Topological orbit equivalence of locally compact Cantor minimal systems *

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Abstract

Minimal homeomorphisms on the locally compact Cantor set are investigated. We prove that scaled dimension groups modulo infinitesimal subgroups determine topological orbit equivalence classes of locally compact Cantor minimal systems. We also introduce several full groups and show that they are complete invariants for orbit equivalence, strong orbit equivalence and flip conjugacy. These are locally compact version of the famous results for Cantor minimal systems obtained by Giordano, Putnam and Skau. Moreover, proper homomorphisms and skew product extensions of locally compact Cantor minimal systems are examined and it is shown that every finite group can be embedded into the group of centralizers trivially acting on the dimension group.

1 Introduction

In this paper, we study minimal homeomorphisms on the locally compact Cantor set. Minimal homeomorphisms on the Cantor set have been studied by several authors by using Bratteli diagrams and dimension groups. The Cantor set is characterized as the topological space which is compact, metrizable, perfect and totally disconnected. We say a topological space X is the locally compact Cantor set if X is homeomorphic to a non-closed open subset of the Cantor set. This is equivalent to saying that X is homeomorphic to the product space of the Cantor set and a countable infinite discrete space. A homeomorphism ϕ from a topological space X to itself is said to be minimal if every orbit $Orb_\phi(x) = \{\phi^n(x); n \in \mathbb{Z}\}$ is dense in X. If X is the Cantor set (resp. the locally compact Cantor set) and ϕ is minimal, we call the pair (X, ϕ) a Cantor minimal system (resp. locally compact Cantor minimal system) or CM system (resp. LCCM system), shortly. When (X, ϕ) is an LCCM system, we write the one-point compactification of (X, ϕ) by $(X \cup \{\infty_X\}, \phi)$. Note that $(X \cup \{\infty_X\}, \phi)$ is not a minimal system but an essentially minimal system in the sense of [HPS, Definition 1.2].

One of our interests is the orbit equivalence of LCCM systems. Let $(X_i, \phi_i), i = 1, 2$ be two topological dynamical systems. If there exists a homeomorphism $F : X_1 \to X_2$ such that $F(Orb_{\phi_1}(x)) = Orb_{\phi_2}(F(x))$ for each $x \in X_1$, these two systems are said to be orbit equivalent. When F gives an orbit equivalence and the systems are aperiodic, the orbit cocycles $n : X_1 \to \mathbb{Z}$ and $m : X_2 \to \mathbb{Z}$ are uniquely determined by, for each $x \in X_1$ and $y \in X_2$,

$$F(\phi_1(x)) = \phi_2^{n(x)}(F(x)),\ F^{-1}(\phi_2(y)) = \phi_1^{m(y)}(F^{-1}(y)).$$

If $(X_i, \phi_i), i = 1, 2$ are CM systems and each of the orbit cocycles has at most one point of discontinuity, these two systems are said to be strong orbit equivalent ([GPS1]). Giordano,

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Putnam and Skau showed that two CM systems are strong orbit equivalent if and only if their associated unital dimension groups are isomorphic to each other in [GPS1]. Danilenko proved the analogous result for LCCM systems in [D]. He defined strong orbit equivalence for LCCM systems and showed that their associated dimension groups are complete invariants of strong orbit equivalence. In this paper, we continue his work and show that various techniques for CM systems can be applied to the locally compact case.

In the case of CM systems the associated dimension group was unital. Indeed, the class of the constant function one was the order unit of the dimension group. But, this is not true for LCCM systems, since the space is non-compact. We have to consider scales of dimension groups instead of order units. Scales were introduced by Elliott to classify non-unital AF-algebras. By using the idea of scales, we can describe σ-finite or finite invariant measures of LCCM systems completely. This is one of the differences between LCCM systems and CM systems. The other difference from the compact case is the fact that LCCM systems have the "bad recurrence" property. In general, topological dynamical systems on non-compact locally compact spaces admit a point whose positive semi-orbit has no accumulation point. Therefore, first return maps can not be defined for compact open sets even if the system is minimal. The "bad recurrence" property, however, is not a bad property. In contrast to the compact case, for LCCM systems, two points which lie in the same orbit always have the same tail in the Bratteli diagram. Moreover every homeomorphism in the topological full group (see Section 4 for the definition) supported on a compact open subset is of finite order. These phenomena are, roughly speaking, due to the fact that LCCM systems have a weaker recurrence property than CM systems.

Now we give an overview of each section below. In Section 2, we recall basic definitions concerning Bratteli diagrams and dimension groups. For an LCCM system (X, ϕ), we denote the C^*-crossed product $C_0(X) \rtimes_\phi \mathbb{Z}$ by $C^*(X, \phi)$. AF-subalgebras of the crossed product C^*-algebras $C^*(X, \phi)$ associated with closed subsets will be introduced, and they play an important role in the argument of the next section. In Section 3, by using the homological algebra theory, we prove that scaled dimension groups modulo infinitesimal subgroups are complete invariants for orbit equivalence of LCCM systems. Our proof is essentially the same as the compact case. Analogue of the results in [GPS2] will be proved for the locally compact system in Section 4. We will introduce seven kinds of (topological) full groups and show that they are complete invariants for orbit equivalence, strong orbit equivalence and flip conjugacy, respectively. It is also proved that their normalizers induce automorphisms of C^*-algebras preserving $C_0(X)$. In Section 5, injective homomorphisms from LCCM systems to CM or LCCM systems are considered. (We use the terminology 'homomorphism' instead of 'factor map' as in [A], because we will sometimes consider non-surjective maps.) This type of homomorphisms is described by embedding of Bratteli diagrams. The strong orbit realization theorem is proved in the locally compact case for ergodic automorphisms on a non-atomic Lebesgue probability space. Section 6 is devoted to the study of homomorphisms and extensions. At first, we will show that proper homomorphisms induce scale preserving order embedding of dimension groups. By considering skew product extensions of LCCM systems associated with finite group valued cocycles, we will prove that every finite group can be included in the group of centralizers acting trivially on the dimension group. This result contrasts with the fact that CM systems admit only cyclic groups as finite subgroups of those centralizers. It is also proved that skew product extensions over non locally finite groups are never minimal.

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2 Bratteli diagrams and dimension groups

In order to describe LCCM systems, we have to introduce almost simple properly ordered Bratteli diagrams. The following definitions can be found in [HPS] and [D].

Definition 2.1. *We say $B = (V, E)$ is a Bratteli diagram, when $V = \bigcup_{n=0}^{\infty} V_n$ and $E = \bigcup_{n=1}^{\infty} E_n$ are disjoint union of finite sets of vertices and edges with source maps $s : E_n \to V_{n-1}$ and range maps $r : E_n \to V_n$ which are both surjective. We always assume V_0 consists of one point v_0 and call it the top vertex. For a Bratteli diagram $B = (V, E)$,*

$$X_B = \{(e_n)_n \ ; \ e_n \in E_n, r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{N}\}$$

is called the infinite path space of B. The space X_B is endowed with the natural product topology. For a finite path p from v_0 to $v \in V_n$, $U(p)$ denotes the clopen set of X_B which consists of all infinite paths whose initial n edges agree with p. We denote the number of paths from v_0 to v by $h_B(v)$.

Simplicity and almost simplicity of Bratteli diagrams are defined in the following way.

Definition 2.2. *Let $B = (V, E)$ be a Bratteli diagram.*

(i) *We say $B = (V, E)$ is simple, when for every $v \in V_n$ there exists $m > n$ such that all vertices of V_m are connected to v.*

(ii) *Let us assume there is an infinite path $(e_n)_n \in X_B$ such that $r^{-1}(r(e_n)) = \{e_n\}$ for all $n \in \mathbb{N}$. The diagram B is said to be almost simple, when for every $v \in V_n \setminus \{r(e_n)\}$ and $v' \in V_n$ there exists $m > n$ such that all vertices of V_m which are connected to v are also connected to v'. We denote the infinite path $(e_n)_n$ by ∞_B.*

We need the idea of ordered Bratteli diagrams to introduce the Bratteli-Vershik systems.

Definition 2.3. *Let $B = (V, E)$ be a Bratteli diagram.*

(i) *When the finite set $r^{-1}(v)$ has a linear order for each $v \in V$, we call B an ordered Bratteli diagram. We define the set of maximal infinite paths and minimal infinite paths by*

$$X_B^{max} = \{(e_n)_n \in X_B \ ; \ e_n \text{ is maximum in } r^{-1}(r(e_n))\}$$

and

$$X_B^{min} = \{(e_n)_n \in X_B \ ; \ e_n \text{ is minimum in } r^{-1}(r(e_n))\}.$$

(ii) *A simple ordered Bratteli diagram $B = (V, E)$ is said to be properly ordered, when each of X_B^{max} and X_B^{min} consists of one point.*

(iii) *An almost simple ordered Bratteli diagram $B = (V, E)$ is said to be properly ordered, when $X_B^{max} = X_B^{min} = \{\infty_B\}$.*

For simple or almost simple properly ordered Bratteli diagram B, we denote the Bratteli-Vershik system of B by (X_B, ϕ_B). In the case of almost simple diagrams, ∞_B is the unique fixed point of ϕ_B.

We refer to [HPS] or [D] for the definition of ϕ_B.

The following theorem is the model theorem of CM systems and LCCM systems.

Theorem 2.4 ([HPS, Theorem 4.6][D, Theorem 3.3]). *When (X, ϕ) is a CM (resp. LCCM) system, there exists a simple (resp. almost simple) properly ordered Bratteli diagram $B = (V, E)$ such that (X, ϕ) (resp. $(X \cup \{\infty_X\}, \phi)$) is isomorphic to (X_B, ϕ_B).*

Remark 2.5. In the locally compact case, we may always assume the following condition for $B = (V, E)$; for every $n \in \mathbb{N}$, $r(s^{-1}(V_n \setminus \{r(e_n)\}))$ equals $V_{n+1} \setminus \{r(e_{n+1})\}$, where $(e_n)_n$ is the infinite path ∞_B. For example, when we set a decreasing sequence $\{U_n\}_n$ of clopen neighborhoods of ∞_X by $U_{n+1} = \phi^{-1}(U_n) \cap U_n \cap \phi(U_n)$ inductively, the corresponding Bratteli diagram B satisfies this condition.

We have to review basic facts about dimension groups. An unperforated ordered group (G, G^+) is called a dimension group, if it satisfies the Riesz interpolation property. When a non-zero positive element $u \in G^+$ is specified, the triple (G, G^+, u) is called a unital dimension group. A Bratteli diagram $B = (V, E)$ determines a unital dimension group. We denote it by $(K_0(V, E), K_0(V, E)^+, 1_B)$. Of course, if B is simple, its dimension group becomes simple. The reader may refer to Chapter 3 and 4 of [Ef] for more details of dimension groups.

Definition 2.6 ([Ef, Chapter 7]). *Let (G, G^+) be a dimension group. A subset $\Sigma \subset G^+$ is called a scale, if the following properties are satisfied;*

(i) *For each $a \in G^+$ there exist $a_1, \dots, a_n \in \Sigma$ with $a = a_1 + \cdots + a_n$.*

(ii) *If $0 \le a \le b \in \Sigma$, then $a \in \Sigma$.*

(iii) *Given $a, b \in \Sigma$, there exists $c \in \Sigma$ with $a, b \le c$.*

When a scale Σ is fixed, we call the triple (G, G^+, Σ) a scaled dimension group. Moreover, if Σ has no maximal element, the scaled dimension group is said to be non-unital. An order homomorphism π from G to another scaled dimension group is called an order contraction, if $\pi(\Sigma)$ is contained in the scale.

We remark that the condition (i) above can be omitted in the case of simple dimension groups.

For a non-unital scaled simple dimension group (G, G^+, Σ), let us consider $G^1 = G \oplus \mathbb{Z}$ and define a positive cone by

$$G^{1+} = \{(a, n) \in G^1 \ ; \ \text{there exists } b \in \Sigma \text{ with } a + nb \ge 0\}.$$

It is easy to see that $(G^1, G^{1+}, (0, 1))$ is a unital dimension group with a unique proper order ideal. We call it the unitization of G.

Definition 2.7 ([D, Definition 3.3]). *Let (G, G^+, u) be a unital dimension group which has a unique proper order ideal J with $G/J \cong \mathbb{Z}$. When u maps to a generator of $\mathbb{Z}$ by the isomorphism $G/J \cong \mathbb{Z}$ and there is no maximal element in*

$$\Sigma = \{a \in J^+ \ ; \ 0 \le a \le u\},$$

G is said to be almost simple.

In the above definition, we can check that Σ is a scale of J (the condition (iii) of Definition 2.6 is proved by applying the Riesz interpolation property for $a + b$ and u). Hence, there exists a bijective correspondence between non-unital scaled simple dimension groups and almost simple dimension groups. The next lemma is easily shown.

