Minimal dynamical systems and approximate conjugacy *

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Abstract

Several versions of approximate conjugacy for minimal dynamical systems are introduced. Relation between approximate conjugacy and corresponding crossed product C^*-algebras is discussed. For the Cantor minimal systems, a complete description is given for these relations via K-theory and C^*-algebras. For example, it is shown that two Cantor minimal systems are approximately τ-conjugate if and only if they are orbit equivalent and have the same periodic spectrum. It is also shown that two such systems are approximately K-conjugate if and only if the corresponding crossed product C^*-algebras have the same scaled ordered K-theory. Consequently, two Cantor minimal systems are approximately K-conjugate if and only if the associated transformation C^*-algebras are isomorphic. Incidentally, this approximate K-conjugacy coincides with Giordano, Putnam and Skau's strong orbit equivalence for the Cantor minimal systems.

1 Introduction

Let X be a compact metric space and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. One obtains two dynamical systems (X, α) and (X, β). This paper studies the relation between these two systems. The two systems are said to be (flip) conjugate if there is a homeomorphism $\sigma : X \to X$ such that $\alpha = \sigma \circ \beta \circ \sigma^{-1}$ (or $\alpha = \sigma \circ \beta^{-1} \circ \sigma^{-1}$). Let $C(X) \times_\alpha \mathbb{Z}$ be the crossed product C^*-algebra associated with the dynamical system (X, α). If X is infinite, the assumption that α is minimal implies that $C(X) \times_\alpha \mathbb{Z}$ is a unital simple C^*-algebra. It follows from a result of J. Tomiyama ([T2]) (based on an earlier result of M. Boyle) that α and β are flip conjugate if and only if there exists an isomorphism $\phi : C(X) \times_\alpha \mathbb{Z} \to C(X) \times_\beta \mathbb{Z}$ such that $\phi \circ j_\alpha = j_\beta$, where $j_\alpha : C(X) \to C(X) \times_\alpha \mathbb{Z}$ and $j_\beta : C(X) \to C(X) \times_\beta \mathbb{Z}$ are the usual embeddings. While this is a very interesting result in the interplay between classical dynamical systems and C^*-algebras, it also suggests that the conjugacy relation is rather too strong.

Inspired by ergodic theory, Giordano, Putnam and Skau ([GPS]) introduced the topological version of orbit equivalences. These notions work nicely when X is the Cantor set. For example, it is shown in [GPS], when X is the Cantor set, then α and β are strong orbit equivalent if and only if the corresponding C^*-algebras $C(X) \times_\alpha \mathbb{Z}$ and $C(X) \times_\beta \mathbb{Z}$ are isomorphic. It is a

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breakthrough result which establishes closer relation between C^*-algebra theory and dynamical systems in the case that X is the Cantor set. No doubt it would be more interesting if similar type results could be established for more traditional topological spaces. It appears that when X is connected, the situation is rather different. The recent work of Q. Lin and N. C. Phillips suggests that C^*-algebra theory and related K-theory may well play an important role in the study of dynamical systems. We believe that recent development in the classification theory for simple amenable C^*-algebras via K-theory may also apply to the dynamical systems and K-theory may be proved to be a useful tool in the study of minimal dynamical systems.

Motivated by this idea, the first named author has been proposing a few versions of approximate conjugacy for more general spaces. A simple version of approximately conjugate is the following: two minimal homeomorphisms are approximately conjugate if there exists a sequence of homeomorphisms $\sigma_n : X \to X$ such that $f \circ \sigma_n \circ \beta^{-1} \circ \sigma_n^{-1} \to f \circ \alpha^{-1}$ for all $f \in C(X)$. When X is not connected, this approximate version of conjugacy is rather weak as we will see in Section 4. When we impose conditions on the conjugating maps σ_n, then the relation becomes more interesting. For example, we introduce the notion of approximate τ-conjugacy in which, we insist, in addition, that each σ_n keeps the trace (translation invariant measure) invariant, and the notion of approximate K-conjugacy in which we require that the conjugating maps σ_n induce the same map on the K-theory. We discuss the general relation of these approximate conjugacies with the corresponding crossed product $C(X) \times_\alpha \mathbb{Z}$. In the special case that X is the Cantor set, we show that α and β are approximately K-conjugate if and only if $C(X) \times_\alpha \mathbb{Z}$ is isomorphic to $C(X) \times_\beta \mathbb{Z}$. So the approximate K-conjugacy is the same as strong orbit equivalence in this case. We also establish the relationship between approximate τ-conjugacy and orbit equivalence in the Cantor set case. These results demonstrate that perhaps appropriate approximate version of conjugacy is a right equivalence relation to study and it may bridge the gulf between the dynamical systems and C^*-algebras.

The paper is organized as follows. In Section 2, we list a number of terminologies used in the paper. We also give a few facts that are relevant to this paper. In Section 3, we give definitions of several versions of approximate conjugate relations among minimal homeomorphisms on compact metric spaces and some consequences. In Section 4, we discuss weakly approximate conjugate minimal homeomorphisms on the Cantor set. In Section 5, we show that approximate conjugate relation is closely related to the orbit equivalence and strong orbit equivalence in the Cantor minimal systems.

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2 Preliminaries

Let X be a compact metric space and $\alpha : X \to X$ be a minimal homeomorphism. The transformation C^*-algebra A_α may be identified with the crossed product $C(X) \times_\alpha \mathbb{Z}$. The unitary obtained from α will be denoted by u_α. Denote by $j_\alpha : C(X) \to A_\alpha$ the canonical unital embedding.

When X is an infinite set, it is well known that A_α is a simple C^*-algebra.

Let A be a unital stably finite C^*-algebra. The tracial state space of A will be denoted by $T(A)$. Let $\mathrm{Aff}(T(A))$ be the space of all real affine functions on $T(A)$. Denote by $\rho_A : K_0(A) \to \mathrm{Aff}(T(A))$ the (positive) homomorphism induced by the tracial states. Let $\tau \in T(A)$. Suppose that $a \in M_k(A)$ for some integer k, we will continue use τ for the extension $\tau \otimes Tr$, Tr is the standard trace (not the normalized one) on M_k.

If $h : A \to B$ is a homomorphism from A to another C^*-algebra B, we denote by $h_{*i} : K_i(A) \to K_i(B)$ $(i = 0, 1)$ the induced map on K-theory. We also use h_* when it is convenient to do so.

If $\sigma : X \to X$ and $f = (f_{ij})_{k\times k} \in M_k(C(X))$, we sometimes write $f \circ \sigma^{-1}$ for $(f_{ij} \circ \sigma^{-1})_{k\times k}$. Denote by $\sigma^* : C(X) \to C(X)$ the induced homomorphism defined by $\sigma^*(f)(x) = f(\sigma^{-1}(x))$ for $x \in X$ and $f \in C(X)$.

We will use $M_\infty(A)$ for the union of $M_k(A)$, $k = 1, 2, \ldots$.

If $A = \lim_{n\to\infty}(A_n, \phi_n)$ is an inductive limit of C^*-algebras, we use $\phi_{n,\infty} : A_n \to A$ for the induced homomorphism from A_n to A.

Definition 2.1. Let A be a separable C^*-algebra and B another C^*-algebra. Let $\phi_n : A \to B$ be a sequence of contractive completely positive linear maps. We say that $\{\phi_n\}$ is a sequential asymptotic morphism, if

$$\lim_{n\to\infty} \|\phi_n(ab) - \phi_n(a)\phi_n(b)\| = 0$$

for all $a, b \in A$.

2.2. For each k, we may use $\phi_n : M_k(A) \to M_k(B)$ for $\phi_n \otimes \mathrm{id}_{M_k}$. If $p \in M_k(A)$ is a projection, then for sufficiently large n, $\phi_n(p)$ is close to a projection $q \in M_k(B)$. Since close projections are equivalent, we may write $[\phi_n(p)] = [q]$. In fact, as in 1.8 of [Ln3] for example, given any finitely generated subgroup $G_i \subset K_i(A)$, when n is sufficiently large, $[\phi_n]|_{G_i}$ are well defined as homomorphisms. We will use this fact whenever it is needed. In fact, one has the following facts: Consider $\Phi : A \to \prod_{n\in\mathbb{N}} B$ by defining $\Phi(a) = \{\phi_n(a)\}$ for $a \in A$. Then $\pi \circ \Phi : A \to \prod_{n\in\mathbb{N}} B/\bigoplus_{n\in\mathbb{N}} B$ is a homomorphism, where $\pi : \prod_{n\in\mathbb{N}} B \to \prod_{n\in\mathbb{N}} B/\bigoplus_{n\in\mathbb{N}} B$. It is well known (and easy to verify) that $\pi_{*i} : K_i(\prod_{n\in\mathbb{N}} B) \to K_i(\prod_{n\in\mathbb{N}} B/\bigoplus_{n\in\mathbb{N}} B)$ is surjective and every finitely generated subgroup of $K_i(\prod_{n\in\mathbb{N}} B/\bigoplus_{n\in\mathbb{N}} B)$ lifts.

Lemma 2.3. *Let A be a separable simple unital C^*-algebra and B be another C^*-algebra. Suppose that $\{\phi_n\} : A \to B$ be a (unital) sequential asymptotic morphism. Then, for any $a \neq 0$ in A,*

$$\lim_{n\to\infty} \|\phi_n(a)\| = \|a\|.$$

Proof. Note that $\|\phi_n(a)\| \le \|a\|$. Suppose that there is a nonzero element $a \in A$ such that

$$\liminf_n \|\phi_n(a)\| < \|a\|.$$

Then there is a subsequence $\{n(k)\}$ such that

$$\lim_{k\to\infty} \|\phi_{n(k)}(a)\| < \|a\|. \tag{e 2.1}$$

Let $\Phi : A \to \prod_{k=1}^\infty B$ be defined by $\Phi(a) = \{\phi_{n(k)}(a)\}_k$. Let $\Pi : \prod_{k=1}^\infty B \to \prod_{k=1}^\infty B/\bigoplus_{k=1}^\infty B$ be the quotient map. Then $\Pi \circ \Phi : A \to \prod_{k=1}^\infty B/\bigoplus_{k=1}^\infty B$ is a homomorphism. Since A is simple, it is a monomorphism. In particular, $\|\Pi \circ \Phi(a)\| = \|a\|$. This contradicts (e 2.1). □

Definition 2.4. Let (X, α) be a Cantor minimal system.

(1) The full group $[\alpha]$ of (X, α) is the subgroup of all homeomorphisms from X to X that preserves every orbit of α. To any $\gamma \in [\alpha]$ is associated a map $n : X \to \mathbb{Z}$, defined by $\gamma(x) = \alpha^{n(x)}(x)$ for $x \in X$.

(2) The topological full group $[[\alpha]]$ of (X, α) is the subgroup of all homeomorphisms $\gamma \in [\alpha]$, whose associated map $n : X \to \mathbb{Z}$ is continuous.

The following can be found in [GW].

Lemma 2.5. *Let X be the Cantor set and $\alpha : X \to X$ be a minimal homeomorphism. Let $p \in C(X)$ be a projection, $x \in K_0(A_\alpha)_+ \setminus \{0\}$ such that $x \le (j_\alpha)_{*0}(p)$. Then there is a projection $q \le p$ such that $(j_\alpha)_{*0}(q) = x$.*

By repeated use of the above lemma, one obtains the following.

Lemma 2.6. *Let X be the Cantor set and $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $x_1, x_2, \dots, x_m \in K_0(A)_+ \setminus \{0\}$ satisfy $\sum_{i=1}^m x_i = [1_A]$. Then there are mutually orthogonal projections $q_1, q_2, \dots, q_m \in C(X)$ such that $\sum_{i=1}^m q_i = 1_{C(X)}$ and*

$$(j_\alpha)_{*0}([q_i]) = x_i \text{ for all } i = 1, 2, \dots, m.$$

Theorem 2.7. *Let X be the Cantor set and $\alpha : X \to X$ be a minimal homeomorphism. Suppose that $\phi : C(X, \mathbb{Z}) \to K_0(A_\alpha)$ is a positive homomorphism for which $\phi(1_{C(X)}) = [1_{A_\alpha}]$.*

(1) Then there exists a unital positive homomorphism $\psi : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ such that

$$(j_\alpha)_{*0} \circ \psi = \phi.$$

(2) If $\phi(C(X, \mathbb{Z})_+ \setminus \{0\}) \subset K_0(A_\alpha)_+ \setminus \{0\}$, then one can require that ψ to be injective.

