots, \lambda_n \in C(I)$ and $\{p_i(t)\}_{i=1}^n \subset M_n$ is a family of mutually orthogonal projections. If $|\lambda_1(t) - \lambda_l(t)| < \varepsilon$ and $|\lambda_1(t) - \lambda_l(t)| < |\lambda_1(t) - \lambda_m(t)|$ for each $l \in \{1, \ldots, k\}$ and $m \in \{k+1, \ldots, n\}$, then $p = \sum_{i=1}^k p_i \in C(I, M_n)$.*

Moreover we can choose a family of mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $q_i(0) = p_i(0)$, $q_i(1) = p_i(1)$ and

$$\left\| pfp - \sum_{i=1}^k \lambda_i q_i \right\| < 2\varepsilon.$$

Proof. We can choose a continuously differentiable function $C \colon I \times \mathbb{T} \to \mathbb{C}$ such that $C(t, \cdot)(= C_t)$ is a simple closed curve with canonical orientation and separates $\{\lambda_1(t), \ldots, \lambda_k(t)\}$ (in its inside) and $\{\lambda_{k+1}(t), \ldots, \lambda_n(t)\}$ (in its outside) for each $t \in I$. Since we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{C_t} \frac{1}{z - f(t)}\, dz = \sum_{i=1}^k p_i(t)$$

for any $t \in I$, this implies the continuity of $p = \sum_{i=1}^{k} p_i$.

By the previous lemma, there are mutually orthogonal projections $q_1, \ldots, q_k \in C(I, M_n)$ such that $p_i(0) = q_i(0)$, $q_i(1) = p_i(1)$ and $p = \sum_{i=1}^{k} q_i$. Then we have

$$\left\| pfp - \sum_{i=1}^{k} \lambda_i q_i \right\| \le \left\| \sum_{i=1}^{k} \lambda_i p_i - \sum_{i=1}^{k} \lambda_i q_i \right\| \le \varepsilon + \left\| \sum_{i=1}^{k} \lambda_1 p_i - \sum_{i=1}^{k} \lambda_i q_i \right\|$$
$$= \varepsilon + \left\| \lambda_1 p - \sum_{i=1}^{k} \lambda_i q_i \right\| < 2\varepsilon. \qquad \square$$

Proposition 1.5. *Let $\varepsilon > 0$ and f be a normal element of $C(I, M_n)$. Then there are $\lambda_1, \ldots, \lambda_n \in C(I)$ and mutually orthogonal projections $q_1, \ldots, q_n \in C(I, M_n)$ such that*

$$\left\| f - \sum_{i=1}^{n} \lambda_i q_i \right\| < \varepsilon.$$

Proof. We can choose $\lambda_1, \ldots, \lambda_n \in C(I)$ such that $f(t) = \sum_{i=1}^{n} \lambda_i(t) p_i(t)$, where $p_1(t), \ldots, p_n(t)$ are mutually orthogonal projections for each $t \in I$. Then there exists $\delta > 0$ such that

$$|t - s| < \delta \Longrightarrow |\lambda_i(t) - \lambda_i(s)| < \varepsilon/2 \quad (i = 1, \ldots, n).$$

For any $t \in I$, we define index sets $I_1(t), \ldots, I_{N(t)}(t)$ as follows:

$$\begin{aligned}
&i_1(t) = 1,\\
&I_1(t) = \{i \in \{1, \ldots, n\} : |\lambda_1(t) - \lambda_i(t)| < \varepsilon/2\},\\
&i_k(t) = \min\left(\{1, \ldots, n\} \setminus \bigcup_{i=1}^{k-1} I_i(t) \right), \quad (k \ge 2)\\
&I_k(t) = \left\{ i \in \{1, \ldots, n\} \setminus \bigcup_{l=1}^{k-1} I_l(t) : \left|\lambda_{i_k(t)}(t) - \lambda_i(t)\right| < \varepsilon/2 \right\}.
\end{aligned}$$

Then we can choose a neighborhood U_t of t satisfying the closure $\overline{U_t}$ of U_t is $[a_t, b_t]$ and $|a_t - b_t| < \delta$ and, for $i \in I_k(t)$, $j \in \{1, \ldots, n\} \setminus \bigcup_{l=1}^{k} I_l(t)$ $(1 \le k \le N(t))$ and $s \in [a_t, b_t]$,

$$\left|\lambda_{i_k(t)}(s) - \lambda_i(s)\right| < \varepsilon/2, \quad \left|\lambda_{i_k(t)}(s) - \lambda_j(s)\right| \ge \varepsilon/4,$$
$$\left|\lambda_{i_k(t)}(s) - \lambda_i(s)\right| < \left|\lambda_{i_k(t)}(s) - \lambda_j(s)\right|.$$

Since $\bigcup_{t \in I} U_t$ is an open covering of I, there exists a finite subcovering of I. We may assume that

$$0 = a_1 < t_1 < a_2 < b_1 < t_2 < a_3 < b_2 < \cdots < b_{K-1} < t_K < b_K = 1,$$
$$\overline{U_{t_l}} = [a_l, b_l] \quad (1 \le l \le K), \quad I = \bigcup_{l=1}^{K} U_{t_l}.$$

For instance, we set $b_0 = a_1$. For each $l = 1, \ldots, K$, applying the previous lemma $N(t_l)$ times, we can find mutually orthogonal projections $q_1^{(l)}, \ldots, q_n^{(l)} \in$

$C([b_{l-1}, b_l], M_n)$ satisfying $q_i^{(l)}(b_{l-1}) = p_i(b_{l-1})$, $q_i^{(l)}(b_l) = p_i(b_l)$ and

$$\left\| f(t) - \sum_{i=1}^{n} \lambda_i(t) q_i^{(l)}(t) \right\| < \varepsilon.$$