Lemma 2.8. *When $B = (V, E)$ is an almost simple Bratteli diagram, the dimension group associated with B is almost simple. Conversely, if the associated dimension group is almost simple, the Bratteli diagram is almost simple.*

Let $B = (V, E)$ be an almost simple Bratteli diagram with $\infty_B = (e_n)_n$. When B satisfies the condition of Remark 2.5, we define a subdiagram $\widetilde{B} = (\widetilde{V}, \widetilde{E})$ of B as follows:

$$\widetilde{V}_0 = \{v_0\},\ \widetilde{V}_n = V_n \setminus \{r(e_n)\},\quad \widetilde{V} = \bigcup_n \widetilde{V}_n$$

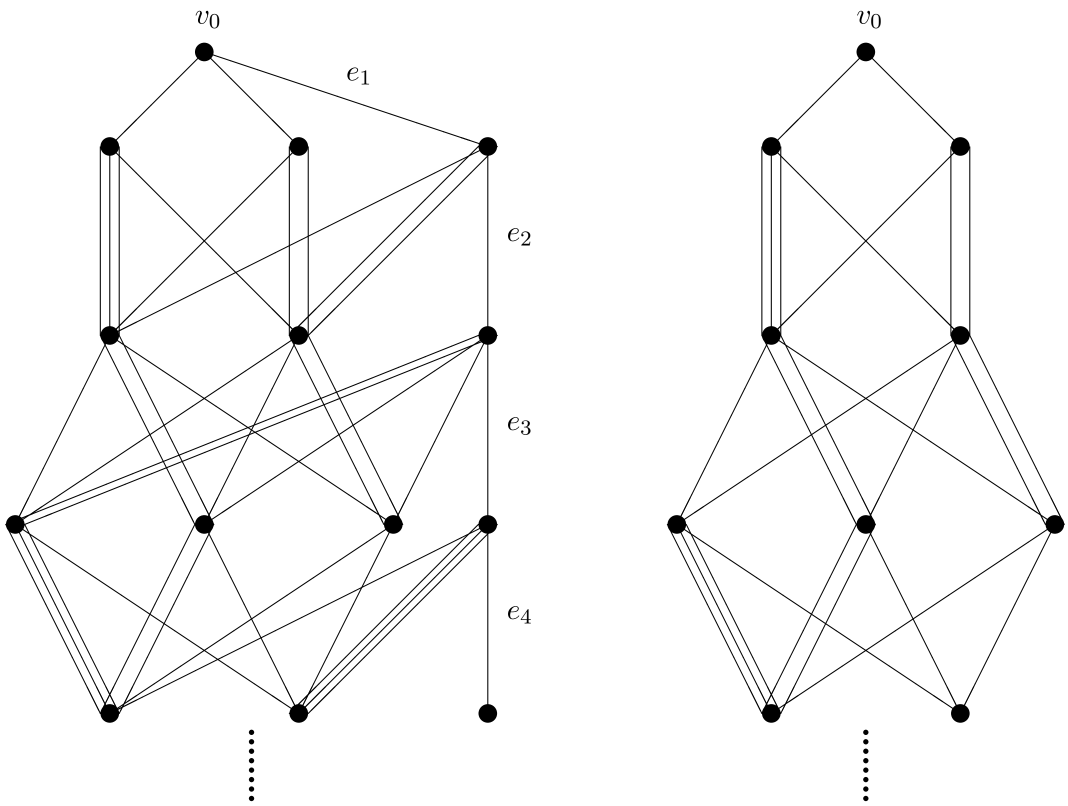
and

$$\widetilde{E}_n = E_n \setminus s^{-1}(s(e_n)),\quad \widetilde{E} = \bigcup_n \widetilde{E}_n.$$

Then, $\widetilde{B}$ becomes a simple Bratteli diagram and the unique proper order ideal of $K_0(V, E)$ is canonically order isomorphic to $K_0(\widetilde{V}, \widetilde{E})$. When we put $u_n = (h_B(v))_v \in \mathbb{Z}^{\widetilde{V}_n}$, the scale is equal to the set of elements which is smaller than v_n for some n.

Definition 2.9. *In the above setting, we call the subdiagram $\widetilde{B}$ the ideal part of B.*

We give an example of an almost simple Bratteli diagram $B = (V, E)$ and its ideal part $\tilde{B} = (\tilde{V}, \tilde{E})$ below.



$B = (V, E)$ $\qquad\qquad$ $\tilde{B} = (\tilde{V}, \tilde{E})$

For a unital dimension group (G, G^+, u),

$$\{\rho \in \mathrm{Hom}(G, \mathbb{R})\ ;\ \rho(u) = 1 \text{ and } \rho(a) \geq 0 \text{ for all } a \in G^+\}$$

is called the state space of G and we denote it by $S_u(G)$. The topology of $S_u(G)$ is induced from the product topology of $\mathbb{R}^G$. For a scaled dimension group (G, G^+, Σ), we write the set of all positive homomorphisms from G to $\mathbb{R}$ by $S(G)$ and call it the quasi-state space. Set

$$S_\Sigma(G) = \{\rho \in S(G)\ ;\ \sup\{\rho(a); a \in \Sigma\} = 1\}.$$

The topology of $S(G)$ and $S_\Sigma(G)$ is induced from $\mathbb{R}^G$, too. In general, $S_\Sigma(G)$ may be empty or non-closed.

Lemma 2.10. *Let (G, G^+, Σ) be a non-unital scaled simple dimension group.*

(i) *$S_\Sigma(G)$ is empty if and only if $\Sigma = G^+$.*

(ii) *For a positive a, $a \in \Sigma$ if and only if $\rho(a) < 1$ for all $\rho \in S_\Sigma(G)$.*

Proof. For (i), we prove the 'only if' part. Take $a \in G^+ \setminus \Sigma$. Let $\{v_n\}_n \subset \Sigma$ be an increasing approximate unit, that is, $v_n \leq v_{n+1}$ and for every $v \in \Sigma$ there is v_n greater than v. Set

$$E_n = \{\rho \in S_a(G)\ ;\ \rho(v_n) \leq \rho(a) = 1\},$$

which is closed and non-empty because a is not in Σ. The compactness of $S_a(G)$ leads the conclusion.

The proof of (ii) is similar. □

Let (X, ϕ) be an LCCM system. We denote by $C_0(X, \mathbb{Z})$ the set of all continuous functions from X to $\mathbb{Z}$ with compact supports. Obviously, $C_0(X, \mathbb{Z})$ is a countable abelian group. Define the coboundary subgroup B_ϕ such as

$$B_\phi = \{f - f \circ \phi^{-1}\ ;\ f \in C_0(X, \mathbb{Z})\},$$

and set $K^0(X, \phi) = C_0(X, \mathbb{Z})/B_\phi$. We write the quotient map by $[\cdot]$. Moreover, we put

$$K^0(X, \phi)^+ = \{[f] \in K^0(X, \phi)\ ;\ f \geq 0\}$$

and

$$\Sigma(X, \phi) = \{[f] \in K^0(X, \phi)\ ;\ 0 \leq f \leq 1\}.$$

Theorem 2.11 ([HPS, Theorem 5.4][Po, Theorem 2.3] [D, Theorem 3.3]). *Let (X, ϕ) be an LCCM system and $B = (V, E)$ be an almost simple properly ordered Bratteli diagram such that $(X \cup \{\infty_X\}, \phi)$ is isomorphic to (X_B, ϕ_B). Then, the triple $(K^0(X, \phi), K^0(X, \phi)^+, \Sigma(X, \phi))$ becomes a non-unital scaled simple dimension group and its unitization is isomorphic to the unital dimension group $(K_0(V, E), K_0(V, E)^+, 1_B)$. The C^*-crossed product $C_0(X) \rtimes_\phi \mathbb{Z}$ is a non-unital simple AF-algebra and its unitization is given by the diagram B.*

Since we can always define a proper order on a given almost simple Bratteli diagram, every non-unital scaled simple dimension group except for $\mathbb{Z}$ can be realized as the associated dimension group of an LCCM system.

We write $C^*(X, \phi) = C_0(X) \rtimes_\phi \mathbb{Z}$ and its unitization by $C^*(X, \phi)^1$. It is clear that the dimension group $K_0(C^*(X, \phi)^1) \cong K_0(V, E)$ is isomorphic to the quotient of $C(X \cup \{\infty\}, \mathbb{Z})$ by the coboundary subgroup B_ϕ.

Let X be the compact or locally compact Cantor set and $\mathcal{B}(X)$ be the Borel field of X generated by all compact subsets. When μ is a positive measure on $\mathcal{B}(X)$ and $\mu(K)$ is finite for every compact subset K, we call μ a Borel measure on X. We denote by $M(X)$ the set of all Borel measures on X. The space $M(X)$ is endowed with the so-called weak-$*$ topology, that is, we say μ_n converges to μ if $\lim \mu_n(U) = \mu(U)$ holds for every compact open set U.

If X is the locally compact Cantor set, we can write X as a countable disjoint union of the Cantor sets X_n, $n \in \mathbb{N}$. For $\mu \in M(X)$, the restriction of μ on each X_n gives an element of $M(X_n)$. Hence, we can identify $M(X)$ with the product space of $M(X_n)$'s. Note that every $\nu \in M(X_n)$ is automatically regular, because X_n is metrizable. For the Cantor set X_n, it is well-known that there exists a canonical bijective correspondence between $M(X_n)$ and the set of positive homomorphisms from $C(X_n, \mathbb{Z})$ to $\mathbb{R}$. Consequently we have the following.

Lemma 2.12. *When X is the locally compact Cantor set, the integration by $\mu \in M(X)$ gives a positive homomorphism ρ_μ from $C_0(X, \mathbb{Z})$ to $\mathbb{R}$, and this map $\mu \mapsto \rho_\mu$ induces a one-to-one correspondence between $M(X)$ and the set of positive homomorphisms from $C_0(X, \mathbb{Z})$ to $\mathbb{R}$.*

Let (X, ϕ) be a CM or LCCM system. We say $\mu \in M(X)$ is ϕ-invariant, if $\mu(E) = \mu(\phi(E))$ holds for every $E \in \mathcal{B}(X)$. Obviously, μ is ϕ-invariant if and only if $\mu(U) = \mu(\phi(U))$ for every compact open set U. We denote by M_ϕ the set of invariant measures and M_ϕ^1 be the set of invariant probability measures. For each $\mu \in M_\phi$, the integration by μ gives a positive homomorphism $\rho_\mu : K^0(X, \phi) \to \mathbb{R}$ evidently. In the compact case, M_ϕ^1 can be identified with $S_{[1]}(K^0(X, \phi))$ by this correspondence ([HPS, Theorem 5.5]). In the locally compact case, we have the following from the above lemma and Theorem 2.11.

Proposition 2.13. *When (X, ϕ) is an LCCM system, $\mu \mapsto \rho_\mu$ gives a homeomorphism from M_ϕ to $S(K^0(X, \phi))$. Moreover, μ is a probability measure if and only if ρ_μ is in $S_{\Sigma(X,\phi)}(K^0(X, \phi))$.*

In the rest of this paper, we will identify M_ϕ with $S(K^0(X, \phi))$. From the definition of the topology of $S(K^0(X, \phi))$, we have μ_n goes to μ in M_ϕ if and only if $\mu_n(U)$ goes to $\mu(U)$ for every compact open subset U.

As pointed out in [D], positive or negative semi-orbits may not be dense in X in the case of locally compact systems. For an LCCM system (X, ϕ), we denote the set of points whose positive (resp. negative) semi-orbit is dense in X by X_+ (resp. X_-). The subsets X_+, X_- and $X_+ \cap X_-$ are dense in X. One can show the following easily.

Lemma 2.14. *When an LCCM system (X, ϕ) is represented by an almost simple properly ordered Bratteli diagram $B = (V, E)$ and B satisfies the condition of Remark 2.5, an infinite path $(f_n)_n \in X_B$ is in X_+ (resp. X_-) if and only if for infinitely many n there is $f \in E_n$ such that $s(f) \neq s(e_n)$ and $f_n < f$ (resp. $f < f_n$), where $\infty_B = (e_n)_n$.*

Let us introduce the notion of first return maps for compact open subsets. Let (X, ϕ) be an LCCM system and $U \subset X$ a compact open subset. It is evident that

$$U_0 = \{x \in U \ ; \ \exists n \in \mathbb{N} \text{ such that } \phi^n(x) \in U\}$$

is open and dense in U. For $x \in U_0$

$$r(x) = \min\{n \in \mathbb{N} \ ; \ \phi^n(x) \in U\}$$

is well-defined, and so $\phi_U(x) = \phi^{r(x)}(x)$ gives a homeomorphism from $U_0 \subset U$ to $\phi_U(U_0) \subset U$. We call ϕ_U the first return map for U. The first return map ϕ_U is a partial homeomorphism on U and $\mu(\phi_U(V)) = \mu(V)$ for every compact open set $V \subset U_0$ and $\mu \in M_\phi$. The $*$-isomorphism $\alpha_U : C_0(U_0) \to C_0(\phi_U(U_0))$ induced by ϕ_U is a partial automorphism on the commutative C^*-algebra $C(U)$ in the sense of [Ex]. The covariance C^*-algebra associated with $(C(U), \alpha_U)$ is clearly isomorphic to $pC^*(X, \phi)p$, where p is the characteristic function of U.

By using the first return map we can prove the following.

Lemma 2.15. *Let (X, ϕ) be an LCCM system.*

(i) *For all* $\mu \in M_\phi$ *we have* $\mu(X_+^c) = \mu(X_-^c) = 0$.

(ii) *If* $V \in \mathcal{B}(X)$ *satisfies* $V \cap \phi^n(V) = \emptyset$ *for all* $n \neq 0$*, then* $\mu(V) = 0$ *for every* $\mu \in M_\phi$.