(3) Suppose that $\beta : X \to X$ is another minimal homeomorphism, $\gamma : K_0(A_\beta) \to K_0(A_\alpha)$ is a unital order isomorphism and $\phi = \gamma \circ (j_\beta)_{*0}$, then one can require that ψ has also an invertible positive homomorphism.

Proof. Write $C(X, \mathbb{Z}) = \lim_{n\to\infty}(\mathbb{Z}^{k(n)}, \zeta_n)$. Here we use the simplicial order on $\mathbb{Z}^{k(n)}$. Each ζ_n is injective, unital and positive. We regard $\mathbb{Z}^{k(n)}$ as the subgroup of $C(X, \mathbb{Z})$. Note that $C(X, \mathbb{Z})_+ = \bigcup_{n=1}^\infty \mathbb{Z}_+^{k(n)}$.

Let $e_1^{(n)} = (1, 0, \dots, 0), e_2^{(n)} = (0, 1, 0, \dots, 0), \dots, e_{k(n)}^{(n)} = (0, \dots, 0, 1)$, $n = 1, 2, \dots$.

Fix n, let $x(n, 1), x(n, 2), \dots, x(n, k(n)) \in K_0(A_\alpha)_+$ such that $\phi(e_i^{(n)}) = x(n, i)$, $i = 1, 2, \dots, k(n)$ (some of them could be zero). By applying Lemma 2.6 above, we obtain a positive homomorphism $h_n : \mathbb{Z}^{k(n)} \to C(X, \mathbb{Z})$ such that

$$(j_\alpha)_{*0} \circ h_n = \phi|_{\mathbb{Z}^{k(n)}}.$$

Moreover, if $\phi(C(X, \mathbb{Z})_+ \setminus \{0\}) \subset K_0(A_\alpha)_+ \setminus \{0\}$, $x(n, i)$ above are non-zero and by Lemma 2.6, we may assume that $h_n(e_i^{(n)})$ are mutually orthogonal projections in $C(X)$ (Here we regard $C(X, \mathbb{Z})$ as a subset of $C(X)$). In particular, in this case, h_n is injective.

We note that in $\mathbb{Z}^{k(n)}$, $1_{C(X)} = \sum_{i=1}^{k(n)} e_i^{(n)}$. Furthermore, we may write

$$\sum_{j \in S(n,i)} e_j^{(n+1)} = e_i^{(n)},$$

where $\{S(n, 1), S(n, 2), \dots, S(n, k(n))\}$ are disjoint and $\bigcup_i S(n, i) = \{1, 2, \dots, k(n+1)\}$. Let $p_i = h_n(e_i^{(n)})$. By applying Lemma 2.5 and using the same argument used in Lemma 2.6, we obtain projections $p(n+1, j) \in C(X, \mathbb{Z})_+$ such that

$$\sum_{j \in S(n,i)} p(n+1, j) = p_i = h_n(e_i^{(n)}) \quad \text{and} \quad (j_\alpha)_{*0}(p(n+1, j)) = \phi(e_j^{(n+1)}).$$

Define $h_{n+1} : \mathbb{Z}^{k(n+1)} \to C(X, \mathbb{Z})$ by $h_{n+1}(e_j^{(n+1)}) = p(n+1, j)$, $j = 1, 2, \dots, k(n+1)$. Clearly $h_{n+1}(1_{C(X)}) = 1_{C(X)}$. More importantly,

$$h_{n+1}|_{\mathbb{Z}^{k(n)}} = h_n, \quad n = 1, 2, \dots.$$

Thus $\{h_n\}$ defines a positive homomorphism $\psi : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ which preserves $1_{C(X)}$. Moreover,

$$(j_\alpha)_{*0} \circ \psi = \phi.$$

Note that in the case (2), each h_n is injective, it follows that ψ is also injective. This proves (1) and (2).

To see (3) holds, we note that, since both $K_0(A_\alpha)$ and $K_0(A_\beta)$ are simple dimension groups, $\phi(C(X, \mathbb{Z})_+ \setminus \{0\}) \subset K_0(A_\alpha)_+ \setminus \{0\}$.

Suppose that h_n has been constructed. We may assume that $h_n(e_i^{(n)}) \subset \mathbb{Z}^{k(n+1)}$, without loss of generality.

We write $\sum_{j \in F(n,i)} e_j^{(n+1)} = h_n(e_i^{(n)})$, where $\{F(n, i)\}$ are mutually disjoint and $\bigcup_j F(n, j) = \{1, 2, \dots, k(n+1)\}$. Suppose that $y(n, j) = \gamma^{-1} \circ (j_\alpha)_{*0}(e_j^{(n+1)})$, $j \in F(n, i)$. Note that $\gamma \circ (j_\beta)_{*0}(e_i^{(n)}) = (j_\alpha)_{*0} \circ h_n(e_i^{(n)})$, $i = 1, 2, \dots, k(n)$. Therefore

$$(j_\beta)_{*0}(e_i^{(n)}) = \sum_{j \in F(n,i)} y(n, j).$$

By Lemma 2.5 and the proof of Lemma 2.6, we obtain mutually orthogonal non-zero projections p_j $(j \in F(n, i))$ such that

$$\sum_{j \in F(n,i)} p_j = e_i^{(n)} \quad \text{and} \quad (j_\beta)_{*0}(p_i) = y(n, j)$$

for some $m > n$. Without loss of generality, we may assume that $m = n + 2$.

By defining $H_n(e_j^{(n+1)}) = p_j$, we obtain a unital positive injective homomorphism $H_n : \mathbb{Z}^{k(n+1)} \to \mathbb{Z}^{k(n+2)} \subset C(X, \mathbb{Z})$ such that

$$(j_\beta)_{*0} \circ H_n = \gamma^{-1} \circ (j_\alpha)_{*0}|_{\mathbb{Z}^{k(n+1)}}.$$

Moreover, $H_n \circ h_n$ is the identity on $\mathbb{Z}^{k(n)}$. In other words, we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}^{k(n)} & \xrightarrow{\zeta_{n,n+2}} & \mathbb{Z}^{k(n+2)} \\ \downarrow_{h_n} & \nearrow_{H_n} & \\ \mathbb{Z}^{k(n+1)} & & \end{array}$$

Exactly the same argument allows us to construct $h_{n+1} : \mathbb{Z}^{k(n+2)} \to Z^{k(n+3)}$ (by perhaps choosing a number larger than $n + 3$) such that $(j_\alpha)_{*0} \circ h_{n+1} = \gamma \circ j_\beta|_{\mathbb{Z}^{k(n+2)}}$ and we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{Z}^{k(n)} & \xrightarrow{\zeta_{n,n+2}} & \mathbb{Z}^{k(n+2)} \\ \downarrow_{h_n} & \nearrow_{H_n} & \downarrow_{h_{n+1}} \\ \mathbb{Z}^{k(n+1)} & \xrightarrow{\zeta_{n+1,n+3}} & Z^{k(n+3)} \end{array}$$

Continuing in this fashion, we obtain $h_{n+m} : \mathbb{Z}^{k(n+2m)} \to \mathbb{Z}^{k(n+2m+1)}$ and $H_{n+m} : \mathbb{Z}^{k(n+2m+1)} \to \mathbb{Z}^{k(n+2m+2)}$ such that $(j_\alpha)_{*0} \circ h_{n+m} = \gamma \circ (j_\beta)_{*0}|_{\mathbb{Z}^{k(n+2m)}}$ and $(j_\beta)_{*0} \circ H_{n+m} = \gamma^{-1} \circ (j_\alpha)_{*0}|_{\mathbb{Z}^{(n+2m+1)}}$.

Furthermore, we have the following intertwining:

$$\begin{array}{ccccccc}
\mathbb{Z}^{k(n)} & \xrightarrow{\zeta_{n,n+2}} & \mathbb{Z}^{k(n+2)} & \xrightarrow{\zeta_{n+2,n+4}} \cdots & \rightarrow \cdots & C(X,\mathbb{Z}) \\
\downarrow h_n & \nearrow H_n & \downarrow h_{n+1} & \nearrow H_{n+1} & & \\
\mathbb{Z}^{k(n+1)} & \xrightarrow{\zeta_{n+1,n+3}} & \mathbb{Z}^{k(n+3)} & \xrightarrow{\zeta_{n+3,n+5}} \cdots & \rightarrow \cdots & C(X,\mathbb{Z})
\end{array}$$

This implies that we have two invertible unital order homomorphisms $\psi, \psi_1 : C(X,\mathbb{Z}) \to C(X,\mathbb{Z})$ such that $\psi_1 = \psi^{-1}$. By the restriction on h_n and H_n, we also have

$$(j_\alpha)_{*0} \circ \psi = \gamma \quad \text{and} \quad (j_\beta)_{*0} \circ \psi^{-1}.$$

$\square$

3 Approximate conjugacy

Let X be a compact metric space and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Recall that α and β are (flip) conjugate if there is a homeomorphism $\sigma : X \to X$ such that $\sigma \circ \alpha \circ \sigma^{-1} = \beta$ (or $\sigma \circ \alpha \circ \sigma^{-1} = \beta^{-1}$). It is in general a rather strong equivalence relation. We propose a few version of approximate conjugacy.

We start with the following:

Definition 3.1. Let X be a compact metric space and $\alpha, \beta : X \to X$ be two homeomorphisms. We say α and β are *weakly approximately conjugate* if there are homeomorphisms $\sigma_n, \gamma_n : X \to X$ such that

$$\lim_{n\to\infty} (\sigma_n \circ \alpha \circ \sigma_n^{-1})^*(f) = \beta^*(f) \quad \text{and} \quad \lim_{n\to\infty} (\gamma_n \circ \beta \circ \gamma_n^{-1})^*(f) = \alpha^*(f)$$

for all $f \in C(X)$.

Proposition 3.2. *Let X be a compact metric space and let α and β be minimal homeomorphisms on X. Let $A = C(X) \times_\alpha \mathbb{Z}$ and $B = C(X) \times_\beta \mathbb{Z}$. If there exist homeomorphisms $\sigma_n : X \to X$ such that $(\sigma_n \circ \alpha \circ \sigma_n^{-1})^*(f) \to \beta^*(f)$ for every $f \in C(X)$, then there exists a unital asymptotic morphism $\{\psi_n\}$ from B to A such that*

$$\lim_{n\to\infty} ||\psi_n(u_\beta) - u_\alpha|| = 0$$

and

$$\lim_{n\to\infty} ||\psi_n(j_\beta(f)) - j_\alpha(f \circ \sigma_n)|| = 0$$

for all $f \in C(X)$. In particular, for every $b \in B$,

$$\lim_{n\to\infty} ||\psi_n(E_B(b)) - E_A(\psi_n(b))|| = 0,$$

where E_A and E_B are the canonical conditional expectations onto $C(X)$ defined on A and B.