We define, for each i, $q_i(t) = q_i^{(l)}(t)$, where $t \in [b_{l-1}, b_l]$. Then we have $q_1, \ldots, q_n \in C(I, M_n)$ as asserted. □

Corollary 1.6. *$C(I, M_n)$ is approximately square root closed.*

Proof. Let $\varepsilon > 0$ and f be a normal element of $C(I, M_n)$. Applying Proposition 1.5, there are $\lambda_1, \ldots, \lambda_n \in C(I)$ and mutually orthogonal projections $q_1, \ldots, q_n \in C(I, M_n)$ such that $\|f - \sum_{i=1}^n \lambda_i q_i\| < \varepsilon$. For each $i = 1, \ldots, n$, we can find $\mu_i \in C(I)$ satisfying $\lambda_i = \mu_i^2$, which means that $C(I, M_n)$ is approximately square root closed. □

A C^*-algebra is called an *AI-algebra* if it is isomorphic to the inductive limit of a sequence $(C(I, F_n), \varphi_n)$, where each F_n is a finite dimensional C^*-algebra and each $\varphi_n : C(I, F_n) \to C(I, F_{n+1})$ is an injective *-homomorphism. A C^*-algebra A is called *stable rank one*, if the set $GL(A)$ of invertible elements of A is dense in A. We remark that each $C(I, F_n)$ has stable rank one.

We need the following lemma. We have been unable to find a suitable reference in the literature, so we include a proof for completeness.

Lemma 1.7. *Let $A = \varinjlim A_n$ be an inductive limit such that each C^*-algebra A_n has stable rank one. Then, for any normal element $x \in A$ and $\varepsilon > 0$, there exists a normal element y in some A_n such that $\|x - y\| < \varepsilon$.*

Proof. For each $n \in \mathbb{N}$, we can find an element $x_n \in A_n$ such that $\|x - x_n\| \to 0$. Then $[(x_n)] := (x_n) + \bigoplus_n A_n$ is a normal element in $\prod_n A_n / \bigoplus_n A_n$ and $C^*([(x_n)])$ is isomorphic to $C(\mathrm{Sp}([(x_n)]))$, where $C^*([(x_n)])$ is the C^*-algebra generated by $[(x_n)]$. Since $\mathrm{Sp}([(x_n)])$ can be embedded in the closed unit disk $\mathbb{D}$, we have a *-homomorphism from $C(\mathbb{D})$ onto $C(\mathrm{Sp}([(x_n)]))$. By using the argument of semi-projectivity [16, Theorem 19.2.7], there exist a natural number m and a normal element $y_n \in A_n$ for $n \geq m$ satisfying

$$[(x_1, \ldots, x_{m-1}, x_m, x_{m+1}, \ldots)] = [(0, \ldots, 0, y_m, y_{m+1}, \ldots)]$$

in $\prod_n A_n / \bigoplus_n A_n$. If we set $y = y_n$ for a sufficiently large n, then y satisfies the desired condition. □

Theorem 1.8. *Every AI-algebra is approximately square root closed.*

Proof. Let $A = \varinjlim A_n$ be an AI-algebra. Since each A_n has stable rank one, we can apply Lemma 1.7. So, for any normal element $a \in A$ and $\varepsilon > 0$, there exists a normal element b in some A_n such that $\|a - b\| < \varepsilon$. By Corollary 1.6, b can be approximated by a square of a normal element. Therefore A is approximately square root closed. □

2. Two-divisibility for K_1

Lemma 2.1. *Let A be a C^*-algebra. If $x \in A$ is normal, then there exists a normal element $y \in A \otimes M_n$ such that $x \otimes 1_n = y^n$.*

Proof. We set

$$y = \frac{x}{|x|^{(n-1)/n}} \otimes e_{1,n} + |x|^{\frac{1}{n}} \otimes e_{2,1} + |x|^{\frac{1}{n}} \otimes e_{3,2} + \cdots + |x|^{\frac{1}{n}} \otimes e_{n,n-1}.$$

Then y becomes a normal element of $A \otimes M_n$ and satisfies $x \otimes 1_n = y^n$. □

Let $M_{2^\infty} = \overline{\bigotimes_{n=1}^{\infty} M_2}$ be the UHF algebra of type 2^∞ and $\gamma\colon M_{2^\infty} \to M_{2^\infty}$ be a unital *-endomorphism defined by $\gamma(x) = 1_2 \otimes x$ ($x \in M_{2^\infty}$). For each n, we choose a unitary $w_n \in \bigotimes_{i=1}^{n} M_2 \subset M_{2^\infty}$ such that

$$\operatorname{Ad} w_n(x_1 \otimes \cdots \otimes x_n) = w_n(x_1 \otimes \cdots \otimes x_n)w_n^* = x_n \otimes x_1 \otimes \cdots \otimes x_{n-1}$$

for any $x_1 \otimes \cdots \otimes x_n \in \bigotimes_{i=1}^{n} M_2$. Then we have

$$\lim_{n\to\infty} \|\gamma(x) - \operatorname{Ad} w_n(x)\| = 0 \quad \text{for all } x \in M_{2^\infty}.$$

Theorem 2.2. *If A is a unital C^*-algebra, then $A \otimes M_{2^\infty}$ is approximately square root closed.*

Proof. We consider the *-endomorphism $\alpha = \mathrm{id} \otimes \gamma$ of $A \otimes M_{2^\infty}$. It is easy to see that $\alpha(x) = \lim_{n\to\infty} \operatorname{Ad}(1 \otimes w_n)(x)$ for all $x \in A \otimes M_{2^\infty}$.