Proof. (i) It is sufficient to prove the assertion only for X_-^c. Take a compact open subset U and let $\phi_U : U_0 \to \phi_U(U_0)$ be the first return map. The subset

$$V = \{x \in U \ ; \ \phi^{-n}(x) \notin U \text{ for all } n \in \mathbb{N}\}$$

is compact and $\phi_U^n(V)$ is contained in U_0 for every $n \geq 0$. Since $\phi_U^n(V)$'s are mutually disjoint and every compact set has finite measure, $\mu(V)$ must be zero for all $\mu \in M_\phi$. Now we get the conclusion from $X_-^c = \bigcup_{n\in\mathbb{Z}} \phi^n(V)$.

(ii) By (i) we may assume $V \subset X_-$. We may also assume that V is contained in a compact open subset U. When $\phi_U : U_0 \to \phi_U(U_0)$ is the first return map for U, it is easy to see that $V, \phi_U(V), \phi_U^2(V), \ldots \subset U_0$ are well-defined and mutually disjoint. Hence V has zero measure for every invariant measure. □

For a closed subset $Y \subset X$, let us define a C^*-subalgebra A_Y of $C^*(X,\phi)$ by

$$A_Y = C^*(C_0(X), uC_0(X \setminus Y)),$$

where u means the implementing unitary (see [Pu, Section 3]). We denote the unitization of A_Y by A_Y^1. It is not hard to see that A_Y and A_Y^1 are AF-algebras. We can write down the Bratteli diagram $B = (V,E)$ of A_Y^1 in the same way as Lemma 5.1 of [GPS1]. The diagram B is ordered and we can define the Bratteli-Vershik map ϕ_B so that $(X \cup \{\infty_X\}, \phi)$ is conjugate to (X_B, ϕ_B) and $X_B^{max} \setminus \{\infty_B\}$ (resp. $X_B^{min} \setminus \{\infty_B\}$) corresponds to Y (resp. $\phi(Y)$). The K_0-group of A_Y is isomorphic to the quotient of $C_0(X,\mathbb{Z})$ by

$$B_{Y,\phi} = \{f - f \circ \phi^{-1} \ ; \ f \in C_0(X,\mathbb{Z}), f|Y = 0\}.$$

The positive cone $K_0(A_Y)^+$ and scale $\Sigma(A_Y)$ is described in the obvious fashion.

Lemma 2.16. *In the above setting, the following are equivalent.*

(i) A_Y *is simple.*

(ii) Y *is contained in* $X_+ \cap X_-$ *and* $Y \cap \phi^j(Y)$ *is empty for all* $j \neq 0$.

(iii) $B = (V,E)$ *is an almost simple Bratteli diagram.*

Proof. (i)⇒(ii). If the positive semi-orbit of $y \in Y$ is not dense in X, $Z = \{\phi^n(y); n > 0\}$ is a closed set of X. Set

$$I = C^*(C_0(X \setminus Z), uC_0(X \setminus (Y \cup Z))).$$

Then, I is a non-trivial ideal of A_Y and A_Y/I is isomorphic to the C^*-algebra of compact operators. The other cases are proved similarly.

(ii)⇒(iii). We can identify X with $X_B \setminus \{\infty_B\}$. By assumption, for every $x \in X_B \setminus \{\infty_B\}$, the set of infinite paths which have the same tail as x is dense in X_B. Therefore, one get the almost simplicity.

(iii)⇒(i). Because $K_0(A_Y)$ is isomorphic to the unique ideal of $K_0(V,E)$, the assertion is clear. □

The kernel of the natural homomorphism from $K_0(A_Y)$ to $K^0(X,\phi)$ is described by the following exact sequence.

Lemma 2.17. *Let (X,ϕ) be an LCCM system and $Y \subset X$ be closed. Then, the sequence*

$$0 \to C_0(Y,\mathbb{Z}) \xrightarrow{\delta} K_0(A_Y) \xrightarrow{q} K^0(X,\phi) \to 0$$

is exact, where the map q is the natural one. Moreover, for every $a \in K^0(X,\phi)^+$ there is $b \in K_0(A_Y)^+$ such that $q(b) = a$.

Proof. See [Pu, Theorem 4.1]. □

In order to prove the orbit equivalence theorem, we need the following lemma.

Lemma 2.18. *Let (X,ϕ) be an LCCM system and Y be a closed subset of X satisfying the condition* (ii) *of Lemma 2.16. Then, the natural homomorphism q from $K_0(A_Y)$ to $K^0(X,\phi)$ induces an isomorphism between their quasi-state spaces, that is, the image of δ is contained in the infinitesimal subgroup* $\mathrm{Inf}(K_0(A_Y))$*. Moreover we have $\Sigma(A_Y) \setminus \{0\} = q^{-1}(\Sigma(X,\phi) \setminus \{0\})$.*

Proof. Take a quasi-state on $K_0(A_Y)$. By Lemma 2.12, it comes from $\mu \in M(X)$ satisfying $\mu(f) = 0$ for all $f \in B_{Y,\phi}$. By a similar argument to the proof of Lemma 2.15(ii), we obtain $\mu(Y) = \mu(\phi(Y)) = 0$. On account of this, one concludes that μ is ϕ-invariant. See [Pu, Corollary 5.7] for details.

The inclusion $\Sigma(A_Y) \setminus \{0\} \subset q^{-1}(\Sigma(X,\phi) \setminus \{0\})$ is obvious. The other inclusion follows from $\mathrm{Im}(\delta) \subset \mathrm{Inf}(K_0(A_Y))$, since $\Sigma(A_Y) \setminus \{0\} + \mathrm{Inf}(K_0(A_Y)) = \Sigma(A_Y) \setminus \{0\}$. □

3 Topological orbit equivalence

In this section, we show the orbit equivalence theorem for LCCM systems. We begin by recalling the bounded orbit equivalence theorem and the strong orbit equivalence theorem. We say two homeomorphisms ϕ_1 and ϕ_2 are flip conjugate, if ϕ_1 is conjugate to ϕ_2 or ϕ_2^{-1}.

Theorem 3.1 ([BT, Corollary 2.7]). *Let (X_i,ϕ_i), $i = 1,2$ be two LCCM systems. The following are equivalent.*

(i) *There exists a homeomorphism $F : X_1 \to X_2$ which gives an orbit equivalence and one of the orbit cocycles is continuous and bounded.*

(ii) *(X_1,ϕ_1) and (X_2,ϕ_2) are flip conjugate.*

Two LCCM systems are said to be strong orbit equivalent if these systems are orbit equivalent and the associated orbit cocycles are continuous. The following is the main theorem of [D].

Theorem 3.2 ([D, Theorem 4.2]). *For LCCM systems (X_i,ϕ_i), $i = 1,2$, the following are equivalent.*

(i) *(X_1,ϕ_1) and (X_2,ϕ_2) are strong orbit equivalent.*

(ii) *The C^*-algebras $C^*(X_1,\phi_1)$ and $C^*(X_2,\phi_2)$ are isomorphic.*

(iii) *$(K^0(X_1,\phi_1), K^0(X_1,\phi_1)^+, \Sigma(X_1,\phi_1))$ and $(K^0(X_2,\phi_2), K^0(X_2,\phi_2)^+, \Sigma(X_2,\phi_2))$ are isomorphic as scaled dimension groups.*

Let (G, G^+, Σ) be a non-unital scaled simple dimension group and π be the quotient map from G to $G/\mathrm{Inf}(G)$. Then, $(\pi(G), \pi(G^+), \pi(\Sigma))$ is also a non-unital scaled simple dimension group. We denote it by $(G, G^+, \Sigma)/\mathrm{Inf}$, simply.

Theorem 3.3. *When (X_i, ϕ_i), $i = 1, 2$ are two LCCM systems, the following are equivalent.*

(i) *(X_1, ϕ_1) and (X_2, ϕ_2) are orbit equivalent.*

(ii) *There exists a homeomorphism $F : X_1 \to X_2$ which induces bijections from M_{ϕ_1} to M_{ϕ_2} and from $M^1_{\phi_1}$ to $M^1_{\phi_2}$.*

(iii) *$(K^0(X_i, \phi_i), K^0(X_i, \phi_i)^+, \Sigma(X_i, \phi_i))/\operatorname{Inf}$, $i = 1, 2$, are isomorphic as scaled dimension groups.*

We need a series of lemmas to prove the implication (iii)⇒(i).

Lemma 3.4. *Let (G_i, G_i^+, Σ_i), $i = 1, 2$ be two non-unital scaled simple acyclic dimension groups and assume $(G_i, G_i^+, \Sigma_i)/\operatorname{Inf}$, $i = 1, 2$ are isomorphic. Then, there exist a non-unital scaled simple dimension group (H, H^+, Σ), homomorphisms $\iota_i : \mathbb{Z}^\infty \to H$ and order contractions $\pi_i : H \to G_i$ such that*

$$0 \to \mathbb{Z}^\infty \xrightarrow{\iota_i} H \xrightarrow{\pi_i} G_i \to 0$$

is exact and

$$\operatorname{Im} \iota_i \subset \operatorname{Inf}(H), \quad \Sigma \setminus \{0\} = \pi_i^{-1}(\Sigma_i \setminus \{0\})$$

for $i = 1, 2$.

The proof is similar to that of Lemma 5.4 of [GPS1].

Let $B = (V, E)$ be an almost simple Bratteli diagram. As in Lemma 10.2 of [GPS1], the exact sequence

$$0 \to \mathbb{Z}^V \xrightarrow{\partial} \mathbb{Z}^V \longrightarrow K_0(V, E) \to 0$$

gives a projective resolution of $K_0(V, E)$, where $\partial(v) = v - \sum_{s(e)=v} r(e)$. Hence, every element of $\operatorname{Ext}(K_0(V, E), \mathbb{Z}^\infty)$ has a representative in $\operatorname{Hom}(\mathbb{Z}^V, \mathbb{Z}^\infty)$. We write the representative as a map ρ from $V \times \mathbb{N}$ to $\mathbb{Z}$. The next lemma is used to replace the map ρ to the appropriate one.

Lemma 3.5. *Let (G, G^+, Σ) be a non-unital scaled simple dimension group and $\mathcal{E}$ be an element of $\operatorname{Ext}(G^1, \mathbb{Z}^\infty)$. Then, there exist an almost simple Bratteli diagram $B = (V, E)$, a map $\rho : V \times \mathbb{N} \to \mathbb{Z}$ and a strictly increasing sequence of natural numbers $\{r_n\}_n$, which satisfy the following.*

- *The diagram B satisfies the property of Remark 2.5 and $K_0(V, E) \cong G^1$ as a unital dimension group.*
- *The map ρ is a representative of $\mathcal{E}$.*

Put $\infty_B = (e_n)_n, v_n = r(e_n)$ and denote the number of edges from v to v' by $E(v, v')$.

- *For every $n \in \mathbb{N}$, V_n contains two vertices v_n and u_n, and $V_{n+1} \setminus \{v_{n+1}, u_{n+1}\}$ contains r_n vertices $v_1^{(n+1)}, v_2^{(n+1)}, \ldots, v_{r_n}^{(n+1)}$.*
- *$\rho(v, i) = 0$ for every $v \in V_n$ and $i > r_n$.*
- *For every $v \in V_n \setminus \{v_n, u_n\}$ and $i \leq r_n$, $0 \leq \rho(v, i) \leq E(v, v_i^{(n+1)}) - 1$.*
- *For every $i \leq r_n$, $1 \leq \rho(v_n, i) \leq E(v_n, v_i^{(n+1)}) - 2$.*
- *For every $i \leq r_n$, $1 \leq \rho(u_n, i) \leq E(u_n, v_i^{(n+1)}) - 1$.*

By using the above lemma, we can define an appropriate order on the Bratteli diagram.

Lemma 3.6. *Let $B = (V, E)$ be an almost simple Bratteli diagram and assume that B, $\rho : V \times \mathbb{N} \to \mathbb{Z}$ and $\{r_n\}_n$ satisfy the condition of Lemma 3.5. Then, we can define a linear order on each $r^{-1}(v)$ so that the following hold.*

- *$B = (V, E)$ is properly ordered.*
- *There exists a closed subset $Y = \{y_i\}_{i \in \mathbb{N}} \subset X_B \setminus \{\infty_B\}$ satisfying the condition* (ii) *of Lemma 2.16.*
- *For every $v \in V_n$ and $i \in \mathbb{N}$,*

$$\rho(v, i) = \#\{e \in E_{n+1} \ ; \ s(e) = v, \ y_i(n+1) < e\}.$$

Proof. The proof goes in a similar way as the compact case. The last condition of Lemma 3.5 is needed to arrange the linear order on $r^{-1}(v_i^{(n+1)})$ in such a way that there exist $e, e' \in r^{-1}(v_i^{(n+1)})$ with $e < y_i(n+1) < e'$ and $s(e) = s(e') = u_n$. By Lemma 2.14, the positive and negative semi-orbits of y_i are dense in X. $\square$

Now we are ready to prove the theorem.

Proof of Theorem 3.3. The proof of (i)$\Rightarrow$(ii) is straightforward. From Lemma 2.10 (ii) and Proposition 2.13, we can prove (ii)$\Rightarrow$(iii).