Proof. Let us denote the quotient map from $\prod_{n\in\mathbb{N}} A$ to $\prod_{n\in\mathbb{N}} A / \bigoplus_{n\in\mathbb{N}} A$ by π. Define $\Psi(j_\beta(f)) = \pi((j_\alpha(f \circ \sigma_n))_n)$ for $f \in C(X)$ and $\Psi(u_\beta) = \pi((u_\alpha)_n)$. One checks that Ψ gives a homomorphism from B to $\prod_{n\in\mathbb{N}} A / \bigoplus_{n\in\mathbb{N}} A$, because

$$u_\alpha j_\alpha(f \circ \sigma_n) u_\alpha^* - j_\alpha(f \circ \beta^{-1}\sigma_n) = j_\alpha(f \circ \sigma_n \alpha^{-1} - f \circ \beta^{-1}\sigma_n) \to 0.$$

Since B is nuclear, there exists a unital completely positive lifting of Ψ ([CE]), that is, we can find a sequence $\{\psi_n\}$ of unital completely positive linear maps from B to A such that

$$\lim_{n\to\infty}\left\|\psi_n\left(\sum_{k=-N}^{L} f_k u_\beta^k\right)-\sum_{k=-N}^{L}(f_k\circ\sigma_n)u_\alpha^k\right\|=0$$

for all $\sum_{k=-N}^{L} f_k u_\beta^k\in B$. Because $\pi\circ\Psi$ is a homomorphism, $\{\psi_n\}$ is an asymptotic morphism. $\square$

Let $Q_A=\prod_{n\in\mathbb{N}}A/\bigoplus_{n\in\mathbb{N}}A$, $Q_{C(X)}=\prod_{n\in\mathbb{N}}C(X)/\bigoplus_{n\in\mathbb{N}}C(X)$. Denote by $\bar\alpha$ the induced action on $Q_{C(X)}$. Let $Q_\alpha=Q_{C(X)}\times_{\bar\alpha}\mathbb{Z}$. It is easy to see that there is an obvious unital embedding from Q_α to Q_A.

Corollary 3.3. *In the situation of Proposition 3.2, one has the following commutative diagram:*

$$\begin{tikzcd}
K_0(C(X)) \arrow[rr, "\mathrm{id}_*-\beta_*"] \arrow[dr, "(\pi\circ\tilde\sigma)_*"] & & K_0(C(X)) \arrow[rr, "(j_\beta)_*"] \arrow[d, "(\pi\circ\tilde\sigma)_*"'] & & K_0(B) \arrow[dl, "\Psi_*"'] \arrow[ddd] \\
& K_0(Q_{C(X)}) \arrow[r] & K_0(Q_{C(X)}) \arrow[r] & K_0(Q_\alpha) \arrow[d] & \\
& K_1(Q_\alpha) \arrow[u] & K_1(Q_{C(X)}) \arrow[l] & K_1(Q_{C(X)}) \arrow[l] & \\
K_1(B) \arrow[uuu] \arrow[ur, "\Psi_*"] & & K_1(C(X)) \arrow[ll, "(j_\beta)_*"] \arrow[u, "(\pi\circ\tilde\sigma)_*"] & & K_1(C(X)) \arrow[ll, "\mathrm{id}_*-\beta_*"'] \arrow[ul, "(\pi\circ\tilde\sigma)_*"']
\end{tikzcd}$$

where $\tilde\sigma: C(X)\to\prod_{n\in\mathbb{N}}C(X)$ is defined by $\tilde\sigma(f)=(f\circ\sigma_n)_n$, the map from $K_i(Q_{C(X)})$ to $K_i(Q_{C(X)})$ is $\mathrm{id}-\bar\alpha_*$ and the map from $K_i(Q_{C(X)})$ to $K_i(Q_\alpha)$ is induced by the natural inclusion.

Moreover, we also have the following commutative diagram:

$$\begin{tikzcd}
K_0(C(X)) \arrow[rr, "\mathrm{id}_*-\beta_*"] \arrow[dr, "\theta_0\circ(\pi\circ\tilde\sigma)_*"] & & K_0(C(X)) \arrow[rr, "(j_\beta)_*"] \arrow[d, "\theta_0\circ(\pi\circ\tilde\sigma)_*"] & & K_0(B) \arrow[dl] \arrow[ddd] \\
& Q_{K_0(C(X))} \arrow[r] & Q_{K_0(C(X))} \arrow[r] & Q_{K_0(A)} \arrow[d] & \\
& Q_{K_1(A)} \arrow[u] & Q_{K_1(C(X))} \arrow[l] & Q_{K_1(C(X))} \arrow[l] & \\
K_1(B) \arrow[uuu] \arrow[ur] & & K_1(C(X)) \arrow[ll, "(j_\beta)_*"] \arrow[u, "\theta_1\circ(\pi\circ\tilde\sigma)_*"] & & K_1(C(X)) \arrow[ll, "\mathrm{id}_*-\beta_*"'] \arrow[ul, "\theta_1\circ(\pi\circ\tilde\sigma)_*"']
\end{tikzcd}$$

where $Q_{K_i(C(X))}$ denotes $\prod_{n\in\mathbb{N}}K_i(C(X))/\bigoplus_{n\in\mathbb{N}}K_i(C(X))$, $Q_{K_i(A)}$ denotes $\prod_{n\in\mathbb{N}}K_i(A)/\bigoplus_{n\in\mathbb{N}}K_i(A)$ and θ_* is the natural homomorphisms from $K_*(Q_{C(X)})$ to $Q_{K_*(C(X))}$.

Proof. By Proposition 3.2, $\pi\circ\tilde\sigma$ induces a homomorphism from B to Q_α. Since the Pimsner-Voiculescu exact sequences behave naturally for a covariant homomorphism, the commutativity of the first diagram follows.

We will show the commutativity of the second diagram. Let us consider the following diagram:

$$\begin{tikzcd}
K_*(Q_{C(X)}) \arrow[r, "\mathrm{id}-\alpha_*"] \arrow[d, "\theta_*"'] & K_*(Q_{C(X)}) \arrow[d, "\theta_*"] \\
Q_{K_*(C(X))} \arrow[r] & Q_{K_*(C(X))},
\end{tikzcd}$$

where the horizontal map below is defined by

$$(x_n)_n + \bigoplus_{n\in\mathbb{N}} K_*(C(X)) \mapsto (x_n - \alpha_*(x_n))_n + \bigoplus_{n\in\mathbb{N}} K_*(C(X)).$$

Then the commutativity is clear.

Next, let us consider

$$\begin{array}{ccc} K_*(Q_{C(X)}) & \longrightarrow & K_*(Q_\alpha) \\ {\scriptstyle\theta_*}\big\downarrow & & \big\downarrow \\ Q_{K_*(C(X))} & \longrightarrow & Q_{K_*(A)}, \end{array}$$

where the top horizontal map is induced by the natural inclusion, the lower horizontal map is induced by the natural map $K_*(C(X)) \to K_*(A)$ and the vertical map of the right-hand side is the composition of the natural map from $K_*(Q_\alpha)$ to $K_*(Q_A)$ and the natural map from $K_*(Q_A)$ to $Q_{K_*(A)}$. Then, again, the commutativity is easily verified.

Finally, we consider the diagram:

$$\begin{array}{ccc} K_*(Q_\alpha) & \longrightarrow & K_{1-*}(Q_{C(X)}) \\ \big\downarrow & & \big\downarrow \\ Q_{K_*(A)} & \longrightarrow & Q_{K_{1-*}(C(X))}. \end{array}$$

The vertical maps and the top horizontal map have been already defined. The lower horizontal map is induced by the map from $K_*(A)$ to $K_{1-*}(C(X))$ appeared in Pimsner-Voiculescu exact sequence. Therefore we can check the commutativity. □

Definition 3.4. Let X be a compact metric space and $\alpha : X \to X$ be a minimal homeomorphism. A positive element $f \in M_k(C(X))$ is said to be α-projection valued, if there is a projection $p = \sum_{i\in\mathbb{Z}} f_n U_\alpha^n \in M_k(A_\alpha)$ such that $f_0 = f$, where $U_\alpha = \mathrm{diag}(u_\alpha, u_\alpha, \dots, u_\alpha)$.

Proposition 3.5. *If $f \in M_n(C(X))$ is α-projection valued, then there is a projection $p \in M_n(A_\alpha)$ such that $\tau(p) = \tau(f)$ for every α-invariant tracial state τ.*

If furthermore, $(j_\alpha)_{*0}(K_0(C(X))_+) \supset K_0(A_\alpha)_+$, then there is a projection $q \in M_k(C(X))$ (for some integer k) such that $\tau(f) = \tau(q)$ for all α-invariant tracial states τ.

Proof. Suppose that $p = \sum_{j\in\mathbb{Z}} f_j U_\alpha^j$ for which $f_0 = f$. Let τ be the tracial state on A_α and E be the canonical faithful expectation from A_α onto $j_\alpha(C(X))$. Then $E(p) = f_0 = f$ and, consequently, $\tau(p) = \tau(f)$.

To see the second part of the statement, we let $p = \sum_{j\in\mathbb{Z}} f_j U_\alpha^j$ be a projection in $M_n(A_\alpha)$ with $f_0 = f$. Since $(j_\alpha)_{*0} : K_0(C(X))_+ \to K_0(A_\alpha)_+$ is surjective, there is a projection $q \in M_k(C(X))$ for some integer $k > 0$ such that $(j_\alpha)_{*0}([q]) = [p]$. It follows that there is an integer $m > 0$ and a partial isometry $v \in M_{k+m}(A_\alpha)$ such that

$$v^*v = p \oplus 1_m \quad \text{and} \quad vv^* = q \oplus 1_m.$$

It follows that $\tau(p) + m = \tau(q) + m$ for all $\tau \in T(A_\alpha)$. Thus $\tau(p) = \tau(q)$ for all $\tau \in T(A_\alpha)$. □

Comparing with the definition of conjugacy, one may impose some restriction on the maps σ_n so that σ_n and σ_{n+1} may have something in common. In the following definition, we require that the conjugating maps σ_n keep the invariant measures unchanged. Later in Definition 5.3, we will give another version of approximate conjugacy where we require that the conjugating maps preserve the K-theory.

Definition 3.6. We say that α and β are *approximately τ-conjugate*, if α and β are weakly approximately conjugate, and, if in addition, for any projection $f \in M_k(C(X))$ there exists N such that

$$\tau'(f\circ\gamma_n) = \tau'(f\circ\gamma_{n+1}),\quad \tau(f\circ\sigma_n) = \tau(f\circ\sigma_{n+1}),$$
$$\tau(f\circ\gamma_n\circ\sigma_m) = \tau(f)\quad\text{and}\quad \tau'(f\circ\sigma_n\circ\gamma_m) = \tau'(f)$$

for every α-invariant measure τ, β-invariant measure τ' and all $m \geq n \geq N$.

Moreover, for any α-projection valued function $f \in M_\infty(C(X))$ and β-projection valued function $g \in M_\infty(C(X))$, there exist a β-projection valued function $f' \in M_\infty(C(X))$ and an α-projection valued function $g' \in M_\infty(C(X))$ such that

$$\lim_{n\to\infty} \tau'(f\circ\gamma_n) = \tau'(f'),\quad \lim_{n\to\infty} \tau(g\circ\sigma_n) = \tau(g')$$

and

$$\lim_{n\to\infty}\limsup_{m\to\infty} |\tau(f\circ\gamma_n\circ\sigma_m) - \tau(f)| = 0,\ \lim_{n\to\infty}\limsup_{m\to\infty} |\tau'(g\circ\sigma_n\circ\gamma_m) - \tau'(g)| = 0$$

uniformly for α-invariant normalized measures τ and β-invariant normalized measures τ'.

Proposition 3.7. *Let X be a compact metric space, $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Let $A = A_\alpha$ and $B = A_\beta$. Suppose that α and β are approximately τ-conjugate. Then there exist two unital sequential asymptotic morphisms $\{\phi_n\} : A \to B$ and $\{\psi_n\} : B \to A$ such that, for each projection $p \in M_\infty(A)$ and projection $q \in M_\infty(B)$, there exist projections $p' \in M_\infty(B)$ and $q' \in M_\infty(A)$ such that*

$$\lim_{n\to\infty} \tau'(\phi_n(p)) = \tau'(p'),\quad \lim_{n\to\infty} \tau(\psi_n(q)) = \tau(q') \tag{e 3.2}$$

and

$$\lim_{n\to\infty}\limsup_{m\to\infty} |\tau(\psi_m\circ\phi_n(p)) - \tau(p)| = 0,\quad \lim_{n\to\infty}\limsup_{m\to\infty} |\tau'(\phi_m\circ\psi_n(q)) - \tau'(q)| = 0 \tag{e 3.3}$$

uniformly for $\tau \in T(A)$ and $\tau' \in T(B)$.