For any normal element $x \in A \otimes M_{2^\infty}$, we can see $\alpha(x)$ like as $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$. So there exists a normal element $y \in A \otimes M_{2^\infty}$ such that $y^2 = \alpha(x)$ by Lemma 2.1. It follows that

$$\left\|(\operatorname{Ad}(1 \otimes w_n^*)(y))^2 - x\right\| = \left\|y^2 - \operatorname{Ad}(1 \otimes w_n)(x)\right\| \to \left\|y^2 - \alpha(x)\right\| = 0,$$

which means that $A \otimes M_{2^\infty}$ is approximately square root closed. □

For a C^*-algebra A, we say that $K_1(A)$ is 2-*divisible* if any $[x] \in K_1(A)$ has an element $[y] \in K_1(A)$ with $[x] = 2[y]$.

It is known that if a unital C^*-algebra A has stable rank one, then $M_n(A)$ has also stable rank one, and in this case the map from the unitary group of A to $K_1(A)$ is surjective, see [19] for details.

Let A be a unital commutative C^*-algebra ($A \cong C(X)$). It is well-known that A has stable rank one if and only if the covering dimension of the associated compact Hausdorff space X is less than one. In this case $K_1(A)$ is isomorphic to $GL(A)/GL_0(A)$, where $GL_0(A)$ is the connected component containing the identity in $GL(A)$.

Proposition 2.3. *Let A be a C^*-algebra with stable rank one.*

(1) *If A is approximately square root closed, then $K_1(A)$ is 2-divisible.*
(2) *If A is commutative and $K_1(A)$ is 2-divisible, then A is approximately square root closed.*

Proof. (1) Let u be a unitary in A. There exists a normal element $a \in A$ such that $\left\|u - a^2\right\| < 1$. In particular, a is invertible. Then we have $[u] = [a^2] = 2[a]$ in $K_1(A)$.

(2) Since A has stable rank one, it suffices to show that any invertible element in A can be approximated by a square of a normal element of A. For $a \in GL(A)$, there exists an invertible $b \in A$ such that $[a] = 2[b] = [b^2]$ in $K_1(A)$. Therefore a is connected to b^2 in $GL(A)$. So we can choose $h_1, \ldots, h_n \in A$ such that

$$a = e^{h_1} \cdots e^{h_n} b^2.$$

It follows that $a = (e^{(h_1+\cdots+h_n)/2}b)^2$. □

Since $K_1(C(\mathbb{T})) = \mathbb{Z}$, we can see that $C(\mathbb{T})$ is not approximately square root closed. We define $A_n = C(\mathbb{T})$ $(n = 1, 2, \ldots)$ and a *-homomorphism φ_n from A_n to A_{n+1} by

$$\varphi_n(f)(z) = f(z^2) \quad (f \in C(\mathbb{T}) = A_n, z \in \mathbb{T}).$$

Then the inductive limit A of this system (A_n, φ_n) is a commutative C^*-algebra with stable rank one and has $K_1(A) \cong \mathbb{Z}[\frac{1}{2}] = \{\frac{m}{2^n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$. In fact, A is approximately square root closed.

We take a sequence $\{x_n\}$ of a compact Hausdorff space X and an increasing sequence $\{k_n\}$ of positive integers such that k_n divides k_{n+1} for each n. For each n, we define a *-homomorphism φ_n from $C(X, M_{k_n})$ to $C(X, M_{k_{n+1}})$ by

$$\varphi_n(f)(x) = \mathrm{diag}(\underbrace{f(x), \ldots, f(x)}_{s(n)}, f(x_n), \ldots, f(x_n))$$

for $f \in C(X, M_{k_n})$ and $x \in X$. Then we call the inductive limit A of the inductive system $(C(X, M_{k_n}), \varphi_n)$ a *Goodearl type algebra over* X. We note that if $\{x_n\}$ is dense in X, then A becomes simple and is called a Goodearl algebra [10]. But, in our setting $\{x_n\}$ is not necessarily dense in X.

Theorem 2.4. *Let A be a Goodearl type algebra over $\mathbb{T}$. Then the following are equivalent.*

(1) *A is approximately square root closed.*
(2) *For any $n \in \mathbb{N}$, there exists $m \geq n$ such that $s(m)$ is even.*
(3) *$K_1(A)$ is 2-divisible.*

Proof. (1) ⇒ (3). It follows from Proposition 2.3.

(3) ⇒ (2). We remark that $K_1(A_n) \cong \mathbb{Z}$ for each $n \in \mathbb{N}$ and denote by 1_n the unit of $K_1(A_n)$. Then we have $(\varphi_n)_*(1_n) = s(n)1_{n+1} \in K_1(A_{n+1})$. By the assumption we can choose a positive integer $N(> n)$ such that

$$s(N)s(N-1)\cdots s(n)1_{N+1} \in 2K_1(A_{N+1}).$$

This means that $s(m)$ is even for some $m \in \{n, \ldots, N\}$.

(2) ⇒ (1). Let f be a normal element in A and $\varepsilon > 0$. Since each A_n has stable rank one, by the same argument in Theorem 1.8, we can choose a number n and a normal element $g \in A_n$ such that $\|f - g\| < \varepsilon$. Then we may assume that $s(n)$ is even. By Lemma 2.1 we can show that

$$\varphi_n(g) = g \otimes 1_{s(n)} \oplus g(x_n) \oplus \cdots \oplus g(x_n)$$

has a square root in A_{n+1}. □

3. Purely infinite simple unital C^*-algebras

Let A be a unital simple C^*-algebra. We say that A is *purely infinite*, if every non-zero hereditary C^*-subalgebra of A contains an infinite projection. The simplicity and the pure infiniteness of A ([20]) implies that A has *real rank zero*, i.e., the invertible self-adjoint elements are dense in the set of the self-adjoint elements of A. It is also known that the following are equivalent:

(i) A has real rank zero.
(ii) A has the property (HP), i.e., every non-zero hereditary C^*-subalgebra B of A has an approximate identity of projections in B ([3]).