Let us show (iii)$\Rightarrow$(i). One can find (H, H^+, Σ), ι_i and π_i as in Lemma 3.4. By the two lemmas above and Lemma 10.3 of [GPS1], there exist almost simple properly ordered Bratteli diagrams $B_i = (V^{(i)}, E^{(i)})$ and closed countable subsets $Y_i \subset X_{B_i}$ such that the following exact sequences hold.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}^\infty \arrow[r, "\iota_i"] \arrow[d, equal] & H^1 \arrow[r, "\pi_i"] \arrow[d, "\eta_i", "\cong"'] & K^0(X_i, \phi_i)^1 \arrow[r] \arrow[d, "\gamma_i", "\cong"'] & 0 \\
0 \arrow[r] & \mathbb{Z}^\infty \arrow[r] & K_0(A_{Y_i})^1 \arrow[r, "q_i"] & K_0(V^{(i)}, E^{(i)}) \arrow[r] & 0
\end{tikzcd}$$

Since γ_i is a unital order isomorphism, (X_i, ϕ_i) and $(X_{B_i} \setminus \{\infty_{B_i}\}, \phi_{B_i})$ are strong orbit equivalent for $i = 1, 2$ by Theorem 3.2. By adjusting η_i by an element of $\mathbb{Z}^\infty$, we may assume $\eta_i(0, 1) = (0, 1)$. Thanks to Lemma 3.4 and 2.18, the map η_i is revealed to be a unital order isomorphism. Because [GPS1, Lemma 5.2] is valid for locally compact systems, we get the conclusion. $\square$

4 Full groups and their normalizers

The aim of this section is to show results analogous to those of [GPS2] in the locally compact setting. We adopt the notation of [GPS2].

Let X be the locally compact Cantor set and $\Gamma \subset \mathrm{Homeo}(X)$ be a subgroup. We say Γ is of class F when Γ has the following properties.

(Γ1) For compact open subsets U and V with $U \sim_\Gamma V$, there exists $\gamma \in \Gamma$ such that $\gamma(U) = V$, $\gamma^2 = 1$ and $\gamma|_{(U \cup V)^c} = 1$.

(Γ2) For a compact open set U and a clopen set V, there exists a clopen partition $U_1, \dots, U_n$ of U such that $U_i \prec_\Gamma V$ for every i.

(Γ3) There is no non-trivial Γ-invariant closed set.

(Γ4) If $\gamma(U) = U$ for a compact open U and $\gamma \in \Gamma$, there exists $\gamma_0 \in \Gamma$ with $\gamma_0|U = \gamma|U$ and $\gamma_0|U^c = 1$.

Several kinds of full groups, which are defined later, will satisfy the above properties.

Let us assume Γ is of class F. When $H \subset \Gamma$ is a subgroup, we write

$$H^{\perp} = \{\gamma \in \Gamma \ ; \ \gamma\eta = \eta\gamma \text{ for all } \eta \in H\}.$$

For an open set $O \subset X$, let Γ_O be the set of homomorphisms $\gamma \in \Gamma$ such that $\gamma(x) = x$ for each $x \in O^c$. In order to characterize a subgroup of Γ of the form Γ_U with a compact open U, we introduce the following conditions for a non-trivial subgroup $H \subset \Gamma$.

(D1) $H^{\perp\perp} = H$ and $H \cap H^{\perp} = \{1\}$.

(D2) If N is a non-trivial normal subgroup of H (resp. $H^{\perp}$), then $N^{\perp} = H^{\perp}$ (resp. $N^{\perp} = H$).

(D3) If H' is a subgroup which contains H properly and satisfies (D1) and (D2), then the subgroup of Γ generated by H' and $H^{\perp}$ is equal to Γ.

(D4) For all $\alpha \in \Gamma \setminus HH^{\perp}$, there exists $\eta \in H \setminus \{1\}$ such that $\alpha\eta\alpha^{-1} \in H^{\perp}$.

(D6) If a subgroup L of H satisfies (D1), (D2) and (D3), there exist $\eta_1, \eta_2, \ldots, \eta_n \in \Gamma$ such that

$$H \cap L^{\perp} \cap (\eta_1 L \eta_1^{-1})^{\perp} \cap \cdots \cap (\eta_n L \eta_n^{-1})^{\perp} = \{1\}.$$

Since (D1) and (D4) imply the condition (D5) of [GPS2] automatically, we omit it (see [GPS2, Lemma 3.26]). Although (D1), (D2) and (D4) are symmetric for H and $H^{\perp}$, (D3) and (D6) are not. The condition (D6) is new.

At first, we can prove the next two lemmas by exactly the same way as Proposition 3.13 and Lemma 3.21 of [GPS2].

Lemma 4.1. *If O is a non-trivial regular open set of X, then Γ_O satisfies* (D1) *and* (D2)*.*

Lemma 4.2. *Let O be a non-trivial regular open set. If, for any subgroup H which contains Γ_O properly and satisfies* (D1) *and* (D2)*, the subgroup of Γ generated by H and $\Gamma_O^{\perp}$ is equal to Γ, then O is clopen.*

The next lemma corresponds to [GPS2, Lemma 3.23].

Lemma 4.3. *Let $H \subset \Gamma$ be a proper subgroup satisfying the conditions* (D1), (D2) *and* (D3)*. Then, either of the following holds.*

(i) *P_H is clopen and $H = \Gamma_{P_H}$.*

(ii) *$P_H = P_{H^{\perp}} = X$.*

Proof. Set $O = (P_H)^o$. Then O is a non-empty regular open set and $H \subset \Gamma_O$. If H equals Γ_O, the conclusion (i) follows because of the above lemma and (D3).

Assume H is a proper subgroup of Γ_O. The condition (D3) implies that Γ is generated by Γ_O and $H^{\perp}$. Since P_H is Γ_O and $H^{\perp}$-invariant, we have $P_H = X$. Put $U = (P_{H^{\perp}})^o$. As $H^{\perp} \subset \Gamma_U$, we get $\Gamma_{U^{\perp}} \subset H$. For $U^{\perp} = (P_{H^{\perp}})^c$ is H-invariant, $\Gamma_{U^{\perp}}$ is a normal subgroup of H. If $U^{\perp}$ is not empty, by (D2), $\Gamma_{U^{\perp}}$ is equal to H. Because $P_H = X$, $U^{\perp}$ must be the whole space X and it contradicts the assumption on H. Thereby we verify (ii). □

We need to deny the possibility of (ii) of the lemma above. The proof of the following lemma is similar to [GPS2, Lemma 3.27].

Lemma 4.4. *Let $H \subset \Gamma$ be a subgroup satisfying* (D1), (D2) *and* (D4)*. When $P_H = P_{H^{\perp}} = X$ and a non-empty open set O is H or $H^{\perp}$-invariant, then O is dense in X.*

We use the non-compactness of X in the next lemma.

Lemma 4.5. *Let $H \subset \Gamma$ be a subgroup satisfying* (D1), (D2) *and* (D4). *When $P_H = P_{H^\perp} = X$, the complement of $O = \{x \in X; \eta(x) = \kappa(x)\}^o$ is not compact for $\eta \in H \setminus \{1\}$ and $\kappa \in H^\perp$.*

Proof. Take $\gamma \in H$ such that $\gamma\eta\gamma^{-1} \neq \eta$. Since γ commutes with κ

$$\gamma(O) = \{x \in X \ ; \ \gamma\eta\gamma^{-1}(x) = \kappa(x)\}^o.$$

If O^c is compact, $O \cap \gamma(O)$ is not empty. Hence,

$$U = \{x \in X \ ; \ \eta(x) = \gamma\eta\gamma^{-1}(x)\}^o$$

is not empty and $H^\perp$-invariant. By the above lemma, U is dense in X, and so we have a contradiction. $\square$

Lemma 4.6. *Let $H \subset \Gamma$ be a subgroup satisfying* (D1), (D2), (D3) *and* (D4). *Then P_H is clopen and $H = \Gamma_{P_H}$.*

Proof. Let us assume the case (ii) of Lemma 4.3. Take $\gamma \in \Gamma \setminus \{1\}$ such that the complement of the γ-fixed point set $X^\gamma = \{x \in X; \gamma(x) = x\}$ is compact. If $\gamma \in HH^\perp$, the above lemma implies $\gamma \in H^\perp$. Therefore, X^γ is H-invariant and we get a contradiction from Lemma 4.4. So, γ is not in $HH^\perp$. By (D4), there is $\eta \in H \setminus \{1\}$ with $\gamma\eta\gamma^{-1} \in H^\perp$. But, for $x \in X^\gamma \cap \eta^{-1}(X^\gamma)$, we have $\eta(x) = \gamma\eta\gamma^{-1}(x)$. By using the above lemma again, we obtain a contradiction. $\square$

One can show the following as in the compact case.

Lemma 4.7. *When O is a compact open non-empty subset, then Γ_O satisfies* (D3) *and* (D4).

The following proposition gives the algebraic characterization of local subgroups.

Proposition 4.8. *The maps $O \mapsto \Gamma_O$ and $H \mapsto P_H$ induce a bijective correspondence between the set of non-empty compact open subsets of X and the set of subgroups of Γ satisfying* (D1), (D2), (D3), (D4) *and* (D6).

Proof. For a clopen set O, it is easy to see that Γ_O satisfies (D6) if and only if O is compact. By virtue of Lemma 4.1, 4.6 and 4.7, we get the conclusion. $\square$

Theorem 4.9. *Let X_i, $i = 1, 2$ be two locally compact Cantor sets and $\Gamma^{(i)} \subset \mathrm{Homeo}(X_i)$ be subgroups of class F. If $\alpha : \Gamma^{(1)} \to \Gamma^{(2)}$ is an isomorphism, then there exists a homeomorphism $\pi : X_1 \to X_2$ such as $\alpha(\gamma) = \pi\gamma\pi^{-1}$ for all $\gamma \in \Gamma^{(1)}$.*

Proof. For a point $x \in X_1$, let $\{O_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of compact open sets of X_1 such that $\bigcap O_n = \{x\}$. Set $H_n = \Gamma^{(1)}_{O_n}$ and $O'_n = P_{\alpha(H_n)}$. Then, $\{O'_n\}_n$ forms a decreasing sequence of compact open subsets of X_2 and the intersection is a one-point set $\{y\}$. Define the map π by $\pi(x) = y$. One can check that π is a well-defined homeomorphism from X_1 to X_2. It is not hard to see that $\alpha(\gamma) = \pi\gamma\pi^{-1}$ for all $\gamma \in \Gamma^{(1)}$. $\square$

Now we would like to introduce several kinds of full groups.

Definition 4.10. *Let (X, ϕ) be an LCCM system. Define the largest full group by*

$$[\phi] = \{\gamma \in \mathrm{Homeo}(X) \ ; \ \gamma(x) \in Orb_\phi(x) \ \ \textit{for all } x \in X\}.$$

If γ is an element of $[\phi]$, the orbit cocycle $n_\gamma : X \to \mathbb{Z}$ is determined by $\gamma(x) = \phi^{n_\gamma(x)}(x)$. Set

$$[\phi]_c = \{\gamma \in [\phi] \ ; \ \exists K \ \textit{a compact open set such that} \ \ n_\gamma|K^c \ \textit{is zero}\}.$$

We can check that these two full groups are of class F because of the next lemma.

Lemma 4.11. *Let (X,ϕ) be an LCCM system and U, V be two compact open sets. Then 1_U is equivalent to 1_V in $K^0(X,\phi)$ modulo infinitesimals if and only if there is $\gamma \in [\phi]_c$ with $\gamma(U) = V$, $\gamma^2 = 1$ and $\gamma|(U \cup V)^c = 1$.*

See [GW, Proposition 2.6] for the proof.

Next, let us define topological full groups.

Definition 4.12. *For an LCCM system (X,ϕ), we set*

$$\tau[\phi] = \{\gamma \in [\phi] \ ; \ n_\gamma \text{ is continuous on } X\},$$

$$\tau[\phi]_b = \{\gamma \in \tau[\phi] \ ; \ n_\gamma \text{ is bounded on } X\},$$

$$\tau[\phi]_u = \{\gamma \in \tau[\phi] \ ; \ \exists K \quad \text{a compact open set such that } n_\gamma \text{ is constant on } K^c\}$$

and $\tau[\phi]_c = \tau[\phi] \cap [\phi]_c$.

It is clear that these topological full groups are of class F. We also remark that every element of $\tau[\phi]_c$ is of finite order, since $\tau[\phi]_c$ is isomorphic to an inductive limit of finite groups.

Theorem 4.13. *Let (X,ϕ) be an LCCM system.*

(i) *$[\phi]$ is a complete invariant for the orbit equivalence class of (X,ϕ).*

(ii) *$[\phi]_c$ is a complete invariant for the orbit equivalence class of (X,ϕ).*

(iii) *$\tau[\phi]$ is a complete invariant for the strong orbit equivalence class of (X,ϕ).*

(iv) *$\tau[\phi]_b$ is a complete invariant for the flip conjugacy class of (X,ϕ).*

(v) *$\tau[\phi]_u$ is a complete invariant for the flip conjugacy class of (X,ϕ).*

(vi) *$\tau[\phi]_c$ is a complete invariant for the strong orbit equivalence class of (X,ϕ).*

Proof. The assertion (i) and (iii) are clear from Theorem 4.9. By Theorem 4.9 and 3.1, (iv) and (v) can be shown.

We prove (ii). Let ϕ and ψ be minimal homeomorphisms on the locally compact Cantor sets X and $[\phi]_c \cong [\psi]_c$. From Theorem 4.9, we may assume $[\phi]_c = [\psi]_c$. Then, it can be shown that each ϕ-orbit is equal to ψ-orbit, and so $[\phi] = [\psi]$.