Proof. Proposition 3.2 applies and yields asymptotic morphisms $\{\phi_n\}$ and $\{\psi_n\}$. Let E_A (resp. E_B) be the canonical conditional expectation from A (resp. B) to $C(X)$. Let $p \in M_\infty(A)$ be a projection. Then $E_A(p)$ is an α-projection valued function. By Definition 3.6 and 3.4, there is a projection $q \in M_\infty(B)$ such that

$$\lim_{n\to\infty} \tau'(E_A(p)\circ\gamma_n) = \tau'(E_B(q)) = \tau'(q)$$

uniformly for $\tau' \in T(B)$. Since

$$\begin{aligned}&||E_A(p)\circ\gamma_n - E_B(\phi_n(p))||\\ &= ||E_A(p)\circ\gamma_n - \phi_n(E_A(p))|| + ||\phi_n(E_A(p)) - E_B(\phi_n(p))|| \to 0,\end{aligned}$$

we get the desired equality.

Moreover, by Definition 3.6 and Proposition 3.2, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$\sup_{\tau\in T(A)}\limsup_{m\to\infty} |\tau(E_A(p)\circ\gamma_n\circ\sigma_m) - \tau(p)| < \varepsilon$$

and

$$||E_A(p)\circ\gamma_n - E_B(\phi_n(p))|| < \varepsilon,$$

for all $n \geq N$. Then we have

$$\begin{aligned}
&\max_{\tau \in T(A)} \limsup_{m \to \infty} |\tau(\psi_m(\phi_n(p))) - \tau(p)| \\
&= \max_{\tau \in T(A)} \limsup_{m \to \infty} |\tau(E_A(\psi_m(\phi_n(p)))) - \tau(p)| \\
&= \max_{\tau \in T(A)} \limsup_{m \to \infty} |\tau(\psi_m(E_B(\phi_n(p)))) - \tau(p)| \\
&\leq \max_{\tau \in T(A)} \limsup_{m \to \infty} |\tau(\psi_m(E_A(p) \circ \gamma_n)) - \tau(p)| + \varepsilon \\
&= \max_{\tau \in T(A)} \limsup_{m \to \infty} |\tau(E_A(p) \circ \gamma_n \circ \sigma_m) - \tau(p)| + \varepsilon < 2\varepsilon
\end{aligned}$$

for all $n \geq N$. The other equalities are shown in the same way. □

Theorem 3.8. *Let X be a compact metric space, $\alpha, \beta : X \to X$ be minimal homeomorphisms. Suppose that α and β are approximately τ-conjugate. Then there is an order unit preserving order isomorphism between $\rho_A(K_0(A_\alpha))$ and $\rho_B(K_0(A_\beta))$.*

Proof. Let $A = A_\alpha$ and $B = A_\beta$.

Let $\{\phi_n\}$ and $\{\psi_n\}$ be unital sequential asymptotic morphisms from A to B as described in Proposition 3.7.

To see that $\rho_A(K_0(B))$ is order isomorphic to $\rho_B(K_0(B))$, we define a map $H : \rho_A(K_0(A)) \to \rho_B(K_0(B))$ by

$$H(\hat{p})(\tau') = \lim_{n \to \infty} \tau' \circ \phi_n(p)$$

for each projection $p \in M_\infty(A)$ and $\tau' \in T(B)$.

To show that H is a well defined map from $\rho_A(K_0(A))$ to $\mathrm{Aff}(T(B))$, we first need to verify that if q is another projection such that $\tau(p) = \tau(q)$ for all $\tau \in T(A)$, then $H(\hat{p}) = H(\hat{q})$.

Let $\tau' \in T(B)$. Define $s_n = \tau' \circ \phi_n$. Then s_n is a state of the unital C^*-algebra A. Let s be a weak limit of $\{s_n\}$. Then it is easy to check that s gives a tracial state of A. It follows that $s(p) = s(q)$. Thus

$$\lim_{n \to \infty} (s_n(p) - s_n(q)) = 0.$$

This shows that $H(\hat{p}) = H(\hat{q})$. By (e 3.2), we see that the image of H is in $\rho_B(K_0(B))$. In other words, H maps $\rho_A(K_0(A))$ to $\rho_B(K_0(B))$. It is then easy to check that H is a positive homomorphism and preserves the order unit. Similarly,

$$H'(\hat{q})(\tau) = \lim_{n \to \infty} \tau \circ \psi_n(q)$$

defines a positive homomorphism from $\rho_B(K_0(B))$ to $\rho_A(K_0(A))$ which preserves the order unit.

We now claim that if $p_1, p_2 \in M_k(B)$ are two projections such that $\sup\{|\tau'(p_1 - p_2)| : \tau' \in T(B)\} < d$, then, for sufficiently large m,

$$\sup\{|\tau(\psi_m(p_1 - p_2))| : \tau \in T(A)\} < d.$$

To see this, assuming that for infinitely many m, $\tau_m \in T(A)$ satisfies $|\tau_m(\psi_m(p_1 - p_2))| \geq d$, let s be a weak limit of $\{\tau_m \circ \psi_m\}$. As above, s is a tracial state on B. It follows that

$$|s(p_1 - p_2)| \geq d,$$

which is a contradiction.

Fix a projection $p \in M_k(A)$ and $\varepsilon > 0$. By (e 3.3), there is a natural number N_1 such that

$$\limsup_{m \to \infty} |\tau(\psi_m \circ \phi_n(p)) - \tau(p)| < \varepsilon/2$$

holds for all $\tau \in T(A)$ and for all $n \geq N_1$.

Let $q \in M_\infty(B)$ so that $\tau'(q) = H(\hat{p})(\tau')$ for all $\tau' \in T(B)$. There is an integer $N_2 > 0$ such that

$$|\tau'(q) - \tau'(\phi_n(p))| < \varepsilon/8$$

for all $\tau' \in T(B)$ and $n \geq N_2$. We may also assume that there are projections $p_n \in M_k(B)$ such that

$$\|\phi_n(p) - p_n\| < \varepsilon/8 \quad \text{for all} \quad n \geq N_2.$$

In particular,

$$|\tau'(\phi_n(p)) - \tau'(p_n)| < \varepsilon/8 \quad \text{for all} \quad \tau' \in T(B).$$

Hence, if $n \geq N_2$,

$$|\tau'(q) - \tau'(p_n)| < \varepsilon/4 \quad \text{for all} \quad \tau' \in T(B).$$

Put $n = \max\{N_1, N_2\}$. It follows from the claim that there exists a natural number N_3 greater than N_1 such that

$$|\tau(\psi_m(q) - \psi_m(p_n))| < \varepsilon/4 \quad \text{for all} \quad \tau \in T(A)$$

and $m \geq N_3$.

Then, for sufficiently large m, we have

$$\begin{aligned}
&|\tau(\psi_m(q)) - \tau(p)| \\
&= |\tau(\psi_m(q)) - \tau(\psi_m(p_n))| + |\tau(\psi_m(p_n)) - \tau(\psi_m(\phi_n(p)))| + |\tau(\psi_m(\phi_n(p))) - \tau(p)| \\
&< \varepsilon/4 + \varepsilon/8 + \varepsilon/2 < \varepsilon
\end{aligned}$$

for all $\tau \in T(A)$, which means that

$$\hat{p} = H'(\hat{q}) = H'H(\hat{p}).$$

It follows that $H' \circ H = \mathrm{id}_{\rho_A(K_0(A))}$. Similarly $H \circ H' = \mathrm{id}_{\rho_B(K_0(B))}$. ☐

Proposition 3.9. *If $(j_{\alpha *})(K_0(C(X))_+) = K_0(A_\alpha)_+$ and $(j_{\beta *})(K_0(C(X))_+) = K_0(A_\beta)_+$, then α and β are approximately τ-conjugate if and only if they are weakly approximately conjugate and in addition, for any projection $f \in M_k(C(X))$ there exists N such that*

$$\tau'(f \circ \gamma_n) = \tau'(f \circ \gamma_{n+1}), \quad \tau(f \circ \sigma_n) = \tau(f \circ \sigma_{n+1}),$$

$$\tau(f \circ \gamma_n \circ \sigma_m) = \tau(f) \quad \textit{and} \quad \tau'(f \circ \sigma_n \circ \gamma_m) = \tau'(f)$$

for every α-invariant measure τ, β-invariant measure τ' and all $m \geq n \geq N$.

Proof. The 'only if' part is obvious. Let us prove the 'if' part.

Let f be an α-projection valued function. By Lemma 3.5, there is a projection $p \in M_\infty(C(X))$ such that $\tau(f) = \tau(p)$ for every $\tau \in T(A_\alpha)$. By the assumption, there exists N such that

$$\tau'(p \circ \gamma_n) = \tau'(p \circ \gamma_{n+1})$$

and

$$\tau(p \circ \gamma_n \circ \sigma_m) = \tau(p)$$

for every α-invariant measure τ, β-invariant measure τ' and all $m \geq n \geq N$. Put $q = p \circ \gamma_n$. Then q is a projection of $M_\infty(C(X))$, and so is β-projection valued. Since

$$\begin{aligned}
&|\tau'(f \circ \gamma_n) - \tau'(q)| \\
&\leq |\tau'(f \circ \gamma_n - \phi_n(f))| + |\tau'(\phi_n(f)) - \tau'(\phi_n(p))| + |\tau'(\phi_n(p) - p \circ \gamma_n)| + |\tau'(p \circ \gamma_n) - \tau'(q)|,
\end{aligned}$$

we have

$$\lim_{n\to\infty} |\tau'(f\circ\gamma_n) - \tau'(q)| = 0$$

uniformly on $T(A_\beta)$. Similarly we estimate that

$$\begin{aligned}&\sup_{\tau\in T(A_\alpha)} \limsup_{m\to\infty} |\tau(f\circ\gamma_n\circ\sigma_m) - \tau(f)|\\ &\le \sup_{\tau\in T(A_\alpha)} \limsup_{m\to\infty}(|\tau(\psi_m(f\circ\gamma_n)) - \tau(\psi_m(p\circ\gamma_n))|\\ &\quad + |\tau(\psi_m(p\circ\gamma_n) - p\circ\gamma_n\circ\sigma_m)| + |\tau(p\circ\gamma_n\circ\sigma_m) - \tau(p)|)\end{aligned}$$

converges to zero as $n\to\infty$. □

4 Cantor minimal systems

Lemma 4.1. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be minimal homeomorphisms. Let $G \subset C(X,\mathbb{Z})$ be a finitely generated subgroup containing $1_{C(X)}$. Then there exists a finitely generated subgroup $F \supset G$ satisfying the following: If $h : (j_\alpha)_*(F) \to K_0(B)$ is a homomorphism for which*

$$h([1_A]) = [1_B], \quad h((j_\alpha)_*(F_+)) \subset K_0(B)_+ \quad \text{and} \quad h(x)\neq 0 \text{ for any } x\in (j_\alpha)_*(F_+)\setminus\{0\},$$

then there is a homeomorphism $\sigma : X\to X$ such that

$$h(j_\alpha)_*|_G = (j_\beta\circ\sigma^*)_*|_G.$$

If one does not assume that, for each $x\in (j_\alpha)_*(F_+)\setminus\{0\}$, $h(x)\neq 0$, one can find a homomorphism $\phi : C(X)\to C(X)$ such that

$$\phi_*|_G = h.$$

Proof. Let $\{p_1, p_2, \ldots, p_m\}$ be nonzero mutually orthogonal projections in $C(X)$ such that $\sum_{i=1}^m p_i = 1_{C(X)}$ and the subgroup F of $C(X,\mathbb{Z})$ generated by $\{p_1, p_2, \ldots, p_m\}$ contains G. Suppose that $h : (j_\alpha)_*(F) \to K_0(B)$ is a homomorphism for which $h([1_A]) = [1_B]$, $h((j_\alpha)_*(F_+)) \subset K_0(B)_+$ and $h(x)\neq 0$ if $x\in (j_\alpha)_*(F_+)\setminus\{0\}$. Since $\sum_{i=1}^m h([j_\alpha(p_i)]) = [1_B]$, by Lemma 2.6, there are nonzero mutually orthogonal projections $\{d_1, d_2, \ldots, d_m\}$ in $C(X)$ such that $\sum_{i=1}^m d_i = 1_{C(X)}$ and $(j_\beta)_*(d_i) = h([j_\alpha(p_i)])$, $i = 1, 2, \ldots, m$. Since non-empty clopen subsets are homeomorphic, there is a homeomorphism $\sigma : X\to X$ such that $\sigma^*(p_i) = d_i$. It is then easy to verify that σ meets the requirements.