(iii) A has the property weak (FU), i.e., for any $u \in U_0(A)$ and $\varepsilon > 0$, there exists a unitary $v \in U_0(A)$ with finite spectrum such that $\|u - v\| < \varepsilon$, where $U_0(A)$ is the connected component containing the identity in the set of unitaries $U(A)$ ([12]).

Proposition 3.1. *Let A be a unital simple purely infinite C^*-algebra. When $u \in A$ is a unitary and $[u]$ is 2-divisible in $K_1(A)$, for any $\varepsilon > 0$ there exists a unitary $v \in A$ such that*

$$\|u - v^2\| < \varepsilon.$$

Proof. We denote the unit circle in $\mathbb{C}$ by $\mathbb{T}$. If $\mathrm{Sp}(u)$ is not the whole of $\mathbb{T}$, then u has a square root. Therefore we may assume $\mathrm{Sp}(u) = \mathbb{T}$. Let $F \subset \mathbb{T}$ be an ε-dense finite subset of $\mathbb{T}$, that is, for any $\xi \in \mathbb{T}$ there exists $\eta \in F$ such that $|\xi - \eta| \leq \varepsilon$. Since A has real rank zero, applying [13, Lemma 2], there exist a unitary $u_0 \in A$ and a family of mutually orthogonal nonzero projections $\{e_\eta\}_{\eta\in F}$ such that $\|u - u_0\| < \varepsilon$ and

$$e_\eta u_0 = u_0 e_\eta = \eta e_\eta$$

for all $\eta \in F$. Let $e = 1 - \sum_{\eta\in F} e_\eta$ and $B = eAe$. Then $u_1 = u_0 e$ is a unitary of B. Note that $[u_1 + 1 - e]$ is equal to $[u]$ in $K_1(A)$. Hence there exists a unitary $v \in B$ such that $[u_2] = -2[v]$ in $K_1(B) \cong K_1(A)$. Since $M_2(B)$ has the property weak (FU), there exist projections $q_1, q_2, \ldots, q_n \in M_2(B)$ and $\xi_1, \xi_2, \ldots, \xi_n \in \mathbb{T}$ such that

$$\sum_{i=1}^n q_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\left\| \begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix} - \sum_{i=1}^n \xi_i q_i \right\| < \varepsilon.$$

Because F is ε-dense in $\mathbb{T}$, for each $i = 1, 2, \ldots, n$, there exists $\eta_i \in F$ such that $|\xi_i - \eta_i| \leq \varepsilon$. It follows that

$$\left\| \begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix} - \sum_{i=1}^n \eta_i q_i \right\| < 2\varepsilon.$$

Since A is simple and purely infinite, there exists a family of mutually orthogonal projections r_i in A such that $r_i \leq e_{\eta_i}$ and $[r_i] = [q_i]$ in $K_0(A)$. Put $r = \sum_{i=1}^n r_i$. Then we have

$$u_0 r = \sum_{i=1}^n \eta_i r_i,$$

and so we can find a unitary $u_2 \in rAr$ which is a copy of the unitary

$$\begin{bmatrix} u_1 & 0 \\ 0 & v^2 \end{bmatrix},$$

and $\|u_2 - u_0 r\|$ is less than 2ε. It follows that $u_3 = u_1 + u_2 + u_0(1 - e - r)$ is a unitary of A and $\|u_3 - u\|$ is less than 3ε. Moreover $u_1 + u_2$ looks like

$$\begin{bmatrix} u_1 & 0 & 0 \\ 0 & u_1 & 0 \\ 0 & 0 & v^2 \end{bmatrix},$$

which is a square of

$$\begin{bmatrix} 0 & u_1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & v \end{bmatrix}.$$

Because $u_0(1-e-r)$ has finite spectrum, the proof is completed. □

Corollary 3.2. *Let A be a unital simple purely infinite C^*-algebra. Suppose that $K_1(A)$ is 2-divisible. If $x \in A$ is a normal element and $\mathrm{Sp}(x)$ is homeomorphic to the circle, then for any $\varepsilon > 0$ there exists a normal element $y \in A$ such that*

$$\|x - y^2\| < \varepsilon.$$

Proof. Since the circle is one-dimensional, by perturbing x a little bit, we may assume that x is invertible. Let $f : \mathbb{T} \to \mathrm{Sp}(x)$ be a homeomorphism. Because f is a homeomorphism onto $\mathrm{Sp}(x)$, the rotation number of f is -1 or 0 or 1. If the rotation number of f is zero, then x has a square root. Hence, without loss of generality, we may assume that the rotation number of f is one. We denote the inverse of f by $f^{-1} : \mathrm{Sp}(x) \to \mathbb{T}$.

There exists $\delta > 0$ such that if $u, v \in A$ are unitaries with $\|u - v\| < \delta$ then $\|f(u) - f(v)\| < \varepsilon$. Applying Proposition 3.1 to the unitary $f^{-1}(x)$, we get a unitary $v \in A$ such that

$$\|f^{-1}(x) - v^2\| < \delta,$$

which means that

$$\|x - f(v^2)\| < \varepsilon.$$

Since the rotation number of the function

$$\mathbb{T} \ni \xi \to f(\xi^2) \in \mathbb{C}$$

is two, we can find a continuous function $g \colon \mathbb{T} \to \mathbb{C}$ such that

$$g^2(\xi) = f(\xi^2)$$

for all $\xi \in \mathbb{T}$. Put $y = g(v)$. Then y is a normal element and $y^2 = g^2(v) = f(v^2)$, which completes the proof. □

Let a and b be two elements of a C^*-algebra and $\varepsilon > 0$. We write $a \overset{\varepsilon}{\approx} b$, if $\|a - b\| < \varepsilon$.