For (vi), we need Theorem 3.2. It is not hard to see $\tau[\phi]_c$ is an invariant for strong orbit equivalence. Assume $\tau[\phi]_c = \tau[\psi]_c$ for two minimal homeomorphisms ϕ and ψ on X. Let us consider the Hilbert space $\ell^2(X)$ spanned by the complete orthonormal basis $\{\xi_x\}_{x\in X}$. The commutative C^*-algebra $C_0(X)$ canonically acts on $\ell^2(X)$ by multiplications. Every $\gamma \in \tau[\phi]_c = \tau[\psi]_c$ induces a unitary v_γ such that $v_\gamma(\xi_x) = \xi_{\gamma(x)}$. We denote the C^*-algebra generated by $C_0(X)$ and v_γ's by A. Then, A is isomorphic to $C^*(X,\phi)$ and $C^*(X,\psi)$. By Theorem 3.2, ϕ and ψ are strong orbit equivalent. □

A minimal homeomorphism ϕ on the locally compact Cantor set can be extended to a homeomorphism on the one-point compactification $X \cup \{\infty_X\}$ or on the Stone-Cech compactification βX. We denote the corresponding C^*-crossed products of $C(X \cup \{\infty_X\})$ and $C(\beta X)$ by $C^*(X \cup \{\infty_X\}, \phi)$ and $C^*(\beta X, \phi)$, respectively. Then, we get the inclusion of the C^*-algebras

$$C_0(X) \subset C^*(X,\phi)^1 \subset C^*(X \cup \{\infty_X\}, \phi) \subset C^*(\beta X, \phi) \subset \mathcal{M}(C^*(X,\phi)).$$

We write all implementing unitaries by the same symbol u. For a C^*-algebra inclusion $B \subset A$, we write the set of unitary normalizers of B in A by $UN(A,B)$.

Proposition 4.14. *When (X,ϕ) is an LCCM system, the following diagram is commutative, and all horizontal rows are exact and split.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(C(X\cup\{\infty_X\})) & \longrightarrow & UN(C^*(X,\phi)^1, C_0(X)) & \longrightarrow & \tau[\phi]_c & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(C(X\cup\{\infty_X\})) & \longrightarrow & UN(C^*(X\cup\{\infty_X\},\phi), C_0(X)) & \longrightarrow & \tau[\phi]_u & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(C(\beta X)) & \longrightarrow & UN(C^*(\beta X,\phi), C_0(X)) & \longrightarrow & \tau[\phi]_b & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(C(\beta X)) & \longrightarrow & UN(\mathcal{M}(C^*(X,\phi)), C_0(X)) & \longrightarrow & \tau[\phi] & \longrightarrow & 0
\end{array}$$

Proof. Since the relative commutant $C^*(\beta X,\phi)\cap C_0(X)'$ is equal to $C(\beta X)$ and $C^*(X,\phi)$ is an ideal of $C^*(\beta X,\phi)$, the set of unitary normalizers $UN(C^*(\beta X,\phi), C_0(X))$ coincides with $UN(C^*(\beta X,\phi), C(\beta X))$. Hence, the third sequence is exact and splits by Theorem 1 of [T]. The same argument also holds for the second sequence. It is easy to see the split exactness of the top sequence.

Let us consider the bottom sequence. The dual action α on $C^*(X,\phi)$ can be extended to the multiplier algebra $\mathcal{M}(C^*(X,\phi))$ and

$$E : a \mapsto \int_{\mathbb{T}} \alpha_t(a)dt$$

gives a norm one projection from $\mathcal{M}(C^*(X,\phi))$ to $C(\beta X)$. Take $v \in UN(\mathcal{M}(C^*(X,\phi)), C_0(X))$ arbitrarily and put $p_n = E(u^{-n}v)$ for $n \in \mathbb{Z}$. The same argument as that in [Pu, Lemma 5.1] shows that p_n is a projection of $C(\beta X)$ and $\{p_n\}_n$ induces a clopen partition of X. If p is a projection of $C_0(X)$ and $p \le p_m$, we have $E(u^{-m}vp) = \delta_{m,n}p$. Therefore, when the support of $f \in C_0(X)$ is compact, we get

$$vfv^* = \sum_{n\in\mathbb{Z}} u^n f p_n u^{-n},$$

where the sum is actually a finite sum. Hence, $f \mapsto vfv^*$ induces a homeomorphism which is contained in $\tau[\phi]$ and the homomorphism from $UN(\mathcal{M}(C^*(X,\phi)), C_0(X))$ to $\tau[\phi]$ is well-defined. Now the split exactness can be shown in the same way. □

We would like to define the index map. Obviously the quotient of $\tau[\phi]_u$ by $\tau[\phi]_c$ is the integers and it corresponds to the fact $K_1(C^*(X\cup\{\infty_X\},\phi)) \cong \mathbb{Z}$. In the same way, by Pimsner-Voiculescu exact sequence, the K_1-group of $C^*(\beta X,\phi)$ is also the integers, and so we obtain a map I from $\tau[\phi]_b$ from $\mathbb{Z}$. We call this homomorphism the index map. It is needless to say that $I(\phi) = 1$ and $\tau[\phi]_c \subset \ker I$.

Lemma 4.15. *Let (X,ϕ) be an LCCM system and $\gamma \in \tau[\phi]_b$. For every point $x \in X$,*

$$I(\gamma) = \#(Orb_\phi^+(x)\cap\gamma(Orb_\phi^-(x))) - \#(Orb_\phi^-(x)\cap\gamma(Orb_\phi^+(x))),$$

where $Orb_\phi^+(x)$ means $\{\phi^n(x); n>0\}$ and $Orb_\phi^-(x)$ means $\{\phi^n(x); n\le 0\}$.

Proof. Note that the right hand side determines a well-defined homomorphism I_x from $\tau[\phi]_b$ to $\mathbb{Z}$. We will construct a representation π of $C^*(\beta X,\phi)$ on the Hilbert space $\ell^2(\mathbb{Z})$. Let $\{\xi_n\}_n$ be the orthonormal basis of $\ell^2(\mathbb{Z})$ and $\pi(u)$ be the bilateral shift. For $f \in C(\beta X)$, define $\pi(f)(\xi_n) = f(\phi^n(x))\xi_n$. Then, π is a well-defined representation. When we denote by

P the orthogonal projection to the closed subspace spanned by $\{\xi_n\}_{n\in\mathbb{N}}$, it is easy to see that $P\pi(a)-\pi(a)P$ is a compact operator for every $a \in C^*(\beta X, \phi)$. So the Fredholm index of $P\pi(v)P$ is well-defined for each unitary $v \in C^*(\beta X, \phi)$, which implies the right hand side of the above equality is homotopy invariant. Hence, the kernels of I and I_x agree. □

Remark 4.16. For a point $x \in X$, we can define the subgroup $\tau[\phi]_{b,x}$ of $\tau[\phi]_b$ such as the set of homeomorphisms which preserve each of $Orb^+_\phi(x)$ and $Orb^-_\phi(x)$. However, this group always has an element of infinite order. We do not know if there exists a surjective homomorphism from $\tau[\phi]_b$ to $\mathbb{Z}$ which differs from the index map I.

We denote the kernel of the index map I by $\tau[\phi]_0$. It is easily seen that $\tau[\phi]_0$ is of class F.

Lemma 4.17. *When (X, ϕ) is an LCCM system, there exists a clopen partition $\{U_i\}_{i=1}^n$ of X such that $\phi(U_i) \cap U_i$ is empty for all $i = 1, 2, \ldots, n$.*

Proof. Existence of such a partition is equivalent to the fact that the system $(\beta X, \phi)$ has no fixed point. This follows from Theorem 5.11. □

Theorem 4.18. *The kernel $\tau[\phi]_0$ of the index map I is a complete invariant for the flip conjugacy class of (X, ϕ).*

Proof. When $\tau[\phi]_0 \cong \tau[\psi]_0$ for two minimal homeomorphisms ϕ and ψ acting on X, we may assume $\tau[\phi]_0 = \tau[\psi]_0$ from Theorem 4.9. By the above lemma, we can take a clopen partition $\{U_i\}_{i=1}^n$ of X such that $\phi(U_i)\cap U_i$ is empty for all i. Define $\gamma_i \in \tau[\phi]_0 = \tau[\psi]_0$ such as $\gamma_i(x) = \phi(x)$ for $x \in U_i$, $\gamma_i(x) = \phi^{-1}(x)$ for $x \in \phi(U_i)$ and $\gamma_i(x) = x$ for all $x \in (U_i \cup \phi(U_i))^c$. Then, ϕ is expressed by

$$\phi(x) = \gamma_i(x) \quad \text{for } x \in U_i,$$

which means $\phi \in \tau[\psi]_b$. Similarly, we can show $\psi \in \tau[\phi]_b$. By Theorem 3.1, ϕ and ψ are flip conjugate. □

For these full groups, we denote by $N(\cdot)$ the set of normalizers in $\mathrm{Homeo}(X)$.

Lemma 4.19. *If (X, ϕ) is an LCCM system, we have*

$$N([\phi]_c) = N([\phi])$$

and

$$N(\tau[\phi]_u) \subset N(\tau[\phi]_0) = N(\tau[\phi]_b) \subset N(\tau[\phi]) = N(\tau[\phi]_c).$$

Proof. It is obvious that $N([\phi])$ is contained in $N([\phi]_c)$. Let $\gamma \in N([\phi]_c)$ and $\tau \in [\phi]$. For every $x \in X$, we can choose $\tau' \in [\phi]_c$ so that $\tau(\gamma^{-1}(x)) = \tau'(\gamma^{-1}(x))$. Then, $\gamma\tau\gamma^{-1}(x) = \gamma\tau'\gamma^{-1}(x) = \tau''(x)$ for some $\tau'' \in [\phi]_c$. Because $\gamma\tau\gamma^{-1}(x)$ lies in the orbit of x, we get $\gamma\tau\gamma^{-1} \in [\phi]$. By a similar argument, one readily verifies $N(\tau[\phi]_u) \subset N(\tau[\phi]_b) \subset N(\tau[\phi]) = N(\tau[\phi]_c)$.

Let us prove $N(\tau[\phi]_0) = N(\tau[\phi]_b)$. If $\gamma \in N(\tau[\phi]_b)$, by the proposition below, we obtain an automorphism of $C^*(\beta X, \phi)$. Hence, γ preserves the kernel of the index map. Conversely, let us assume $\gamma \in N(\tau[\phi]_0)$. For every $\tau \in \tau[\phi]_b$, we can find a clopen partition $\{U_i\}_{i=1}^n$ in the same way as Lemma 4.17 such that $\tau(U_i) \cap U_i = \emptyset$. Therefore, τ can be written as an element of $\tau[\phi]_0$ locally, which finishes the proof. □

For an inclusion of C^*-algebras $B \subset A$, we denote by $\mathrm{Aut}(A, B)$ the set of automorphisms which preserve B globally.

Proposition 4.20. *When (X, ϕ) is an LCCM system, the following diagram is commutative and all horizontal rows are exact and split.*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(C(X \cup \{\infty_X\})) & \longrightarrow & \mathrm{Aut}(C^*(X \cup \{\infty_X\}, \phi), C_0(X)) & \longrightarrow & N(\tau[\phi]_u) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(C(\beta X)) & \longrightarrow & \mathrm{Aut}(C^*(\beta X, \phi), C_0(X)) & \longrightarrow & N(\tau[\phi]_b) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U(C(\beta X)) & \longrightarrow & \mathrm{Aut}(\mathcal{M}(C^*(X, \phi)), C_0(X)) & \longrightarrow & N(\tau[\phi]) & \longrightarrow & 0
\end{array}$$

Proof. See the proof of [GPS2, Proposition 2.4] for the top and middle sequence. One can replace the bottom sequence with

$$0 \to U(C(\beta X)) \to \mathrm{Aut}(C^*(X, \phi), C_0(X)) \to N(\tau[\phi]_c) \to 0,$$

and so the split exactness is clear by the same argument as that in [GPS2, Proposition 2.4]. □

We would like to conclude this section by giving a semi-direct product decomposition of $N(\tau[\phi]_u)$ and $N(\tau[\phi]_b)$. Define

$$C^\epsilon(\phi) = \{\gamma \in \mathrm{Homeo}(X) \ ; \ \gamma\phi\gamma^{-1} \ = \ \phi \text{ or } \phi^{-1}\}$$

for an LCCM system (X, ϕ). It is not hard to see $C^\epsilon(\phi) \subset N(\tau[\phi]_u)$.

Proposition 4.21. *Let (X, ϕ) be an LCCM system. Then, the sequences*

$$0 \to \mathbb{Z} \xrightarrow{i} \tau[\phi]_u \rtimes C^\epsilon(\phi) \xrightarrow{\pi} N(\tau[\phi]_u) \to 0$$

and

$$0 \to \mathbb{Z} \xrightarrow{i} \tau[\phi]_b \rtimes C^\epsilon(\phi) \xrightarrow{\pi} N(\tau[\phi]_b) \to 0$$

are exact, where the map i is given by $n \mapsto (\phi^n, \phi^{-n})$ and the map π is given by $(\tau, \gamma) \mapsto \tau\gamma$. Moreover, we have $\tau[\phi]_c \rtimes C^\epsilon(\phi) \cong N(\tau[\phi]_u)$ and $\tau[\phi]_0 \rtimes C^\epsilon(\phi) \cong N(\tau[\phi]_b)$.

Proof. Apply [BT, Corollary 2.7] for $(X \cup \{\infty_X\}, \phi)$ or $(\beta X, \phi)$. See also [GPS2, Proposition 5.11]. □

5 Injective homomorphisms from LCCM systems

The aim of this section is to construct injective homomorphisms from LCCM systems by using embedding of Bratteli diagrams.