If we do not assume that $h(x)\neq 0$ for all $x\in (j_\alpha)_*(F_+)\setminus\{0\}$, then in the above, we may assume $q_i = 0$ for $0 < k < i \le m$. Therefore, we may assume that $d_i = 0$ for $0 < k < i \le m$. We then fix a point ξ_i in the clopen set corresponding to p_i $(i\le k)$ and define $\phi(f) = \sum_{i=1}^k f(\xi_i)d_i$ for all $f\in C(X)$. One then sees that ϕ meets the requirements. □

Lemma 4.2. *Let $A = C(X)\times_\alpha\mathbb{Z}$ and $B = C(X)\times_\beta\mathbb{Z}$, where α and β are minimal homeomorphisms on the Cantor set X. Let G_n be an increasing sequence of finitely generated subgroups of $C(X,\mathbb{Z})$ which contains 1_X so that $\bigcup_{n=1}(j_\alpha)_*(G_n) = K_0(A)$. Let $\lambda : K_1(A)\to K_1(B)$ be an isomorphism. Suppose that there is a sequence of homomorphisms $h_n : (j_\alpha)_*(G_n)\to K_0(B)$ for which $h_n((j_\alpha)_*((G_n)_+)) \subset K_0(B)_+$ such that $h_n([1_A]) = [1_B]$. Then there exists a sequential asymptotic morphism $\{\phi_n\} : A\to B$ such that, for each n, $[\phi_n]|_{(j_\alpha)_*(G_n)}$ and $[\phi_n]|_{K_1(A)}$ are well defined and*

$$[\phi_n]|_{(j_\alpha)_*(G_n)} = h_n \quad \text{and} \quad [\phi_n]|_{K_1(A)} = \lambda$$

Moreover, there are homomorphisms $\phi_n' : j_\alpha(C(X)) \to B$ such that

$$\lim_{n\to\infty} \|\phi_n' \circ j_\alpha(f) - \phi_n \circ j_\alpha(f)\| = 0$$

for all $f \in C(X)$.

Proof. We write $A = \lim_{n\to\infty}(A_n, \phi_{n,n+1})$ where A_n is a circle algebra. Let $\phi_{n,\infty} : A_n \to A$ be the homomorphism induced by the inductive limit. It follows from 1.1 of [NT] that we may assume that $\phi_{n,\infty}$ is injective. We may assume that $(j_\alpha)_*(G_n) \supset (\phi_{n,\infty})_{*0}(K_0(A_n))$. Note that $K_1(A) = \mathbb{Z} = K_1(B)$. So we may also assume that $(\phi_{n,\infty})_{*1}(K_1(A_n)) \supset K_1(A)$. Fix a finite subset $\mathcal{F} \subset A$. We may assume that $\mathcal{F} \subset \phi_{n,\infty}(A_n)$. It should be noted that $\bigcup_{n=1}^\infty (j_\alpha)_*((G_n)_+) = K_0(A)_+$. Note that $K_0(A_n)_+$ are finitely generated. We may also assume that

$$(j_\alpha)_*((G_n)_+) \supset (\phi_{n,\infty})_*(K_0(A_n)_+).$$

It follows from [El] that there exists a unital homomorphism $\psi_n : A_n \to B$ such that

$$(\psi_n)_{*0} = h_n \circ (\phi_{n,\infty})_{*0} \quad \text{and} \quad (\psi_n)_{*1} = \lambda \circ (\phi_{n,\infty})_{*1},$$

where $\lambda : K_1(A) \to K_1(B)$ is an isomorphism. Since both A and B are nuclear, by identifying A_n with $\phi_{n,\infty}(A_n)$, for any $\varepsilon > 0$, there is a unital contractive completely positive map $\phi_n : A \to B$ such that

$$\|\phi_n(a) - \psi_n(a)\| < \varepsilon$$

for all $a \in A_n$ (see 3.2 of [Ln3]). Note if $a, b \in A_n$ with $\|a\|, \|b\| \le 1$,

$$\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \le \|\phi_n(ab) - \psi_n(ab)\| + \|\psi_n(a)\psi_n(b) - \phi_n(a)\phi_n(b)\|$$
$$\le \varepsilon + 2\varepsilon = 3\varepsilon.$$

For each non-zero projection $e \in B$, since eBe is a unital a simple non-elementary $A\mathbb{T}$-algebra with real rank zero, there is a unital C^*-subalgebra $C \subset eBe$ which is a simple AF-algebra with the same ordered K_0-group (for example, see 2.9 of [Ln1]). It is well known that there is a unital commutative C^*-subalgebra $D \subset C$ such that $D = C(Y)$, where Y is a Cantor set. Since $\mathbb{C}^m$ as a C^*-algebra is semiprojective, combinied with the above fact, one obtains a homomorphism $\phi_n' : j_\alpha(C(X)) \to B$ such that

$$\lim_{n\to\infty} \|\phi_n'(j_\alpha(f)) - \phi_n(j_\alpha(f))\| = 0 \quad \text{for all} \quad f \in C(X).$$

$\square$

Lemma 4.3. *Let A be a unital simple C^*-algebra with stable rank one and let X be the Cantor set. Suppose that $h_1, h_2 : C(X) \to A$ are two monomorphisms such that $h_i(1_{C(X)})$ are both unital or both are not unital.*

Let $\varepsilon > 0$ and $\mathcal{F} \subset C(X)$ be a finite subset. There is $\delta > 0$, and a finite subset $\mathcal{P}$ of $K_0(C(X))$ for which the following holds: if

$$(h_1)_*|_{\mathcal{P}} = (h_2)_*|_{\mathcal{P}},$$

then there is a unitary $u \in A$ such that

$$\|u^* h_1(f) u - h_2(f)\| < \varepsilon \quad \textit{for all} \quad f \in \mathcal{F}.$$

Proof. Without loss of generality, we may assume that $e = h_1(1_{C(X)}) = h_2(1_{C(X)})$. By replacing A by eAe, we may further assume that both h_1 and h_2 are unital.

There exists a finite subset of mutually orthogonal projections $\{p_1, p_2, \dots, p_m\}$ in $C(X)$ with $\sum_{i=1}^m p_i = 1_{C(X)}$ and finitely many points $x_1, x_2, \dots, x_m \in X$ such that

$$\|f - \sum_{i=1}^m f(x_i)p_i\| < \varepsilon/2 \quad \text{for all} \quad f \in \mathcal{F}.$$

Let $\mathcal{P} = \{p_1, p_2, \dots, p_m\}$. Assume $[h_1(p_i)] = [h_2(p_i)]$ in $K_0(A)$. Since A has stable rank one, it follows that there are partial isometries $u_i \in A$ such that

$$u_i^* h_1(p_i) u_i = h_2(p_i), \quad i = 1, 2, \dots, m.$$

Since $1 = \sum_{i=1}^m h_1(p_i) = \sum_{i=1}^m h_2(p_i)$, there exists a unitary $u \in A$ such that

$$u^* h_1(p_i) u = h_2(p_i), \quad i = 1, 2, \dots, m.$$

Then we have

$$\|u^* h_1(f) u - h_2(f)\| \le \|u^* h_1(f) u - \sum_{i=1}^m f(x_i) u^*(h_1(p_i)) u\| +$$

$$\|\sum_{i=1}^m f(x_i) u^*(h_1(p_i)) u - \sum_{i=1}^m f(x_i) h_2(p_i))\| + \|\sum_{i=1}^m f(x_i) h_2(p_i)) - h_2(f)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $f \in \mathcal{F}$. ☐

Recall that two maps $h_1, h_2 : A \to B$ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} \| \operatorname{ad} u_n \circ h_1(a) - h_2(a)\| = 0$$

for all $a \in A$.

Lemma 4.4. *Let A be a unital simple C^*-algebra with stable rank one and let X be the Cantor set. Suppose that $h_1, h_2 : C(X) \to A$ are two monomorphisms. Then h_1 and h_2 are approximately unitarily equivalent if and only if $(h_1)_* = (h_2)_*$ on $K_0(C(X))$.*

Proof. This is an immediate consequence of the previous lemma. ☐

Lemma 4.5. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Let $A = C(X) \times_\alpha \mathbb{Z}$ and $B = C(X) \times_\beta \mathbb{Z}$. Let $\{F_n\}$ and $\{G_n\}$ be increasing sequences of finitely generated subgroups of $C(X, \mathbb{Z})$ such that $\bigcup_{n=1}^\infty F_n = C(X, \mathbb{Z})$ and $\bigcup_{n=1}^\infty G_n = C(X, \mathbb{Z})$, respectively. Suppose that there exist sequences of homomorphisms $\xi_n : (j_\alpha)_*(F_n) \to K_0(B)$ and $\zeta_n : (j_\beta)_*(G_n) \to K_0(A)$ for which $\xi_n (j_\alpha)_*((F_n)_+) \subset K_0(B)_+$ and $\zeta_n (j_\beta)_*((G_n)_+) \subset K_0(A)_+$ such that $\xi_n([1_A]) = [1_B]$ and $\zeta_n([1_B]) = [1_A]$.*

Then there exist sequential asymptotic morphisms $\{\phi_n\} : A \to B$ and $\{\psi_n\} : B \to A$ and homomorphisms $\phi_n' : j_\alpha(C(X)) \to B$, $\psi_n' : j_\beta(C(X)) \to A$, homomorphisms $\Phi_n, \Psi_n : C(X) \to C(X)$ with

$$\lim_{n\to\infty} \|\phi_n(j_\alpha(f)) - \phi_n'(j_\alpha(f))\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\phi_n(j_\beta(f)) - \psi_n'(j_\beta(f))\| = 0$$

for all $f \in C(X)$, and

$$\lim_{n\to\infty} \|\phi_n(j_\alpha(f)) - j_\beta(\Phi_n(f))\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\psi_n(j_\beta(f)) - j_\alpha(\Psi_n(f))\| = 0 \qquad \text{(e 4.4)}$$

for all $f \in C(X)$. Furthermore, if $\xi_n((j_\alpha)_*((F_n)_+) \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$ and $\zeta((j_\beta)_*((G_n)_+) \setminus \{0\}) \subset K_0(A) \setminus \{0\}$, then one can require that Φ_n and Ψ_n to be isomorphisms.

Proof. It follows from Lemma 4.2 that only (e 4.4) needs to be proved. Note also as in 4.2, one has

$$[\phi_n]|_{(j_\alpha)_{*0}(F_n)} = \xi_n \quad \text{and} \quad [\psi_n]|_{(j_\beta)_{*0}(G_n)} = \zeta_n.$$

By applying Lemma 4.1 and 4.3, one obtains a sequence of homomorphisms $\Phi_n : C(X) \to C(X)$ and a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} \| \operatorname{ad} u_n \circ \phi_n' \circ j_\alpha(f) - j_\beta(\Phi_n(f))\| = 0 \quad \text{for all} \quad f \in C(X).$$

We may replace ϕ_n by $\operatorname{ad} u_n \circ \phi_n$ and ϕ' by $\operatorname{ad} u_n \circ \phi_n'$. We then apply the same argument to ψ_n. For the very last of the statement, we note that, by Lemma 4.1, under the assumption that $\xi_n((j_\alpha)_*((F_n)_+) \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$ and $\zeta((j_\beta)_*((G_n)_+) \setminus \{0\}) \subset K_0(A) \setminus \{0\}$, one can require that Φ_n and Ψ_n to be isomorphisms. □

Lemma 4.6. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of $\{1, 2, \ldots, N\}$ and $\pi : \mathcal{P} \to \mathcal{Q}$ be a bijection with $\#U = \#\pi(U)$ for all $U \in \mathcal{P}$. If, for every non-empty proper subset $\mathcal{F}$ of $\mathcal{P}$, we have*

$$\bigcup_{U\in\mathcal{F}} U \neq \bigcup_{U\in\mathcal{F}} \pi(U),$$

then there exists a cyclic permutation σ on $\{1, 2, \ldots, N\}$ of order N such that $\sigma(U) = \pi(U)$ for all $U \in \mathcal{P}$, that is, $\sigma(i) \in \pi(U)$ whenever $i \in U \in \mathcal{P}$.