Lemma 3.3. *Let A be a unital C^*-algebra and $x \in A$ be a normal element. Suppose that there exist $\zeta \in \mathrm{Sp}(x)$ and closed subsets $G_0, G_1 \subset \mathrm{Sp}(x)$ such that $\mathrm{Sp}(x) = G_0 \cup G_1$ and $G_0 \cap G_1 = \{\zeta\}$. Then, for any $\varepsilon > 0$, there exist normal elements $x_0, x_1 \in A$ and a unitary $u \in M_2(A)$ such that $\mathrm{Sp}(x_i) = G_i$ and*

$$\left\| u \begin{bmatrix} x & \\ & \zeta \end{bmatrix} u^* - \begin{bmatrix} x_0 & \\ & x_1 \end{bmatrix} \right\| < \varepsilon.$$

Proof. We can identify $C(\mathrm{Sp}(x))$ with the abelian C^*-subalgebra of A which is generated by x and $1 \in A$. Put

$$O = \{\xi \in \mathbb{C} : |\xi - \zeta| < \varepsilon/2\}.$$

Since $G_0 \setminus O$ and $G_1 \setminus O$ are disjoint, there exists a unitary $u \in M_2(C(\mathrm{Sp}(x))) \cong C(\mathrm{Sp}(x), M_2)$ such that

$$u(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } \xi \in G_0 \setminus O \quad \text{and} \quad u(\xi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ for } \xi \in G_1 \setminus O.$$

Define $x_i \in C(\mathrm{Sp}(x))$ by

$$x_i(\xi) = \begin{cases} \xi & \xi \in G_i \\ \zeta & \xi \in G_{1-i}. \end{cases}$$

If $\xi \notin O$, then we can check

$$u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix} u(\xi)^* = \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.$$

If $\xi \in G_0 \cap O$, then

$$u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix} u(\xi)^* \overset{\varepsilon/2}{\approx} u(\xi) \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} u(\xi)^* = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta \end{bmatrix} \overset{\varepsilon/2}{\approx} \begin{bmatrix} \xi & 0 \\ 0 & \zeta \end{bmatrix} = \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.$$

When $\xi \in G_1 \cap O$, we can obtain the same estimate. □

We put

$$H_+ = \{a + b\sqrt{-1} \in \mathbb{C} : b \geq 0\}$$

and

$$H_- = \{a + b\sqrt{-1} \in \mathbb{C} : b \leq 0\}.$$

We identify the real line $\mathbb{R}$ with $H_+ \cap H_-$.

Lemma 3.4. *Let A be a unital C^*-algebra and $x \in A$ be a normal element. Suppose that there exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{R}) \cap \mathrm{Sp}(x) = f([-1, 1])$. Then, for any $\varepsilon > 0$, there exist normal elements $x_0, x_1, a \in A$ and a unitary $u \in M_2(A)$ such that*

$$\mathrm{Sp}(x_0) = f(H_+) \cap \mathrm{Sp}(x), \quad \mathrm{Sp}(x_1) = f(H_-) \cap \mathrm{Sp}(x), \quad \mathrm{Sp}(a) = f([-1, 1])$$

and

$$\left\| u \begin{bmatrix} x & \\ & a \end{bmatrix} u^* - \begin{bmatrix} x_0 & \\ & x_1 \end{bmatrix} \right\| < \varepsilon.$$

Proof. We identify $C(\mathrm{Sp}(x))$ with the abelian C^*-subalgebra of A which is generated by x and $1 \in A$. We first deal with the case that $f : \mathbb{C} \to \mathbb{C}$ is the identity map. Let $h_0 : H_+ \to [-1, 1]$ and $h_1 : H_- \to [-1, 1]$ be continuous functions such that $h_i(\xi) = \xi$ for $\xi \in [-1, 1]$. Define $a, x_0, x_1 \in C(\mathrm{Sp}(x))$ by

$$a(\xi) = \begin{cases} h_0(\xi) & \xi \in H_+ \\ h_1(\xi) & \xi \in H_- \end{cases},$$

$$x_0(\xi) = \begin{cases} \xi & \xi \in H_+ \\ h_1(\xi) & \xi \in H_- \end{cases}$$

and

$$x_1(\xi) = \begin{cases} h_0(\xi) & \xi \in H_+ \\ \xi & \xi \in H_-. \end{cases}$$

Since $\mathrm{Sp}(x) \cap \mathbb{R} = [-1, 1]$, there exists $\delta > 0$ such that if $\xi = s + t\sqrt{-1} \in \mathrm{Sp}(x)$ with $|t| < \delta$, then $|h_i(\xi) - \xi| < \varepsilon/2$ for each $i = 0, 1$. We can find a unitary $u \in M_2(C(\mathrm{Sp}(x))) \cong C(\mathrm{Sp}(x), M_2)$ such that

$$u(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $\xi = s + t\sqrt{-1} \in \mathrm{Sp}(x)$ with $t \geq \delta$ and

$$u(\xi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

for $\xi = s + t\sqrt{-1} \in \mathrm{Sp}(x)$ with $t \leq -\delta$. If $|t| \geq \delta$, then for $\xi = s + t\sqrt{-1} \in \mathrm{Sp}(x)$ we can check

$$u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & a(\xi) \end{bmatrix} u(\xi)^* = \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.$$

If $|t| < \delta$, then for $\xi = s + t\sqrt{-1} \in \mathrm{Sp}(x)$ we can also check

$$u(\xi) \begin{bmatrix} \xi & 0 \\ 0 & a(\xi) \end{bmatrix} u(\xi)^* \overset{\varepsilon/2}{\approx} u(\xi) \begin{bmatrix} a(\xi) & 0 \\ 0 & a(\xi) \end{bmatrix} u(\xi)^* = \begin{bmatrix} a(\xi) & 0 \\ 0 & a(\xi) \end{bmatrix} \overset{\varepsilon/2}{\approx} \begin{bmatrix} x_0(\xi) & 0 \\ 0 & x_1(\xi) \end{bmatrix}.$$