Definition 5.1. *Let $D = (W, F)$ be a simple Bratteli diagram and $B = (V, E)$ be a simple or almost simple Bratteli diagram. A graph homomorphism $\rho : D \to B$ is called an embedding of the Bratteli diagram D, if $\rho|W$ and $\rho|F$ are injective and $\rho(W_n) \subset V_n$ for every $n \geq 0$. We denote the associated continuous embedding from X_D to X_B by the same symbol ρ.*

We would like to characterize a compact subset of X_B of the form $\rho(X_D)$.

Definition 5.2. *Let (X, ϕ) be a CM or LCCM system. A compact subset $A \subset X$ is said to be ϕ-homogeneous if the following two conditions are satisfied.*

(i) *For every $x \in A$, the subset $A \cap Orb_\phi(x)$ is dense in A.*

(ii) *If x and $\phi^n(x)$ are contained in A, there exists a clopen neighborhood U of x such that $\phi^n(A \cap U) = A \cap \phi^n(U)$.*

Lemma 5.3. *When (X, ϕ) is a CM system and $A \subset X$ is a compact subset, the following are equivalent.*

(i) *The compact subset A is ϕ-homogeneous.*

(ii) *There exists a simple properly ordered Bratteli diagram $B = (V, E)$ and a conjugacy $\pi : (X, \phi) \to (X_B, \phi_B)$ such that $\pi(A)$ equals $\rho(X_D)$ for some embedding $\rho : D \to B$ of a simple Bratteli diagram D.*

A similar statement also holds for LCCM systems.

Proof. The proof of (ii)⇒(i) is obvious. Let us show the converse. Assume

$$\mathcal{P} = \{X(k, j) \ ; \ k = 1, 2, \dots, K, \ j = 1, 2, \dots, J_k\}$$

is a Kakutani-Rohlin partition for (X, ϕ). By using (ii) of Definition 5.2, we can divide each tower so that

$$A \cap X(k, i) \neq \emptyset, \ A \cap X(k, j) \neq \emptyset \quad \Rightarrow \quad \phi^{j-i}(A \cap X(k, i)) = A \cap X(k, j)$$

holds for all $k = 1, 2, \dots, K$ and $i, j = 1, 2, \dots, J_k$. Then, edges corresponding to (k, j) with $A \cap X(k, j)$ non-empty are chosen as edges coming from a Bratteli diagram D. By repeating this procedure, we obtain a simple properly ordered Bratteli diagram $B = (V, E)$ which represents (X, ϕ), a Bratteli diagram D and an embedding $\rho : D \to B$. It is clear that $\rho(X_D)$ equals A under the identification of X with X_B. The simplicity of D is derived from (i) of Definition 5.2. The proof of the locally compact case is similar. □

Let (X, ϕ) be a CM or LCCM system and $A \subset X$ be a ϕ-homogeneous compact subset. We moreover assume that A is nowhere dense and perfect. Let us consider the disjoint union of A and one point $*$. The countable infinite product $(A \cup \{*\})^{\mathbb{Z}}$ endowed with the product topology is the Cantor set. Let $\eta : X \to (A \cup \{*\})^{\mathbb{Z}}$ be the map defined for $x \in X$ by

$$\eta(x)(n) = \begin{cases} \phi^n(x) & \phi^n(x) \in A \\ * & \phi^n(x) \notin A \end{cases} \quad \text{for } n \in \mathbb{Z}.$$

Then, the range of η is a shift invariant subset containing the point $*^\infty$, because $\bigcup_n \phi^n(A)$ is not equal to X by the Baire category theorem. We set $Y = \eta(X) \setminus \{*^\infty\}$ and denote the subshift on Y by ψ.

Lemma 5.4. *In the above setting, (Y, ψ) is a LCCM system and there exists a canonical injective homomorphism π from (Y, ψ) to (X, ϕ). A similar statement holds for an LCCM system (X, ϕ), too.*

Proof. We will write $*^\infty$ by ∞_Y. Let us show that $Y \cup \{\infty_Y\}$ is a closed set. Take $\{x_k\}_k \subset X$ and assume $\lim_k \eta(x_k) = \xi$ for some $\xi \in (A \cup \{*\})^{\mathbb{Z}}$. If ξ equals ∞_Y, we have nothing to do. Hence, we may assume $\xi(n) = x$ for some $n \in \mathbb{Z}$ and $x \in A$. Since $\eta(x_k)(n)$ goes to x, we have $\phi^n(x_k) \in A$ eventually. If $\xi(m) = *$, then $\phi^m(x_k)$ is not in A eventually. From the ϕ-homogeneity of A, we get $\phi^{m-n}(x) \notin A$. When $\xi(m) \in X$, it is clear that $\phi^{m-n}(x) = \xi(m) \in A$. Therefore, ξ is equal to $\eta(\phi^{-n}(x))$, which implies $Y \cup \{\infty_Y\}$ is closed. By using the ϕ-homogeneity of A, one can also check the minimality of (Y, ψ) easily. When we define $\pi(\eta(x)) = x$ for $\eta(x) \in Y$, the map π is a continuous and injective homomorphism. □

We call $\pi : (Y, \psi) \to (X, \phi)$ in the above lemma the LCCM extension arising from A.

Lemma 5.5. *Let (X, ϕ) be a CM system and $A \subset X$ a ϕ-homogeneous subset. Suppose (X, ϕ) is strong orbit equivalent to another CM system (X', ϕ') by a homeomorphism $F : X \to X'$. When the orbit cocycle $n : X \to \mathbb{Z}$ is continuous on all points in A, the closed subset $F(A)$ is ϕ'-homogeneous. Then two LCCM systems arising from A and $F(A)$ are strong orbit equivalent. If (X, ϕ) and (X', ϕ') are LCCM systems, we also get a similar statement.*

Proof. As F gives an orbit equivalence, the condition (i) of Definition 5.2 follows immediately. For every $x \in A$ the orbit cocycle n is continuous at x, and so one verifies the condition (ii) of Definition 5.2.

Let $\pi : (Y, \psi) \to (X, \phi)$ and $\pi' : (Y', \psi') \to (X', \phi')$ be the LCCM extensions arising from A and $F(A)$, respectively. Then, $\pi'^{-1} F \pi$ is a well-defined homeomorphism from Y to Y', and two LCCM systems (Y, ψ) and (Y', ψ') are strong orbit equivalent by this homeomorphism. □

Proposition 5.6. *Let $B = (V, E)$ be a simple (resp. almost simple) properly ordered Bratteli diagram and $\rho : D \to B$ be an embedding of a simple Bratteli diagram $D = (W, F)$. Assume $\rho(X_D)$ does not contain the unique maximal or minimal infinite paths. If $\pi : (Y, \psi) \to (X_B, \phi_B)$ (resp. $(X_B \setminus \{\infty_B\}, \phi_B)$) is the LCCM extension arising from $\rho(X_D)$, the dimension group $K^0(Y, \psi)$ is order isomorphic to $K_0(W, F)$ and the scale $\Sigma(Y, \psi)$ is given by*

$$\{a \in K_0(W, F)^+ \ ; \ a \le p_n \text{ for some } n \in \mathbb{N}\},$$

where p_n is $(h_B(\rho(w)))_{w \in W_n} \in \mathbb{Z}^{W_n}$.

Proof. Let us construct an almost simple Bratteli diagram $D' = (W', F')$ whose ideal part is equal to D. We add a new vertex w_n to W_n and set $W'_n = W_n \cup \{w_n\}$. For every $w \in W_{n+1}$, we put $h_B(\rho(w)) - h_D(w)$ edges between w_n and w so that $h_{D'}(w)$ equals $h_B(\rho(w))$ for every $w \in W'_{n+1} \setminus \{w_{n+1}\}$. Finally, we connect w_n and w_{n+1} by a single edge. Then, the almost simple Bratteli diagram $D' = (W', F')$ is obtained. By using the linear order in $r^{-1}(\rho(w))$, we can define a linear order in $r^{-1}(w)$ for $w \in W'$ so that D' becomes a properly ordered Bratteli diagram which represents (Y, ψ). The assertion clearly follows from Theorem 2.11. □

The converse of Lemma 5.4 also holds.

Proposition 5.7. *Let $\pi : (Y, \psi) \to (X, \phi)$ be an injective homomorphism from an LCCM system to a CM or LCCM system. When $K \subset Y$ is a non-empty compact open subset, $\pi(K)$ is a compact ϕ-homogeneous subset of X. Moreover, if $\pi' : (Y', \psi') \to (X, \phi)$ is the LCCM extension arising from $\pi(K)$, there exists a homeomorphism $\gamma : Y \to Y'$ such that $\gamma\psi\gamma^{-1} = \psi'$ and $\pi'\gamma = \pi$.*

Proof. Let us show the ϕ-homogeneity of $\pi(K)$. If $y, y' \in K$, by the minimality of ψ, we may assume $\lim \psi^{a_n}(y) = y'$. Because $\psi^{a_n}(y)$ is in K eventually, the condition (i) of Definition 5.2 is proved. To prove the condition (ii), take $y, \psi^n(y) \in K$ arbitrarily. The proof is by contradiction. Assume $\{y_m\}_m \in K$ satisfies $\lim \pi(y_m) = \pi(y)$ and $\phi^n(\pi(y_m)) \notin \pi(K)$. Since K is compact and π is injective, we have $\lim y_m = y$. Then, $\psi^n(y_m) \in K$ for sufficiently large m, which contradicts the assumption.

Let $\eta : X \to Y'$ be the map which is used in Lemma 5.4 to construct (Y', ψ'). The composition $\gamma = \eta\pi$ is a homeomorphism from Y to Y' satisfying $\gamma\psi\gamma^{-1} = \psi'$ and $\pi'\gamma = \pi$. □

Let $\pi : (Y, \psi) \to (X, \phi)$ be a homomorphism from an LCCM system to a CM or LCCM system. If π is surjective and injective, there exists a compact open set $U \subset Y$ such that the image $\pi(U)$ is open, by the Baire category theorem. Then, from the minimality of (Y, ψ) one can show that (X,ϕ) is an LCCM system and $\pi(U)$ is open for every open set $U \subset Y$, thus π is actually a homeomorphism. Notice that there exists a continuous bijection which is not a homeomorphism from the locally compact Cantor set to the Cantor set or the locally compact Cantor set.

Example 5.8. Let $X = \{0,1,2,3,4\}^{\mathbb{N}}$ and (X,ϕ) be the odometer system of type 5^∞ (see [O2, Section 2] for odometer systems). It is well known that the dimension group of (X,ϕ) is order isomorphic to $(\mathbb{Z}[1/5], \mathbb{Z}[1/5]^+)$ and the order unit is one. We set the closed subset A of X by

$$A = \{(x_n)_{n\in\mathbb{N}} \in X \ ; \ x_n \text{ is } 1,2 \text{ or } 3 \text{ for all } n \in \mathbb{N}\}.$$

It is easily seen that A is a ϕ-homogeneous subset. The dimension group of the LCCM extension arising from A is order isomorphic to $(\mathbb{Z}[1/3], \mathbb{Z}[1/3]^+)$ and the scale equals the whole of the positive cone.

In general, when (Y,ψ) and (X,ϕ) are CM or LCCM systems and $\pi : (Y,\psi) \to (X,\phi)$ is a homomorphism, π induces a map $\pi_* : M^1_\psi \to M^1_\phi$. By [DGS, Proposition 3.11] π_* is continuous and surjective, if (Y,ψ) and (X,ϕ) are CM systems. Let us consider the case that (Y,ψ) is an LCCM system. Assume $\{\mu_n\}_n \subset M^1_\psi$ converges to $\mu \in M^1_\psi$, that is, $\mu_n(U)$ goes to $\mu(U)$ for each compact open set $U \subset Y$. Since every clopen subset of Y can be divided as a countable disjoint union of compact open subsets, we see that $\lim \mu_n(U) = \mu(U)$ holds for all clopen subsets. Hence, the map $\pi_* : M^1_\psi \to M^1_\phi$ is always continuous.

Lemma 5.9. *If $\pi : (Y,\psi) \to (X,\phi)$ is an injective homomorphism from an LCCM system (Y,ψ) to a CM or LCCM system (X,ϕ), then π_* gives a continuous injection from M^1_ψ onto $\{\mu \in M^1_\phi \ ; \ \mu(\pi(Y)) = 1\}$, and $\nu \in M^1_\psi$ is an extremal point of M^1_ψ if and only if $\pi_*(\nu)$ is an extremal point of M^1_ϕ. When M^1_ψ is closed, π_* becomes a homeomorphism.*

Proof. If $\mu \in M^1_\phi$ satisfies $\mu(\pi(Y)) = 1$, then it is easy to see that $U \mapsto \mu(\pi(U))$ for a compact open subset U of Y gives the preimage of μ. Therefore, π_* is a bijective correspondence. When M^1_ψ is closed, then it is compact, and so the inverse map is continuous. □

The following theorem shows that there is no K-theoretical obstruction for existence of injective homomorphism from LCCM systems without finite invariant measures.

Theorem 5.10. *Let (Y,ψ) be an LCCM system without finite invariant measures and (X,ϕ) be a CM or LCCM system. Then, there exists an injective continuous map $\pi : Y \to X$ and continuous functions $n, m : Y \to \mathbb{Z}$ such that*

$$\pi(\psi(y)) = \phi^{n(y)}(\pi(y)), \quad \pi(\psi^{m(y)}(y)) = \phi(\pi(y)),$$

for all $y \in Y$. Thus, π gives an orbit embedding.