Proof. It is easy to find a permutation σ satisfying $\sigma(U) = \pi(U)$ for all $U \in \mathcal{P}$. If σ consists of one cycle, then we have nothing to do. Otherwise we will modify σ so that it becomes a cyclic permutation. Let C be a cycle contained in σ. Put $\mathcal{F} = \{U \in \mathcal{P} : U \cap C \neq \emptyset\}$. If $C = \bigcup_{U\in\mathcal{F}} U$, then $\mathcal{F}$ is a proper subset of $\mathcal{P}$ and

$$\bigcup_{U\in\mathcal{F}} U = C = \sigma(C) = \bigcup_{U\in\mathcal{F}} \pi(U),$$

which is a contradiction. Therefore we can find $U \in \mathcal{P}$ such that $U \cap C \neq \emptyset$ and U is not contained in C. Take $i \in U \cap C$ and $j \in U \setminus C$. By sending i to $\sigma(j)$ and j to $\sigma(i)$, we obtain a new σ. Then the number of its cycles is less than that of the original one. By repeating this, we get finally a cyclic permutation σ. □

Lemma 4.7. *Let X be the Cantor set and α, β be minimal homeomorphisms. Let $\mathcal{P}$ be a clopen partition of X. If $[1_U] = [1_{\beta(U)}]$ in $K^0(X, \alpha)$ for all $U \in \mathcal{P}$, then we can find a homeomorphism $\sigma \in [[\alpha]]$ such that $\sigma\alpha\sigma^{-1}(U) = \beta(U)$ for all $U \in \mathcal{P}$.*

Proof. Let $\mathcal{Q}$ denote the clopen partition $\{\beta(U) : U \in \mathcal{P}\}$. We use the Bratteli-Vershik model for α (see [HPS, Theorem 4.2]). Let

$$\mathcal{P}_n = \{E(n, v, k) : v \in V_n,\ k = 1, 2, \ldots, h(v)\}$$

be a sequence of Kakutani-Rohlin partitions such that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ for all $n \in \mathbb{N}$, $\bigcup_n \mathcal{P}_n$ generate the topology of X and the roof sets

$$U_n = \bigcup_{v\in V_n} E(n, v, h(v))$$

shrink to a single point in X. By taking a sufficiently large n, we may assume that $\mathcal{P}_n$ is finer than $\mathcal{P}$ and $\mathcal{Q}$. Moreover, for every $U \in \mathcal{P}$ and $v \in V_n$,

$$\#\{k : E(n, v, k) \subset U\} = \#\{k : E(n, v, k) \subset \beta(U)\} \geq 1$$

since $[1_U] = [1_{\beta(U)}]$ by the assumption. Note that these properties hold for every m larger than n. Because α is minimal, we can find $m > n$ so that for every $v \in V_n$, $k \in \{1, 2, \ldots, h(v)\}$ and $w \in V_m$, there exists $j \in \{1, 2, \ldots, h(w)\}$ such that $E(m, w, j) \subset E(n, v, k)$.

Define a map $p_n : \mathcal{P}_n \to \mathcal{P}$ so that $p(U) \supset U$ for all $U \in \mathcal{P}$. Define $q_n : \mathcal{P}_n \to \mathcal{Q}$ in the same way. Let $\mathcal{F}$ be a non-trivial proper subset of $\mathcal{P}$ and put $\mathcal{G} = \{\beta(U) \in \mathcal{Q} : U \in \mathcal{F}\}$. Since β has no non-trivial invariant clopen sets, we have

$$\bigcup_{U \in \mathcal{F}} U \neq \bigcup_{V \in \mathcal{G}} V.$$

So there exist $v \in V_n$ and an integer k such that either $p_n(E(n, v, k))$ belongs to $\mathcal{F}$ but $q_n(E(n, v, k))$ does not belong to $\mathcal{G}$, or $p_n(E(n, v, k))$ does not belong to $\mathcal{F}$ but $q_n(E(n, v, k))$ belongs to $\mathcal{G}$. Hence, by the choice of m, for every $w \in V_m$, we get

$$\bigcup_{p_m(E(m,w,j)) \in \mathcal{F}} E(m, w, j) \neq \bigcup_{q_m(E(m,w,j)) \in \mathcal{G}} E(m, w, j).$$

By applying Lemma 4.6, we obtain a cyclic permutation σ_w of order $h(w)$ such that $q_m(E(m, w, \sigma_w(j))) = \beta(p_m(E(m, w, j)))$ holds for all $j = 1, 2, \ldots, h(w)$.

Choose $U_0 \in \mathcal{P}$ and $j_w \in \{1, 2, \ldots, h(w)\}$ for all $w \in V_m$ such that $p_m(E(m, w, j_w)) = U_0$. For $w \in V_m$ and $j \in \{1, 2, \ldots, h(w)\}$, put $r(w, j) = \sigma_w^j(j_w) - j$. Define a homeomorphism σ by $\sigma(x) = \alpha^{r(w,k)}(x)$ for $x \in E(m, w, j)$. It is clear that σ belongs to $[[\alpha]]$. One can check $\sigma\alpha\sigma^{-1}(E(m, w, k)) = E(m, w, \sigma_w(k))$ for $k \neq j_w$. Furthermore we have

$$\bigcup_{w \in V_m} \sigma\alpha\sigma^{-1}(E(m, w, j_w)) = \bigcup_{w \in V_m} E(m, w, \sigma_w(j_w)).$$

Consequently we get $\sigma\alpha\sigma^{-1}(U) = \beta(U)$ for all $U \in \mathcal{P}$. $\square$

Theorem 4.8. *Let X be the Cantor set and let $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Let $A = A_\alpha$ and $B = A_\beta$. Then the following are equivalent:*

(1) *α and β are weakly approximately conjugate;*

(2) *There exist two unital sequential asymptotic morphisms $\{\phi_n\} : A \to B$ and $\{\psi_n\} : B \to A$;*

(3) *There are two increasing sequences of finitely generated subgroups $\{F_n\}$ and $\{G_n\}$ of $K_0(C(X))$ with $\bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty G_n = K_0(C(X))$ and two sequences of homomorphisms $\xi_n : (j_\alpha)_*(F_n) \to K_0(B)$ with $\xi_n((j_\alpha)_*((F_n)_+) \subset K_0(B)_+$ and $\zeta_n : (j_\beta)_*(G_n) \to K_0(A)$ with $\zeta_n((j_\beta)_*((G_n)_+) \subset K_0(A)_+$ for which $\xi_n([1_A]) = [1_B]$, $\zeta_n([1_B]) = [1_A]$, and for any $x \in K_0(A)_+ \setminus \{0\}$ and $y \in K_0(B)_+ \setminus \{0\}$, there is $N > 0$ such that $\xi_n(x) \neq 0$ and $\zeta_n(y) \neq 0$ for all $n \geq N$;*

(4) *there are two increasing sequences of finitely generated subgroups $\{F_n\}$ and $\{G_n\}$ of $K_0(C(X))$ with $\bigcup_{n=1}^\infty F_n = \bigcup_{n=1}^\infty G_n = K_0(C(X))$ and two sequences of positive homomorphisms $\xi_n : (j_\alpha)_*(F_n) \to K_0(B)$ with $\xi_n((j_\alpha)_*((F_n)_+) \subset K_0(B)_+$ and $\zeta_n : (j_\beta)_*(G_n) \to K_0(A)$ with $\zeta_n((j_\beta)_*((G_n)_+) \subset K_0(A)_+$ for which $\xi_n([1_A]) = [1_B]$ and $\zeta_n([1_B]) = [1_A]$, respectively, and two sequences of positive isomorphisms $(\sigma_n)_*, (\gamma_n)_* : K_0(C(X)) \to K_0(C(X))$ with $(\sigma_n)_*([1_{C(X)}]) = [1_{C(X)}]$, $(\gamma_n)_*([1_{C(X)}]) = [1_{C(X)}]$ such that, for each $x \in K_0(C(X))$,*

$$\xi_n \circ (j_\alpha)_*(x) = (j_\beta)_* \circ (\sigma_n)_*(x) \quad \text{and} \quad \zeta_n \circ (j_\beta)_*(x) = (j_\alpha)_* \circ (\gamma_n)_*(x)$$

for all large n;

(5) *there exist two sequential asymptotic morphisms* $\{\phi_n\} : A \to B$ *and* $\{\psi_n\} : B \to A$ *and homomorphisms* $\phi_n' : j_\alpha(C(X)) \to B$ *and* $\psi_n' : j_\beta(C(X)) \to A$ *such that*

$$\lim_{n\to\infty} \|\phi_n \circ j_\alpha(f) - \phi_n' \circ j_\alpha(f)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\psi_n \circ j_\beta(f) - \psi_n' \circ j_\beta(f)\| = 0$$

for all $f \in C(X)$, *and there are unital injective homomorphisms* $\Phi_n, \Psi_n : C(X) \to C(X)$ *such that*

$$\lim_{n\to\infty} \|\phi_n' \circ j_\alpha(f) - j_\beta \circ \Phi_n(f)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\psi_n' \circ j_\beta(f) - j_\alpha \circ \Psi_n(f)\| = 0$$

for all $f \in C(X)$;

Proof. (1)⇒(2) follows from Proposition 3.2.

(2)⇒(3) follows from Definition 2.2. But note that, when A (and B) are simple, by Lemma 2.3 and by Definition 2.2, for any nonzero projection p, $[\phi_n]([p])$ cannot be zero in the simple dimension group $K_0(B)_+$.

(3)⇒(4) follows from Lemma 4.1.

(3)⇒(5) follows from Lemma 4.5.

(4)⇒(3). Note that the requirements that $\xi_n(x) > 0$ eventually follows from the fact that $K_0(A)$ is a simple dimension group and $mx \geq [1_A]$ if $x \in K_0(A)_+ \setminus \{0\}$.