Now let us turn to the general case. Because $K = f^{-1}(\mathrm{Sp}(x)) = \mathrm{Sp}(f^{-1}(x))$ is compact, there exists $\delta > 0$ such that if y_0 and y_1 are normal elements in some C^*-algebra B with $\mathrm{Sp}(y_i) \subset K$ and $\|y_0 - y_1\| < \delta$, then $\|f(y_0) - f(y_1)\| < \varepsilon$. Applying the first part of this proof to $f^{-1}(x)$ and δ, we get

$$\left\| u \begin{bmatrix} f^{-1}(x) & \\ & a \end{bmatrix} u^* - \begin{bmatrix} x_0 & \\ & x_1 \end{bmatrix} \right\| < \delta.$$

By the choice of δ, we obtain

$$\left\| u \begin{bmatrix} x & \\ & f(a) \end{bmatrix} u^* - \begin{bmatrix} f(x_0) & \\ & f(x_1) \end{bmatrix} \right\| < \varepsilon,$$

thereby completing the proof. □

We define I_0 and I_1 by

$$I_0 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq a \leq 1, b = 0\}$$

and

$$I_1 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \leq b \leq 1, a = 0\}.$$

Let G be a compact subset of $\mathbb{C}$. We say that G is a lattice graph, if there exist finite subsets F_0 and F_1 of $\mathbb{Z} + \mathbb{Z}\sqrt{-1}$ such that

$$G = \bigcup_{i=0,1} \bigcup_{\zeta \in F_i} I_i + \zeta.$$

We call each point in $G \cap (\mathbb{Z} + \mathbb{Z}\sqrt{-1})$ a vertex of G and each $I_i + \zeta$ contained in G an edge of G. We denote by $|G|$ the number of edges of G.

Proposition 3.5. *For any nonempty connected lattice graph G, there exists a natural number $N(G) \in \mathbb{N}$ such that the following holds: Let A be a unital C^*-algebra and $x \in A$ be a normal element with $\mathrm{Sp}(x) = G$. For any $\varepsilon > 0$, there exist a natural number $N \leq N(G)$, normal elements $a_1, a_2, \ldots, a_N, x_0, x_1, \ldots, x_N \in A$, and a unitary $u \in M_{N+1}(A)$ such that the following are satisfied.*

1. $\|u\,\mathrm{diag}(x, a_1, a_2, \ldots, a_N)u^* - \mathrm{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon$.
2. $\mathrm{Sp}(x_i)$ *is contained in* G.
3. $\mathrm{Sp}(x_i)$ *is homeomorphic to the closed interval* $[-1, 1]$ *or the circle.*
4. $\mathrm{Sp}(a_i)$ *is contained in* G.
5. $\mathrm{Sp}(a_i)$ *is a single point or homeomorphic to the closed interval* $[-1, 1]$.

Proof. The proof goes by induction concerning $|G|$. If $|G| = 1$, then G is homeomorphic to the closed interval, and so we have nothing to do.

We may assume that the assertion has been proved for all G with $|G| < L$. Let us consider a connected lattice graph G with $|G| = L$. We would like to show that

$$N(G) = 2\max\{N(G_0) : G_0 \text{ is a connected lattice graph with } G_0 \subsetneq G\} + 1$$

does the work. Suppose that A is a unital C^*-algebra and $x \in A$ is a normal element with $G = \mathrm{Sp}(x)$. Take $\varepsilon > 0$.

Suppose that there exists a vertex $\zeta \in G$ such that $G \setminus \{\zeta\}$ is not connected. We can find nonempty connected lattice graphs G_0 and G_1 such that $G = G_0 \cup G_1$ and $G_0 \cap G_1 = \{\zeta\}$. Applying Lemma 3.3 to G_0, G_1, ζ and $\varepsilon/2$, we obtain normal elements $x_0, x_1 \in A$ and a unitary $u \in M_2(A)$ such that $\mathrm{Sp}(x_i) = G_i$ and

$$\left\| u \begin{bmatrix} x & \\ & \zeta \end{bmatrix} u^* - \begin{bmatrix} x_0 & \\ & x_1 \end{bmatrix} \right\| < \frac{\varepsilon}{2}.$$

By the induction hypothesis, there exists $N_i \le N(G_i)$ such that the assertion holds for x_i and $\varepsilon/2$. Hence $N = N_0 + N_1 + 1 \le N(G)$ works for x and ε.

Therefore we may assume that $G \setminus \{\zeta\}$ is connected for all vertices ζ in G. Let O be the unbounded connected component of $\mathbb{C} \setminus G$ and ∂O be the boundary of O in $\mathbb{C}$. Then $\partial O \subset G$ is homeomorphic to the circle. If $G = \partial O$, then we have nothing to do. Let us assume that $G \neq \partial O$. We can find an edge $e \subset G$ such that e is not contained in ∂O and an endpoint ζ_0 of e belongs to ∂O. Let ζ_1 be the other endpoint of e. Since $G \setminus \{\zeta_0\}$ is connected, we can find a path in G from ζ_1 to a vertex $\zeta_2 \in \partial O$ which is distinct from ζ_0. Let P be the union of this path and e. Then $P \subset G$ is homeomorphic to the closed interval $[-1, 1]$ and its endpoints are ζ_0 and ζ_2. There exists a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $f(\mathbb{R}) \cap G = P$ and $f([-1, 1]) = P$. Applying Lemma 3.4 to f and $\varepsilon/2$, we obtain normal elements $x_0, x_1, a \in A$ and a unitary $u \in M_2(A)$ such that

$$\mathrm{Sp}(x_0) = f(H_+) \cap \mathrm{Sp}(x), \quad \mathrm{Sp}(x_1) = f(H_-) \cap \mathrm{Sp}(x), \quad \mathrm{Sp}(a) = f([-1, 1])$$

and

$$\left\| u \begin{bmatrix} x & \\ & a \end{bmatrix} u^* - \begin{bmatrix} x_0 & \\ & x_1 \end{bmatrix} \right\| < \frac{\varepsilon}{2}.$$