Proof. We assume (X,ϕ) is a CM system. The proof of the locally compact case is similar. Let $B = (V,E)$ be a simple properly ordered Bratteli diagram such that (X_B, ϕ_B) is isomorphic to (X,ϕ) and $D = (W,F)$ be a simple Bratteli diagram such that $K_0(W,F)$ is order isomorphic to $K^0(Y,\psi)$. By telescoping B appropriately, we can construct an embedding $\rho : D \to B$ in such a way that $\rho(X_D)$ does not contain the unique maximal or minimal infinite paths and $\mu(\rho(X_D))$ is zero for every $\mu \in M^1_{\phi_B}$ (see [GPS3, Lemma 4.2]). Then, we get the LCCM extension $\pi : (Y',\psi') \to (X_B,\phi_B)$. By Proposition 5.6 and Lemma 5.9, $K^0(Y',\psi')$ is isomorphic to $K^0(Y,\psi)$ as scaled dimension groups. From Theorem 3.2, we get the conclusion. □

The author does not know the necessary and sufficient condition for CM and LCCM systems so that there exists a continuous injection which preserve orbits like the theorem above.

Let us show that every LCCM system has an injective homomorphism to a CM system.

Theorem 5.11. *When (Y, ψ) is an LCCM system, there exist a CM system (X, ϕ) and an injective homomorphism $\pi : (Y, \psi) \to (X, \phi)$. Moreover, when (Y, ψ) has no infinite invariant measure, we can take the above homomorphism π so that the induced map $\pi_* : M^1_\psi \to M^1_\phi$ is surjective.*

Proof. We begin with the proof of the first part. We would like to construct a simple properly ordered Bratteli diagram $B = (V, E)$ and an almost simple properly ordered Bratteli diagram $D = (W, F)$ so that the following conditions are satisfied. We write $\infty_D = (f_n)_n$ and $r(f_n) = w_n$.

(i) The Bratteli-Vershik system (X_D, ϕ_D) is conjugate to $(Y \cup \{\infty_Y\}, \psi)$. (We will identify them in the argument below.)

(ii) For each n, V_n contains two distinguished vertices v_n^0 and v_n^1.

(iii) When we denote the ideal part of D by $\widetilde{D} = (\widetilde{W}, \widetilde{F})$, there exists an embedding $\rho : \widetilde{D} \to B$ such that $\rho(\widetilde{W}_n) = V_n \setminus \{v_n^0, v_n^1\}$ and $h_D(w) = h_B(\rho(w))$ for every $w \in \widetilde{W}$.

(iv) Let $p_1, p_2, \dots, p_N$ and $q_1, q_2, \dots, q_N$ be the ordered list of paths from the top vertex w_0 to w and from the top vertex v_0 to $\rho(w)$, where w is a vertex of W_n and $N = h_D(w)$. Then, if p_i goes through w_{n-1}, q_i goes through v_n^0 or v_n^1. If p_i goes through $w' \in \widetilde{W}_{n-1}$, q_i goes through $\rho(w')$.

(v) For each n, $h_B(v_n^1) - h_B(v_n^0)$ is one. Put $L_n = h_B(v_n^0)^2 + h_B(v_n^0) + 1$.

(vi) When p is the unique path from w_0 to w_n, the clopen subset $U(p)$ of Y satisfies the following: if $y \notin U(p)$ and $\psi(y) \in U(p)$, then $\psi^2(y), \dots, \psi^{L_n}(y) \in U(p)$.

Let us consider the case $n = 1$. We put two edges from v_0 to v_1^1 and one edge from v_0 to v_1^0. Then, L_1 is three. One can choose a clopen neighborhood U of ∞_Y so that $y \notin U$ and $\psi(y) \in U$ imply $\psi^2(y) \in U$ and $\psi^3(y) \in U$. Define a vertex set W_1 which consists of two points corresponding to U^c and U, and put a single edge from the top vertex w_0 to each of them. We can construct V_1 and $\rho : W_1 \setminus \{w_1\} \to V_1$ satisfying (iii).

For the induction step, suppose all the items have been constructed to level $n-1$. At first, for every $v \in V_{n-1}$, we put a single edge between v and each of v_n^0 and v_n^1. Furthermore, we add one more edge between v_{n-1}^i and v_n^i for each $i = 0, 1$. Then, the condition (v) is achieved and the natural number L_n is determined. We can define a linear order on $r^{-1}(v_n^i)$, $i = 0, 1$, so that the maximum edge has the source vertex v_{n-1}^1 and the minimum edge has the source vertex v_{n-1}^0. Let p be the unique path from the top vertex w_0 to w_{n-1}. Set

$$U = \bigcup_{i=0}^{L_n} \psi^i \Big(\bigcap_{j=-L_n-1}^{L_{n-1}} \psi^j(U(p)) \Big).$$

It is easy to check that U is a clopen neighborhood of ∞_Y and contained in $\bigcap_{j=-1}^{L_{n-1}} \psi^j(U(p))$. One can take a sufficiently finer Kakutani-Rohlin partition for (Y, ψ) which has U as the roof set. This partition determines the vertex set W_n which contains w_n, the edge set F_n and the partial order on F_n. Let $\widetilde{W}_n$ and $\widetilde{F}_n$ be the ideal part of W_n and F_n. Put the copy of $\widetilde{W}_n$ and $\widetilde{F}_n$ in the n-th level of the diagram B. Then, the embedding ρ in the n-th level is determined obviously. When we denote by p' the unique path from w_0 to w_n, the clopen set $U(p')$ equals

$U \cap \psi(U)$. By the definition of U, we can check the condition (vi) for $U(p')$. Let w be a vertex of $W_n \setminus \{w_n\}$. From the definition of U and (vi) for $U(p)$, one verifies that edges whose source vertex is w_{n-1} appear in the ordered list of edges in $r^{-1}(w)$ at least L_{n-1} times consecutively . For every natural number $L \geq L_{n-1}$, there exist $m, n \in \mathbb{N}$ such that $L = mh_B(v_n^0) + nh_B(v_n^1)$, and so we can put edges between v_{n-1}^0, v_{n-1}^1 and $\rho(w)$ and define the linear order in $r^{-1}(\rho(w))$ so that the condition (iv) holds. Besides, we may assume that v_{n-1}^0 is the source vertex of the minimum edge and v_{n-1}^1 is the source vertex of the maximum edge. We can do the same thing for all vertices in $W_n \setminus \{w_n\}$. This completes the induction step and all the desired properties are clear.

By using (iv), we can construct an injective homomorphism from (Y, ψ), which is identified with (X_D, ϕ_D), to (X_B, ϕ_B).

Let us prove the second part. Suppose (Y, ψ) has no infinite invariant measure. Note that this is equivalent to assuming that the scale $\Sigma(Y, \psi)$ is bounded above in the ordered group $K^0(Y, \psi)$. Under this assumption we will show that the map π_* induced by the homomorphism π constructed above is a surjection from M_ψ^1 to $M_{\phi_B}^1$. Take $\mu \in M_{\phi_B}^1$ arbitrarily. It suffices to show $\mu(\rho(X_D)) > 0$ from Lemma 5.9. We denote by a the element $(h_D(w))_{w \in W_n}$ in $K^0(Y, \psi)^+$. Since $K^0(Y, \psi)$ is a simple dimension group, there exists a constant $C > 4$ such that $b < Ca$ for all $b \in \Sigma(Y, \psi)$. For $v \in V$ let p_v be a finite path from the top vertex v_0 to v. From the construction we have

$$\begin{aligned} \mu(\rho(X_D)) &= \lim_{n \to \infty} \sum_{w \in W_n} h_D(w) \mu(U(p_{\rho(w)})) \\ &= \lim_{n \to \infty} \sum_{w \in W_n} \sum_{v \in V_{n+1}} h_D(w) E(\rho(w), v) \mu(U(p_v)), \end{aligned}$$

where $E(\cdot, \cdot)$ is the number of edges between two vertices. When $v = v_{n+1}^0$ or $v = v_{n+1}^1$, one has

$$\sum_{w \in W_n} h_D(w) E(\rho(w), v) \mu(U(p_v)) = \sum_{w \in W_n} h_D(w) \mu(U(p_v)) \geq \frac{1}{4} h_B(v) \mu(U(p_v)).$$

When $v = \rho(w')$ for some $w' \in W_{n+1}$, one also gets

$$\sum_{w \in W_n} h_D(w) E(\rho(w), v) \mu(U(p_v)) = h_D(w') \mu(U(p_v)) \geq C^{-1} h_B(v) \mu(U(p_v)).$$

Hence $\mu(X_D) \geq C^{-1} > 0$ and the proof is completed. □

By a topological realization of an automorphism T of a measure space (Y, ν), we mean a homeomorphism ϕ on a topological space X along with a ϕ-invariant measure μ such that (ϕ, μ) is measurably conjugate to (T, ν). In [O1] Ormes answered the following question: when can we realize a given ergodic automorphism on a non-atomic Lebesgue probability space as a minimal homeomorphism on the Cantor set within a given strong orbit equivalence class? He showed that there is no obstruction except for rational discrete spectrum. We can prove an analogous result for LCCM systems.

Theorem 5.12. *Let (G, G^+, Σ) be a non-unital scaled simple dimension group and $\mu \in S_\Sigma(G)$ be an extremal point of $S_\Sigma(G)$. For any ergodic transformation T on a non-atomic Lebesgue probability space (Y, ν), we can find an LCCM system (X, ϕ) such that $K^0(X, \phi)$ is isomorphic to G as scaled dimension groups and (ϕ, μ) is measurably conjugate to (T, ν).*

Proof. Suppose $D = (W, F)$ is an almost simple Bratteli diagram such that the unitization of G is isomorphic to $K_0(W, F)$. One can easily construct a simple Bratteli diagram $B = (V, E)$ and an embedding $\rho : \widetilde{D} \to B$ so that the order unit 1_B of $K_0(V, E)$ is not divisible by any natural numbers (see the proof of Theorem 5.11 for example). Let $\mu' \in S_{1_B}(K_0(V, E))$ be the state corresponding to μ. By Lemma 5.9 μ' is an extremal point. From Theorem 2.5 of [O1] we can telescope the Bratteli diagram B and define a proper order such that the unique maximal and minimal paths are not contained in $\rho(X_{\widetilde{D}})$ and (X_B, ϕ_B, μ') is measurably conjugate to (Y, T, ν). Let $\pi : (X, \phi) \to (X_B, \phi_B)$ be the LCCM extension arising from $\rho(X_{\widetilde{D}})$. Then, $K^0(X, \phi)$ is isomorphic to G as scaled dimension groups by Lemma 5.5 and Proposition 5.6, and (X, ϕ, μ) is measurably conjugate to (X_B, ϕ_B, μ'). $\square$

In the compact case, Sugisaki showed that the strong orbit equivalence class of a CM system contains transformations of all possible topological entropies ([S1],[S2]). The following corollary is a locally compact analogue of this result.

Corollary 5.13. *Let (Y, ψ) be an LCCM system and let $0 \le \alpha \le \infty$. If M^1_ψ consists of one point, or every invariant measure for (Y, ψ) is finite, then there exists an LCCM system (X, ϕ) such that (Y, ψ) is strong orbit equivalent to (X, ϕ) and the topological entropy of $(X \cup \{\infty_X\}, \phi)$ equals α.*

Proof. Suppose $M^1_\psi = \{\mu\}$. Choose an ergodic automorphism T on a non-atomic Lebesgue probability space (Y, ν) which has measure theoretic entropy α. From Theorem 5.12 there exists an LCCM system (X, ϕ) such that (Y, ψ) is strong orbit equivalent to (X, ϕ) and (ϕ, μ) is measurably conjugate to (T, ν). There exist two ergodic ϕ-invariant probability measures on $X \cup \{\infty_X\}$: one is μ and the other is the Dirac measure on ∞_X. By the variational principle (see [DGS, Proposition 18.11]) we get the conclusion. In fact, one needs only $M^1_\psi \neq \emptyset$ in the case of $\alpha = \infty$.

Let us assume that every invariant measure of (Y, ψ) is finite. We can find an injective homomorphism π from (Y, ψ) to a CM system (Z, τ) which induces a surjection $\pi_* : M^1_\psi \to M^1_\tau$. By Sugisaki's results there exists a CM system (Z', τ') such that (Z, τ) is strong orbit equivalent to (Z', τ') and the topological entropy of (Z', τ') is α. From Lemma 5.5 we get an LCCM extension $\pi' : (X, \phi) \to (Z', \tau')$ and (X, ϕ) is strong orbit equivalent to (Y, ψ). For $\mu \in M^1_\phi$ the measurable dynamical system (ϕ, μ) is conjugate to $(\tau', \pi'_*(\mu))$ via π'. Because the set of ergodic probability measures on $(X \cup \{\infty_X\}, \phi)$ consists of the Dirac measure on ∞_X and ergodic probability measures of (X, ϕ), by the variational principle the topological entropy of $(X \cup \{\infty_X\}, \phi)$ is equal to α. $\square$

When an LCCM system (X, ϕ) has no finite invariant measure, the topological entropy of $(X \cup \{\infty_X\}, \phi)$ is always zero because there is no invariant probability measure on $X \cup \{\infty_X\}$ except for the Dirac measure on ∞_X. It should be also pointed out that the topological entropy of $(\beta X, \phi)$ is always infinity when (X, ϕ) is an LCCM system. This is because there exists a homomorphism π from (X, ϕ) to the n-full shift on $\{1, 2, \dots, n\}^{\mathbb{Z}}$ such that the range $\pi(X)$ is dense. Such a homomorphism extends to a surjective homomorphism from βX to the n-full shift. It is well known that the n-full shift has topological entropy $\log n$. Therefore the topological entropy of $(\beta X, \phi)$ is equal to or greater than $\log n$, and so we have the conclusion.