(5)⇒(1). Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset C(X)$. Let $\mathcal{P}$ be a clopen partition for X and $p_1, p_2, \dots, p_m$ be corresponding projections in $C(X)$. We choose $\mathcal{P}$ so that

$$\|f - \sum_{i=1}^m f(x_i)p_i\| < \varepsilon/2 \quad \text{for all} \quad f \in \mathcal{F},$$

where x_i in the clopen set in $\mathcal{P}$ associated with p_i. Let $\mathcal{G} = \mathcal{F} \cup \{p_1, \dots, p_m\}$. Without loss of generality, we may assume that

$$\|\phi_n' \circ j_\alpha(f) - j_\beta \circ \Phi_n(f)\| < \varepsilon/4 \tag{e 4.5}$$

for all $f \in \mathcal{G} \cup \alpha^*(\mathcal{G})$. We have

$$\lim_{n\to\infty} \|\phi_n(u_\alpha)\phi_n'(j_\alpha(f))\phi_n(u_\alpha^*) - \phi_n'(j_\alpha(\alpha^*(f)))\| = 0 \tag{e 4.6}$$

for all $f \in C(X)$. There are unitaries $z_n \in B$ such that

$$\lim_{n\to\infty} \|\phi_n(u_\alpha) - z_n\| = 0. \tag{e 4.7}$$

It follows from this and (e 4.5) and (e 4.6) that

$$\|z_n j_\beta(\Phi_n(p_i))z_n^* - j_\beta(\Phi_n(\alpha^*(p_i)))\| < \varepsilon/4, \quad i = 1, 2, \dots, m. \tag{e 4.8}$$

In particular, (for $0 < \varepsilon < 1/2$)

$$[j_\beta \circ \Phi_n(p_i)] = [j_\beta \circ \Phi_n(\alpha^*(p_i))], \quad i = 1, 2, \dots, m.$$

This implies, by Lemma 4.7, there is a homeomorphism $\sigma \in [[\beta]]$ such that

$$(\sigma\beta\sigma^{-1})^*(\Phi_n(p_i)) = \Phi_n(\alpha^*(p_i)), \quad i = 1, 2, \dots, m.$$

It follows that

$$\|(\sigma\beta\sigma^{-1}) \circ \Phi_n(f) - \Phi_n \circ \alpha^*(f)\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

Thus we obtain a homeomorphism $\sigma' : X \to X$ such that

$$\|(\sigma'\beta(\sigma')^{-1})^*(f) - \alpha^*(f)\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

The same argument shows that there exists a homeomorphism $\gamma : X \to X$ such that

$$\|(\gamma\alpha\gamma^{-1})^*(f) - \beta^*(f)\| < \varepsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

□

Remark 4.9. In the proof of (5)⇒(1), we remark that for a given finitely generated subgroup of $K_0(A)$ (and $K_0(B)$), σ' (and γ) may be chosen so it induces the same map on that subgroup as that of Φ_n (and Ψ_n), since $\sigma \in [[\alpha]]$.

Corollary 4.10. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Suppose that there is a positive homomorphism $h : K^0(X, \alpha) \to K^0(X, \beta)$ with $h([1_A]) = [1_B]$. Then there exists a sequence of homeomorphisms $\sigma_n : X \to X$ such that*

$$\lim_{n\to\infty} (\sigma_n\beta\sigma_n^{-1})^*(f) = \alpha^*(f) \quad \textit{for all} \quad f \in C(X).$$

Moreover, there exists a sequence of $\gamma_n \in [[\beta]]$ and a unital injective homomorphism $\Phi : C(X) \to C(X)$ such that

$$\lim_{n\to\infty} (\gamma_n\beta\gamma_n^{-1})^* \circ \Phi(f) = \Phi \circ \alpha^*(f) \quad \textit{for all} \quad f \in C(X).$$

Moreover, $(j_\beta)_{*0} \circ \Phi_{*0} = h \circ (j_\alpha)_{*0}$.

Proof. Since both $K^0(X, \alpha)$ and $K^0(X, \beta)$ are simple dimension groups, we conclude that

$$h \circ (j_\alpha)_{*0}(C(X, \mathbb{Z})_+ \setminus \{0\}) \subset K^0(X, \beta)_+ \setminus \{0\}.$$

It follows from Theorem 2.7 that there is an injective unital positive homomorphism $\delta : C(X, \mathbb{Z}) \to C(X, \mathbb{Z})$ such that

$$(j_\beta)_{*0} \circ \delta = h \circ (j_\alpha)_{*0}.$$

One obtains an injective unital homomorphism $\Phi : C(X) \to C(X)$ such that $\Phi_{*0} = \delta$. In the proof of Theorem 4.8, one can choose $\Phi_n = \Phi$. □

Definition 4.11. Let (G, G_+, u) be a scaled ordered abelian group. Set

$$D(G, u) = \{n \in \mathbb{N} : ne = u \text{ for some } e \in G_+\}.$$

Definition 4.12. Let (X, α) be a Cantor minimal system. By the periodic spectrum of (X, α) or α, we mean the set of natural numbers p for which there are disjoint clopen sets $U, \alpha(U), \ldots, \alpha^{p-1}(U)$ whose union is X.

It is obvious that p belongs to periodic spectrum of α if and only if $p \in D(K_0(A_\alpha), [1_{A_\alpha}])$.

Theorem 4.13. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. Then the following are equivalent.*

(1) *α and β are weakly approximately conjugate.*

(2) *The periodic spectrum of α and β agree.*

(3) *$D(K_0(A_\alpha), [1_{A_\alpha}]) = D(K_0(A_\beta), [1_{A_\beta}])$.*

Proof. We have seen (2)⇔(3).

(1)⇒(3). Suppose that $p \in D(K_0(A_\alpha), [1_{A_\alpha}])$. Then there is $e \in K_0(A_\alpha)_+$ such that $pe = [1_{A_\alpha}]$. Let $G \subset K_0(A_\alpha)$ generated by e. Suppose that $\{\psi_n\} : A_\alpha \to A_\beta$ is a unital asymptotic morphism given by Proposition 3.2. It follows that there are homomorphisms $h_n : G \to K_0(A_\beta)$ such that $h_n(e) \geq 0$, $h_n(e) \neq 0$ and $h_n([1_{A_\alpha}]) = [1_{A_\beta}]$. Thus $ph_n(e) = [1_\beta]$. This proves that $D(K_0(A_\alpha), [1_{A_\alpha}]) \subset D(K_0(A_\beta), [1_{A_\beta}])$. Applying the same argument to A_β, one obtains $D(K_0(A_\alpha), [1_{A_\alpha}]) = D(K_0(A_\beta), [1_{A_\beta}])$.

(3)⇒(1). Write $A = \lim_{n\to\infty}(A_n, \phi_n)$, where A_n is a circle algebra and ϕ_n are injective. Thus $K_0(A) = \lim_{k\to\infty}(K_0(A_n), (\phi_n)_*)$ where each $K_0(A_n) = \mathbb{Z}^{r_n}$ with the simplicial order. Fix n, we may assume that there is $u_n \in K_0(A_n)_+$ such that $\phi_{n,\infty}(u_n) = [1_A]$. Denote $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_{r_n} = (0, \ldots, 0, 1)$. We may assume that $u_n = \sum_{i=1}^{r_n} m_i e_i$, where $m_i > 0$. Let p be the greatest common divisor of $m_1, m_2, \ldots, m_{r_n}$. Write $m_i = pk_i$, where $k_i \in \mathbb{N}$ and $k_1, k_2, \ldots, k_{r_n}$ are relatively prime. Choose an integer $N > 0$ such that

$$\{n \in \mathbb{N} : n \geq N\} \subset \mathbb{N}k_1 + \mathbb{N}k_2 + \cdots + \mathbb{N}k_{r_n}.$$

Write $B = \lim_{n\to\infty}(B_n, \psi_n)$, where each B_n is a circle algebra and ψ_n are injective. Thus $K_0(B) = \lim_{n\to\infty}(K_0(B_n), (\psi_n)_*)$, where $K_0(B_n) = \mathbb{Z}^{s_n}$ with the usual order. We may assume that $v_n \in K_0(B_n)_+$ such that $\psi_{n,\infty}(v_n) = [1_B]$. Since $K_0(B)$ is a simple dimension group and p divides $[1_B]$ in $K_0(B)$, we may assume that

$$[1_B] = (pd_1, pd_2, \ldots, pd_{s_n}) \text{ with } d_i > N, \quad i = 1, 2, \ldots, s_n.$$

Thus there are $N(i,j) \in \mathbb{N}$ such that

$$d_i = \sum_{j=1}^{r_n} N(i,j)k_i, \quad i = 1, 2, \ldots, s_n$$

(in $K_0(B_n)$). Define a $s_n \times r_n$ matrix $T_n = (N(i,j))_{s_n \times r_n}$. Then T_n gives a positive homomorphism from $\mathbb{Z}^{r_n}$ to $\mathbb{Z}^{s_n}$ which maps u_n to v_n. Note that we may write $A_n = \bigoplus_{i=1}^{r_n} C(S^1, M_{m_i})$. Using a point-evaluation, we obtain a unital homomorphism $\theta_n : \phi_{n,\infty}(A_n) \to \bigoplus_{i=1}^{r_n} M_{m_i}$. One checks that $(\theta_n)_{*0} = \mathrm{id}_{K_0(A_n)}$. Write $B_n = \bigoplus_{j=1}^{s_n} C(S^1, M_{pd_j})$. There is a unital homomorphism $h_n : \bigoplus_{i=1}^{r_n} M_{m_i} \to \bigoplus_{j=1}^{s_n} M_{pd_j}$ such that $(h_n)_{*0} = T$. It follows from 5.2 of [P] (see also 3.2 (1) of [Ln3]) that there is a unital completely positive linear map $L_n : A \to \bigoplus_{j=1}^{s_n} M_{pd_j}$ such that $(L_n)|_{\phi_{n,\infty}(A_n)} = h_n \circ \theta_n$. Now define $\Phi_n = \psi_{n,\infty} \circ L_n$. It follows that $\{\Phi_n\}$ is a unital sequential asymptotic morphism from A to B.

The same argument allows one to construct a unital sequential asymptotic morphism from B to A. It follows from Theorem 4.8 that α and β are weakly approximately conjugate. □

Remark 4.14. There is a direct dynamical proof of the implication from (2) (or (3)) to (1) in the theorem above. See [M].

5 Other notions of approximate conjugacy in the Cantor minimal systems

Lemma 5.1. *Let F_1, F_2 and G be countable abelian torsion free groups with $g \in G$ and $f_i \in F_i$ $(i = 1, 2)$. Suppose that there is a surjective homomorphism $\pi_i : F_i \to G$ such that $\pi_i(f_i) = g$ $(i = 1, 2)$. Suppose that $D(F_1, f_1) = D(F_2, f_2)$. Then, for each finitely generated group $G_1 \subset F_1$ containing f_1, there exists a homomorphism $h : G_1 \to F_2$ such that $\pi_2 \circ h = \pi_1|_{G_1}$ and $h(f_1) = f_2$.*

Proof. Write $G_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (there are r copies of $\mathbb{Z}$). Suppose that $f_1 = (m_1, m_2, \dots, m_r)$, where $m_i \in \mathbb{Z}$. Let p be the greatest common divisor of $m_1, m_2, \dots, m_r$. We may write $m_i = pk_i$, $i = 1, 2, \dots, r$ and $k_1, k_2, \dots, k_r$ are relatively prime. By the assumption there is $f_0 \in F_2$ such that $pf_0 = f_2$. Denote $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots, e_r = (0, \dots, 0, 1)$ in G_1. Since $k_1, k_2, \dots, k_r$ are relatively prime, there are $n_1, n_2, \dots, n_r \in \mathbb{Z}$ such that

$$\sum_{i=1}^{r} k_i n_i = 1.$$

On the other hand, there are $g_i \in F_2$ such that $\pi_2(g_i) = \pi_1(e_i)$, $i = 1, 2, \dots, r$. Since G is torsion free, $f_{00} = \sum_{i=1}^{r} k_i g_i - f_0 \in \ker \pi_2$.

Put $x_i = n_i f_{00}$, $i = 1, 2, \dots, r$. Define $h : G_1 \to F_2$ by $h(e_i) = g_i - x_i$, $i = 1, 2, \dots, r$. Then $\pi_2 \circ h = \pi_1$ and

$$h(f_1) = p \sum_{i=1}^{r} k_i (g_i - x_i) = p[(\sum_{i=1}^{r} k_i g_i) - f_{00}] = pf_0 = f_2.$$

$\square$

Theorem 5.2. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be two minimal homeomorphisms. The following are equivalent.*

(1) *α and β are approximately τ-conjugate.*

(2) *α and β are orbit equivalent and have the same periodic spectrum.*

(3) *$\rho_A(K_0(A_\alpha))$ is unital order isomorphic to $\rho_B(K_0(A_\beta))$ and $D(K_0(A_\alpha), [1_{A_\alpha}]) = D(K_0(A_\beta), [1_{A_\beta}])$.*

(4) *There exist unital sequential asymptotic morphisms $\phi_n : A_\alpha \to A_\beta$ and $\psi_n : A_\beta \to A_\alpha$ such that the following are satisfied: for any projections $p \in A$ and $q \in B$ there exists N such that*

$$\tau([\psi_n]([q])) = \tau([\psi_{n+1}]([q])), \quad \tau'([\phi_n]([p])) = \tau'([\phi_{n+1}]([p])), \tag{e 5.9}$$

$$\tau([\psi_m \circ \phi_n]([p])) = \tau([p]) \quad \textit{and} \quad \tau'([\phi_m \circ \psi_n]([q])) = \tau'([q]) \tag{e 5.10}$$

for every trace τ on A, every trace τ' on B and all $m \geq n \geq N$.