Put $G_i = \mathrm{Sp}(x_i)$ for $i = 0, 1$. Note that G_i is a connected lattice graph. By the induction hypothesis, there exists a natural number $N_i \le N(G_i)$ such that the assertion holds for x_i and $\varepsilon/2$. Hence $N = N_0 + N_1 + 1 \le N(G)$ works for x and ε. □

Lemma 3.6. *Let A be a unital C^*-algebra and $a \in A$ be a normal element. Suppose that $\mathrm{Sp}(a)$ is homeomorphic to the closed interval $[-1, 1]$. For any $\varepsilon > 0$, there exist complex numbers $\xi_1, \xi_2, \dots, \xi_N, \eta_0, \eta_1, \dots, \eta_N \in \mathrm{Sp}(a)$ and a unitary $u \in M_{N+1}(A)$ such that*

$$\|u \,\mathrm{diag}(a, \xi_1, \xi_2, \dots, \xi_N) u^* - \mathrm{diag}(\eta_0, \eta_1, \dots, \eta_N)\| < \varepsilon.$$

Proof. By using Lemma 3.3 repeatedly, we can find $\xi_1, \xi_2, \dots, \xi_N \in \mathrm{Sp}(a)$ and normal elements $x_0, x_1, \dots, x_N \in A$ and a unitary $u \in M_{N+1}(A)$ such that

$$\|u \,\mathrm{diag}(a, \xi_1, \xi_2, \dots, \xi_N) u^* - \mathrm{diag}(x_0, x_1, \dots, x_N)\| < \frac{\varepsilon}{2}$$

and $\mathrm{Sp}(x_i)$ has diameter less than $\varepsilon/2$. Replacing x_i with some $\eta_i \in \mathrm{Sp}(x_i)$, we get the conclusion. □

This lemma together with Proposition 3.5 directly implies the following.

Proposition 3.7. *Let A be a unital C^*-algebra and $x \in A$ be a normal element. Suppose that $G = \mathrm{Sp}(x)$ is a lattice graph. For any $\varepsilon > 0$, there exist $N \in \mathbb{N}$, $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{C}$, normal elements $x_0, x_1, \ldots, x_N \in A$ and a unitary $u \in M_{N+1}(A)$ such that the following are satisfied.*

(1) $\|u\,\mathrm{diag}(x, \xi_1, \xi_2, \ldots, \xi_N)u^* - \mathrm{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon$.
(2) $\mathrm{Sp}(x_i)$ *is contained in* G.
(3) $\mathrm{Sp}(x_i)$ *is a single point or homeomorphic to the closed interval* $[-1, 1]$ *or the circle.*
(4) ξ_i *is contained in* G.

Combining this with Corollary 3.2, we get the following.

Lemma 3.8. *Let A be a unital simple purely infinite C^*-algebra. Suppose that $K_1(A)$ is 2-divisible. If $x \in A$ is a normal element and $\mathrm{Sp}(\varepsilon^{-1}x)$ is a connected lattice graph for some $\varepsilon > 0$, then there exists a normal element $y \in A$ such that*

$$\|x - y^2\| < 2\varepsilon.$$

Proof. Put $G = \mathrm{Sp}(\varepsilon^{-1}x)$ and

$$F = \mathrm{Sp}(x) \cap (\varepsilon\mathbb{Z} + \varepsilon\mathbb{Z}\sqrt{-1}).$$

Thus, $\varepsilon^{-1}F$ is the set of vertices of the lattice graph G. Clearly F is an $\varepsilon/2$-dense finite subset of $\mathrm{Sp}(x)$. As before, we put

$$I_0 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \le a \le 1, b = 0\}$$

and

$$I_1 = \{a + b\sqrt{-1} \in \mathbb{C} : 0 \le b \le 1, a = 0\}.$$

We define a continuous function $f : \mathrm{Sp}(x) \to \mathrm{Sp}(x)$ as follows: If $\xi = a + b\sqrt{-1} \in \mathrm{Sp}(x)$ belongs to $\varepsilon I_0 + \zeta$ with $\zeta = t + b\sqrt{-1} \in F$, then we set

$$f(\xi) = \begin{cases} \zeta & t \le a \le t + \frac{\varepsilon}{3} \\ \zeta + 3(a - t - \frac{\varepsilon}{3}) & t + \frac{\varepsilon}{3} \le a \le t + \frac{2\varepsilon}{3} \\ \zeta + \varepsilon & t + \frac{2\varepsilon}{3} \le a \le t + \varepsilon. \end{cases}$$

If $\xi = a + b\sqrt{-1} \in \mathrm{Sp}(x)$ belongs to $\varepsilon I_1 + \zeta$ with $\zeta = a + t\sqrt{-1} \in F$, then we set

$$f(\xi) = \begin{cases} \zeta & t \le b \le t + \frac{\varepsilon}{3} \\ \zeta + 3(b - t - \frac{\varepsilon}{3})\sqrt{-1} & t + \frac{\varepsilon}{3} \le b \le t + \frac{2\varepsilon}{3} \\ \zeta + \varepsilon\sqrt{-1} & t + \frac{2\varepsilon}{3} \le b \le t + \varepsilon. \end{cases}$$

Define $z = f(x)$. Evidently we have $\|x - z\| \le \varepsilon/3$ and $\mathrm{Sp}(z) = \mathrm{Sp}(x) = \varepsilon G$. For each $\eta \in F$, let $g_\eta : \mathbb{C} \to [0, 1]$ be a continuous function such that $g_\eta(\eta) = 1$ and $g_\eta(\xi) = 0$ if $|\xi - \eta| \ge \varepsilon/3$. Since A has real rank zero, there exists a nonzero projection $e_\eta \in g_\eta(x)Ag_\eta(x)$. It is not hard to see that $e_\eta z = ze_\eta = \eta e_\eta$. Note that $\{e_\eta\}_{\eta \in F}$ is a family of mutually orthogonal projections. Put $e = 1 - \sum_{\eta \in F} e_\eta$, $B = eAe$ and $z_0 = ze$. Then we have

$$z = z_0 + \sum_{\eta \in F} \eta e_\eta,$$

and so the spectrum of z_0 in B is equal to $\varepsilon G = \mathrm{Sp}(z)$.

By applying Proposition 3.7 to $\varepsilon^{-1}z_0 \in B$ and 1, we obtain complex numbers $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{C}$, normal elements $x_0, x_1, \ldots, x_N \in B$ and a unitary $u \in M_{N+1}(B)$ such that

- $\|u \operatorname{diag}(\varepsilon^{-1}z_0, \xi_1, \xi_2, \ldots, \xi_N)u^* - \operatorname{diag}(x_0, x_1, \ldots, x_N)\| < 1$.
- $\operatorname{Sp}(x_i)$ is a single point or homeomorphic to the closed interval $[-1, 1]$ or the circle.
- ξ_i is contained in G.

By replacing ξ_i and x_i with $\varepsilon^{-1}\xi_i$ and $\varepsilon^{-1}x_i$, we get

- $\|u \operatorname{diag}(z_0, \xi_1, \xi_2, \ldots, \xi_N)u^* - \operatorname{diag}(x_0, x_1, \ldots, x_N)\| < \varepsilon$.
- $\operatorname{Sp}(x_i)$ is a single point or homeomorphic to the closed interval $[-1, 1]$ or the circle.
- ξ_i is contained in $\operatorname{Sp}(x)$.

Because F is $\varepsilon/2$-dense in $\operatorname{Sp}(x)$, for each $i = 1, 2, \ldots, N$ we can find $\eta_i \in F$ such that $|\xi_i - \eta_i| \le \varepsilon/2$. It follows that

$$\|u \operatorname{diag}(z_0, \eta_1, \eta_2, \ldots, \eta_N)u^* - \operatorname{diag}(x_0, x_1, \ldots, x_N)\| < \frac{3\varepsilon}{2}.$$

Since A is purely infinite, there exists a family of mutually orthogonal projections q_i such that $q_i \le e_{\eta_i}$ and $[q_i] = [e]$ in $K_0(A)$. Put $q = \sum q_i$. Then we have

$$(e + q)z = z_0 + \sum_{i=1}^{N} \eta_i q_i,$$

and so there exists a normal element $w \in (e + q)A(e + q)$ which is a unitary conjugation of $\operatorname{diag}(x_0, x_1, \ldots, x_N)$ and

$$\|(e + q)z - w\| < \frac{3\varepsilon}{2}.$$

Thanks to Corollary 3.2, we can find a normal element $y_0 \in (e + q)A(e + q)$ such that

$$\|w - y_0^2\| < \frac{\varepsilon}{6}.$$

Since $(1 - e - q)z$ has finite spectrum, it has a square root y_1. Put $y = y_0 + y_1$. Then we have

$$\|z - y^2\| = \|(e + q)z - y_0^2\| < \|w - y_0^2\| + \frac{3\varepsilon}{2} < \frac{3\varepsilon}{2} + \frac{\varepsilon}{6}.$$

This estimate together with $\|x - z\| \le \varepsilon/3$ implies

$$\|x - y^2\| < 2\varepsilon.$$

□

Now we are ready to prove the main result of this section.

Theorem 3.9. *For a unital simple purely infinite C^*-algebra A, the following are equivalent.*

(1) *A is approximately square root closed.*
(2) *$K_1(A)$ is 2-divisible.*

Proof. (1)⇒(2). Since $K_1(A) \cong U(A)/U_0(A)$, it suffices to show that every unitary in A is divided by 2 in $K_1(A)$. Let u be a unitary in A. Then there exists a unitary $v \in A$ such that $\|u - v^2\| < 1$. Therefore $[u] = 2[v]$ in $K_1(A)$.

(2)⇒(1). Take a normal element $x \in A$ and a small real number $\varepsilon > 0$. By [9, Lemma 3.2], there exists a normal element $z \in A$ such that $\|x - z\| < \varepsilon$ and $\operatorname{Sp}(z)$ is contained in

$$\{a + b\sqrt{-1} \in \mathbb{C} : a \in \varepsilon\mathbb{Z} \text{ or } b \in \varepsilon\mathbb{Z}\}.$$

By perturbing z a little bit more, we can find a normal element $w \in A$ such that $\|z - w\| < \varepsilon$ and $G = \varepsilon^{-1} \operatorname{Sp}(w)$ is a lattice graph. Let $G_1, G_2, \ldots, G_n$ be connected components of G. Each G_i is a connected lattice graph. Let h_i be the characteristic function on εG_i and put $w_i = h_i(w)$. Then w is the direct sum of $w_1, w_2, \ldots, w_n$ and $\operatorname{Sp}(w_i) = \varepsilon G_i$. By using the lemma above, we get mutually orthogonal normal elements $y_1, y_2, \ldots, y_n$ such that $\|w_i - y_i^2\| < 2\varepsilon$. Put $y = y_1 + y_2 + \cdots + y_n$. We can easily see that $\|x - y^2\| < 4\varepsilon$. □

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