Proof. (1)⇒(4). This follows immediately from the definition and Proposition 3.7.

(4)⇒(3). The fact that $D(K_0(A_\alpha), [1_{A_\alpha}]) = D(K_0(A_\beta), [1_{A_\beta}])$ follows from Theorem 4.13. Then one applies Theorem 3.8.

(3)⇒(1). Let $A = A_\alpha$ and $B = A_\beta$ and let $\{G_n\}$ be an increasing sequence of finitely generated subgroups of $C(X, \mathbb{Z})$ containing $[1_{C(X)}]$ such that $\bigcup_{n=1}^{\infty} G_n = C(X, \mathbb{Z})$. Let $s : \rho_A(K_0(A)) \to \rho_B(K_0(B))$ be the order isomorphism preserving the unit. It follows from Lemma 5.1, there exists a homomorphism $h_n : G_n \to K_0(B)$ which maps $[1_{C(X)}]$ to $[1_B]$ and $\rho_B \circ h_n = (s \circ \rho_A \circ (j_\alpha)_{*0})|_{G_n}$, $n = 1, 2, \dots$. Then h_n is a positive homomorphism which maps $[1_{C(X)}]$ to $[1_B]$. It follows from Theorem 4.8 ((2)⇒(1)) that there exists a sequence of homeomorphisms $\sigma_n : X \to X$ such that

$$\lim_{n \to \infty} (\sigma_n \circ \alpha \circ \sigma_n^{-1})^*(f) = \beta^*(f) \quad \text{for all} \quad f \in C(X).$$

Similarly, we obtain a sequence of homeomorphisms $\gamma_n : X \to X$ such that

$$\lim_{n \to \infty} (\gamma_n \circ \beta \circ \gamma_n^{-1})^*(f) = \alpha^*(f) \quad \text{for all} \quad f \in C(X).$$

So α and β are weakly approximately conjugate. Moreover, from the definition of ϕ_n and Remark 4.9 we see that they are actually approximately τ-conjugate. For example, if $p \in C(X)$ is a projection, then, by the proof of Theorem 4.8,

$$[j_\beta \circ \Phi_n(p)] = h_n \circ s \circ \rho_A \circ (j_\alpha)_*([p])$$

for all large n. Then

$$\rho_B([j_\beta \circ \Phi_n(p)]) = s \circ \rho_A \circ (j_\alpha)_*([p]),$$

is a constant (for sufficiently large n).

(2)⇔(3). This follows from Theorem 2.3 of [GPS]. □

There is a general version of K-conjugacy. However, we only introduce this notion for the Cantor set for simplicity.

Definition 5.3. Let X be the Cantor set, $\alpha, \beta : X \to X$ be two minimal homeomorphisms. We say α and β are *approximately K-conjugate* if there are homeomorphisms $\sigma_n, \gamma_n : X \to X$ such that

$$\lim_{n\to\infty} (\sigma_n \alpha \sigma_n^{-1})^*(f) = \beta^*(f) \quad \text{and} \quad \lim_{n\to\infty} (\gamma_n \beta \gamma_n^{-1})^*(f) = \alpha^*(f) \quad \text{for all} \quad f \in C(X),$$

in addition, for any $g \in C(X, \mathbb{Z})$, there exists N such that

$$(j_\alpha)_{*0}(g \circ \sigma_n) = (j_\alpha)_{*0}(g \circ \sigma_{n+1}), \quad (j_\beta)_{*0}(g \circ \gamma_n) = (j_\beta)_{*0}(g \circ \gamma_{n+1}),$$

$$(j_\alpha)_{*0}(g \circ \gamma_n \circ \sigma_m) = (j_\alpha)_{*0}(g) \quad \text{and} \quad (j_\beta)_{*0}(g \circ \sigma_n \circ \gamma_m) = (j_\beta)_{*0}(g)$$

for all $m \geq n \geq N$.

Theorem 5.4. *Let X be the Cantor set and $\alpha, \beta : X \to X$ be minimal homeomorphisms. Let $A = A_\alpha$ and $B = A_\beta$. Then the following are equivalent.*

(1) *α and β are approximately K-conjugate.*

(2) $C(X) \times_\alpha \mathbb{Z} \cong C(X) \times_\beta \mathbb{Z}$.

(3) *$(K^0(X, \alpha), K^0(X, \alpha)_+, [1])$ is unital order isomorphic to $(K^0(X, \beta), K^0(X, \beta)_+, [1])$.*

(4) *There exists a homeomorphism $F : X \to X$, and exist sequences $\sigma_n \in [[\alpha]]$ and $\gamma_n \in [[\beta]]$ such that*

$$\lim_{n\to\infty} (F^{-1} \circ \sigma_n \circ \alpha \circ \sigma_n^{-1} \circ F)^*(f) = \beta^*(f) \quad \text{and} \quad \lim_{n\to\infty} (F \circ \gamma_n \circ \beta \circ \gamma_n^{-1} \circ F^{-1})^*(f) = \alpha^*(f)$$

for all $f \in C(X)$.

(5) *α and β are strongly orbit equivalent.*

(6) *There exist asymptotic morphisms $\phi_n : A \to B$ and $\psi_n : B \to A$ such that the following are satisfied: there is a unit preserving order isomorphism $\Phi : (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B)_+, [1_B])$ such that, for any finite subsets $\mathcal{G} \subset K_0(A)$ and $\mathcal{F} \subset K_0(B)$, there exists $N > 0$ for which*

$$[\phi_n]|_{\mathcal{G}} = \Phi|_{\mathcal{G}} \quad \text{and} \quad [\psi_n]|_{\mathcal{F}} = \Phi^{-1}|_{\mathcal{F}},$$

whenever $n \geq N$.

Proof. (1)⇒(6). This follows from the definition of approximate K-conjugacy and Proposition 3.2.

(6)⇒(3) is obvious.

(3)⇒(4). This follows from Theorem 2.7 and the proof of Corollary 4.10. Note this time the map Φ in Corollary 4.10 can be chosen to be isomorphism.

(4)⇒(1). This is obvious.

(5)⇔(2) follows from [GPS].

(2)⇔(3) follows from [El] (see also [GPS]). □

Remark 5.5. Let us consider the case $X = S^1$. It is well-known that every minimal homeomorphism on S^1 is conjugate to an irrational rotation R_α (see [PY, Proposition 6.4] and [KH, Ex. 11.2.4, p. 400], for example). In this case, when are two irrational rotations R_α and R_β weakly approximately conjugate? By the following easy observation, we can see that they are weakly approximately conjugate if and only if they are actually conjugate.

We identify S^1 with $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let us denote the set of orientation preserving homeomorphisms on $\mathbb{T}$ by $\mathrm{Homeo}^+(\mathbb{T})$. For $\phi \in \mathrm{Homeo}^+(\mathbb{T})$ we denote the rotation number of ϕ by $r(\phi) \in \mathbb{T}$. The reader may refer to [PY, Section 6] or [KH, Chapter 11] for the definition and the elementary property of $r(\phi)$.

By [KH, 11.1.3], $r(\sigma\phi\sigma^{-1}) = r(\phi)$ for all $\phi, \sigma \in \mathrm{Homeo}^+(\mathbb{T})$. Furthermore, by [KH, 11.1.6], the map $r : \mathrm{Homeo}^+(\mathbb{T}) \to \mathbb{T}$ is continuous, where $\mathrm{Homeo}^+(\mathbb{T})$ is equipped with the so called uniform topology (see p.710 of [KH]). It is not hard to see that the uniform topology of $\mathrm{Homeo}(X)$ agrees with the topology induced from pointwise convergence in norm on $C(X)$ when X is compact and metrizable.

Suppose that there exists a sequence of homeomorphisms σ_n such that $\sigma_n R_\alpha \sigma_n^{-1}$ converges to R_β. We may assume that σ_n belongs to $\mathrm{Homeo}^+(\mathbb{T})$ for all $n \in \mathbb{N}$ or $\sigma_n \circ \pi$ belongs to $\mathrm{Homeo}^+(\mathbb{T})$ for all $n \in \mathbb{N}$, where π is given by $\pi(t) = -t$. When $\sigma_n \in \mathrm{Homeo}^+(\mathbb{T})$, we have

$$\beta = r(R_\beta) = \lim_{n\to\infty} r(\sigma_n R_\alpha \sigma_n^{-1}) = \lim_{n\to\infty} r(R_\alpha) = \alpha.$$

Thus, R_β is conjugate to R_α. If $\sigma_n \pi \in \mathrm{Homeo}^+(\mathbb{T})$,

$$\beta = r(R_\beta) = \lim_{n\to\infty} r(\sigma_n \pi R_{-\alpha} \pi \sigma_n^{-1}) = \lim_{n\to\infty} r(R_{-\alpha}) = -\alpha,$$

which also means that R_β is conjugate to R_α.

Approximate conjugacy for $X = S^1 \times Y$, where Y is the Cantor set, will be discussed in a subsequent paper.

References

[CE] Choi, M.; Effros, E.; *The completely positive lifting problem for C^*-algebras*, Ann. of Math. (2) 104 (1976), 585–609.

[D] Davidson, K.; *C^*-algebras by example*, Fields Institute Monographs, 6. Amer. Math. Soc., Providence, RI, 1996.

[El] Elliott, G. A.; *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. 443 (1993), 179–219.

[GPS] Giordano, T.; Putnam, I. F.; Skau, C. F.; *Topological orbit equivalence and C^*-crossed products*, J. reine angew. Math. 469 (1995), 51–111.

[GW] Glasner, E.; Weiss, B.; *Weak orbit equivalence of Cantor minimal systems*, Internat. J. Math. 6 (1995), 559–579.

[HPS] Herman, R. H.; Putnam, I. F.; Skau, C. F.; *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. 3 (1992), 827–864.

[KH] Katok, A.; Hasselblatt, B.; *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its Applications, 54. Cambridge University Press, Cambridge, 1995.

[Ln1] Lin, H.; *Skeleton C^*-subalgebras*, Canad. J. Math. 44 (1992), 324–341.

[Ln2] Lin, H.; *Tracially AF C^*-algebras*, Trans. Amer. Math. Soc. 353 (2001), 693–722.

[Ln3] Lin, H.; *Classification of simple tracially AF C^*-algebras*, Canad. J. Math., 53 (2001), 161–194.

[Ln4] Lin, H.; *Classification of simple C^*-algebras and higher dimensional noncommutative tori*, Ann. of Math. (2) 157 (2003), 521–544.

[Ln5] Lin, H.; *Tracially equivalences for C^*-algebras and orbit equivalences for minimal dynamical systems*, preprint.

[M] Matui, H.; *Approximate conjugacy and full groups of Cantor minimal systems*, Publ. Res. Inst. Math. Sci. 41 (2005), 695–722. math.DS/0404224

[NT] Nielsen, K. E.; Thomsen, K.; *Limits of circle algebras*, Exposition. Math. 14 (1996), 17–56.

[P] Paulsen, V. I.; *Completely bounded maps and dilations*, Pitman Research Notes in Mathematics Series, 146. Longman Scientific & Technical, Harlow; John Wiley & Sons, Inc., New York.

[PY] Pollicott, M.; Yuri, M.; *Dynamical systems and ergodic theory*, London Mathematical Society Student Texts, 40. Cambridge University Press, Cambridge, 1998.

[T] Tomiyama, J.; *Invitation to C^*-algebras and Topological dynamics*, World Scientific, 1987, New Jersey/London/Singapore/Hong Kong/Bangalore.

[T2] Tomiyama, J.; *Topological full groups and structure of normalizers in transformation group C^*-algebras*, Pacific J. Math. 173 (1996), 571–583.