6 Proper homomorphisms and skew product extensions

If (X, ϕ) and (Y, ψ) are LCCM systems and $\pi : Y \to X$ is a proper map satisfying $\psi\pi = \pi\phi$, then we call π a proper LCCM homomorphism. A proper LCCM homomorphism π induces a homomorphism π^* sending $[f]$ to $[f \circ \pi]$ from $K^0(X, \phi)$ to $K^0(Y, \psi)$. The homomorphism π also gives a continuous map π_* from M_ψ to M_ϕ.

Lemma 6.1. *Let $\pi : (Y, \psi) \to (X, \phi)$ be a proper LCCM homomorphism.*

(i) *π is surjective.*

(ii) *π_* is surjective and $\pi_*(M_\psi^1) = M_\phi^1$.*

Proof. (i) Take $y \in Y$ and $x \in X$ arbitrarily. Since (X, ϕ) is minimal, we can find a sequence $\{a_n\}_n \subset \mathbb{Z}$ such that $\phi^{a_n}(\pi(y))$ goes to x. When U is a compact open neighborhood of x, $\psi^{a_n}(y)$ is in $\pi^{-1}(U)$ eventually. Because π is proper, $\pi^{-1}(U)$ is compact. Therefore, we may assume $\psi^{a_n}(y)$ converges to a point in $\pi^{-1}(U)$, and this point is a preimage of x.

(ii) In fact, we do not need the minimality in the following argument. For a given $\mu \in M_\phi$, we would like to construct its preimage in M_ψ. One may assume X is a disjoint union of countable Cantor sets $\{X_n\}_{n \in \mathbb{N}}$. Since $\mu(X_n)$ is finite and $\pi^{-1}(X_n)$ is compact open in Y, there exists a Borel measure on $\pi^{-1}(X_n)$ which maps to $\mu|X_n$ by π_*. Hence, we can find $\nu \in M(Y)$ such that $\nu(\pi^{-1}(E)) = \mu(E)$ for every Borel set $E \in \mathcal{B}(X)$. There exists a translation invariant state m on $\ell^\infty(\mathbb{Z})$, because $\mathbb{Z}$ is amenable. For every compact open set U in Y and $k \in \mathbb{Z}$, we have

$$\nu(\psi^k(U)) \leq \mu(\pi(\psi^k(U))) = \mu(\pi(U)),$$

and so $(\nu(\phi^k(U)))_k$ is in $\ell^\infty(\mathbb{Z})^+$. When we put $\nu'(U) = m((\nu(\phi^k(U)))_k)$, it is easy to check that ν' is positive and $\nu'(\pi^{-1}(V)) = \mu(V)$ for each compact open set V in X. The translation invariance of m implies $\nu'(U) = \nu'(\psi(U))$. Therefore, ν' induces a positive homomorphism from $C_0(Y, \mathbb{Z})$ to $\mathbb{R}$ which is ψ-invariant. Thanks to Lemma 2.12, we get the desired Borel measure on Y. □

The following is the well-known characterization of coboundaries. For the proof, see [GH, Theorem 14.11] or [O2, Theorem 2.6].

Lemma 6.2. *When (X, ϕ) is an LCCM system and $f : X \to \mathbb{Z}$ is a continuous function which vanishes at infinity, the following conditions are equivalent.*

(i) *There exists a point $x \in X$ such that its positive semi-orbit is dense in X and the sum $|\sum_{i=0}^{n-1} f(\phi^i(x))|$ is bounded over $n \in \mathbb{N}$.*

(ii) *There exists a constant $C > 0$ with $|\sum_{i=0}^{n-1} f(\phi^i(x))| < C$ and $|\sum_{i=1}^{n} f(\phi^{-i}(x))| < C$ for every $x \in X$ and $n \in \mathbb{N}$.*

(iii) *The function f is a coboundary, i.e. there exists $g \in C_0(X, \mathbb{Z})$ with $f = g - g \circ \phi$.*

By using this characterization, we can prove the next proposition.

Proposition 6.3. *When $\pi : (Y, \psi) \to (X, \phi)$ is a proper LCCM homomorphism, the following conditions hold.*

(i) *The order homomorphism π^* is injective.*

(ii) *For $a \in K^0(X, \phi)$, we have $a \in K^0(X, \phi)^+$ if and only if $\pi^*(a) \in K^0(Y, \psi)^+$.*

(iii) *A positive element $b \in K^0(Y, \psi)^+$ is contained in the scale if and only if there exists $a \in \Sigma(X, \phi)$ with $b \leq \pi^*(a)$.*

The proof is straightforward (see [GW, Propositon 3.1]). An order homomorphism between scaled dimension groups which has the properties of the above proposition is called a scale preserving order embedding.

Let $\pi : (Y, \psi) \to (X, \phi)$ be a proper LCCM homomorphism. We would like to examine the scale preserving order embedding π^* more precisely. If $\{U_n\}_n$ is a decreasing sequence of clopen neighborhoods of ∞_X which shrinks to ∞_X, the sequence $\{\pi^{-1}(U_n)\}_n$ also has the same property. Hence, we can find sufficiently finer Kakutani-Rohlin partitions $\mathcal{P}_n$ for (X, ϕ) and $\mathcal{Q}_n$ for (Y, ψ) so that the following are satisfied.

- The roof sets of $\mathcal{P}_n$ (resp. $\mathcal{Q}_n$) is U_n (resp. $\pi^{-1}(U_n)$).
- For all $n \in \mathbb{N}$, $\mathcal{Q}_n$ is finer than the pull back of $\mathcal{P}_n$.
- For all $n \in \mathbb{N}$, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and $\mathcal{Q}_{n+1}$ is finer than $\mathcal{Q}_n$.

By using these partitions, we can construct almost simple properly ordered Bratteli diagrams $B = (V, E)$ and $D = (W, F)$ having the property of Remark 2.5 so that the following conditions hold.

- The Bratteli-Vershik systems (X_B, ϕ_B) and (X_D, ϕ_D) are isomorphic to $(X \cup \{\infty_X\}, \phi)$ and $(Y \cup \{\infty_Y\}, \psi)$, respectively.
- There exists a graph homomorphism $\rho : D \to B$ such that $\rho(W_n) = V_n$ and $\rho(F_n) = E_n$ for each $n \in \mathbb{N}$.
- Moreover, if $(f_1, f_2, \dots, f_m)$ is an ordered list of edges in $r^{-1}(w)$ for $w \in W_n$, then $(\rho(f_1), \rho(f_2), \dots, \rho(f_m))$ is also the ordered list of edges in $r^{-1}(\rho(w))$.
- The homomorphism induced by ρ coincides with π.

From this observation one obtains the next lemma.

Lemma 6.4. *When $\pi : (Y, \psi) \to (X, \phi)$ is a proper LCCM homomorphism, the quotient of $K^0(Y, \psi)$ by $\pi^*(K^0(X, \phi))$ is torsion free.*

The proof is the same as that of [M2, Lemma 6]. We must remark that Proposition 6.3 can be also shown by using this graph homomorphism ρ.

Definition 6.5. *Let (X, ϕ) be an LCCM system. If a continuous map $\gamma : X \to X$ is proper and preserves the coboundary subgroup B_ϕ, the map $f \mapsto f \circ \gamma$ for $f \in C_0(X, \mathbb{Z})$ gives rise to a homomorphism from $K^0(X, \phi)$ to itself. We denote it by* $\mathrm{mod}(\gamma)$.

It is not hard to see that $\mathrm{mod}(\gamma)$ can be defined for elements of the normalizer group $N(\tau[\phi])$ of Section 4. As in the compact case, we denote by $C(\phi)$ the set of all continuous maps from X to itself which commutes with ϕ and call it the centralizer of (X, ϕ). Although $\mathrm{mod}(\gamma)$ can not be defined for all $\gamma \in C(\phi)$, we write $T(\phi) = C(\phi) \cap \ker \mathrm{mod}$.

The next aim of this paper is to show that every finite group can be contained in $T(\phi)$.

Definition 6.6. *Let (X, ϕ) be an LCCM system and G be a discrete countable group. A continuous map $c : X \to G$ is called a G-valued cocycle. We define a homeomorphism ψ on $Y = X \times G$ by $\psi(x, g) = (\phi(x), gc(\phi(x)))$. When π is the natural projection from Y to X, $\pi : (Y, \psi) \to (X, \phi)$ is called the skew product extension of (X, ϕ) associated with c. The homeomorphism γ_g for $g \in G$ which sends (x, h) to (x, gh) commutes with ψ. We call them the canonical centralizers.*

Needless to say, we are interested in the case that the skew product extension (Y,ψ) becomes an LCCM system. We remark that π is proper if and only if G is finite.

In the same way as the compact case, we can prove the following.

Lemma 6.7. *When (Y,ψ) is an LCCM system and $G \subset C(\psi)$ is a finite subgroup, the quotient system (X,ϕ) of (Y,ψ) by the action of G is also an LCCM system. The LCCM system (Y,ψ) can be written as the skew product extension of (X,ϕ) associated with a G-valued cocycle.*

The proof is similar to that of [M1, Lemma 2.2].

Let G be a finite group. Suppose the skew product extension (Y,ψ) of an LCCM system (X,ϕ) associated with a G-valued cocycle c is an LCCM system. We would like to compute the dimension group of (Y,ψ). From the observation before Lemma 6.4, we may assume that (X,ϕ) and (Y,ψ) are represented by almost simple properly ordered Bratteli diagrams $B=(V,E)$ and $D=(W,F)$, and the proper LCCM homomorphism $\pi : (Y,\psi) \to (X,\phi)$ is given by a graph homomorphism $\rho : D \to B$. The ideal part of B and D is denoted by $\widetilde{B} = (\widetilde{V},\widetilde{E})$ and $\widetilde{D} = (\widetilde{W},\widetilde{F})$. We will identify $(X \cup \{\infty_X\},\phi)$ and $(Y \cup \{\infty_Y\},\psi)$ with (X_B,ϕ_B) and (X_D,ϕ_D), respectively. By taking sufficiently finer Kakutani-Rohlin partitions, we may further assume the following.

- If p is a finite path from the top vertex v_0 to $v \in \widetilde{V}_n$, the cocycle c is constant on the compact open set $U(p)$.
- The vertex set $\widetilde{W}_n$ is canonically identified with $\widetilde{V}_n \times G$ for every $n \in \mathbb{N}$.

As mentioned in Section 2, $K^0(X,\phi)$ is isomorphic to the inductive limit of $\mathbb{Z}^{\widetilde{V}_n}$ with connecting matrices $A_n \in M_{\widetilde{V}_{n-1}\times\widetilde{V}_n}(\mathbb{Z})$ determined by $\widetilde{E}_n$. We denote the canonical basis of $\mathbb{Z}^{\widetilde{V}_n}$ by the same symbols $\{v\ ; v \in \widetilde{V}_n\}$ as vertices. Under the identification of $\mathbb{Z}^{\widetilde{W}_n}$ with $\mathbb{Z}^{\widetilde{V}_n} \otimes \mathbb{Z}[G]$, we write the basis of $\mathbb{Z}^{\widetilde{W}_n}$ by $\{v \otimes g\ ;\ v \in \widetilde{V}_n,\ g \in G\}$. By the same argument as that in Section 2 of [M1], we see that the connecting map from $\mathbb{Z}^{\widetilde{W}_{n-1}}$ to $\mathbb{Z}^{\widetilde{W}_n}$ is given by a $\widetilde{V}_{n-1} \times \widetilde{V}_n$ rectangular matrix whose entries are in the group ring $\mathbb{Z}[G]$. We denote this matrix by B_n. It is clear that if $B_n(v,v') = \sum_{g\in G} \lambda_g g$ for $v \in \widetilde{V}_{n-1}$ and $v' \in \widetilde{V}_n$, then $A_n(v,v') = \sum_{g\in G}\lambda_g$. In contrast to the compact case, we obtain $K^0(Y,\psi)$ directly as the inductive limit of $\mathbb{Z}^{\widetilde{V}_n} \otimes \mathbb{Z}[G]$ with the incidence matrix B_n. The order embedding π^* is given by $v \mapsto \sum_g v \otimes g$. It is not hard to see that the canonical centralizers $\{\gamma_g\}_{g\in G}$ act on the dimension group via $\mathrm{mod}(\gamma_g)(v\otimes h) = v \otimes gh$.

Proposition 6.8. *In the above setting, we have*

$$\pi^*(K^0(X,\phi)) = \mathrm{Im}\,\big(\sum_g \mathrm{mod}(\gamma_g)\big) = \bigcap_g \ker(id - \mathrm{mod}(\gamma_g)).$$

Moreover if G is a cyclic group $\mathbb{Z}/m\mathbb{Z}$,

$$\mathrm{Im}(id-\gamma) = \ker\big(\sum_{i=1}^m \mathrm{mod}(\gamma^i)\big)$$

holds, where γ is a generator of the canonical centralizers.

Proof. For the first equation, it is sufficient to prove that $\bigcap_g \ker(id - \mathrm{mod}(\gamma_g))$ is contained in $\pi^*(K^0(X,\phi))$. But, this fact can be easily shown by the above observations. The second equation is proved by the similar way. ∎

The above equalities concerning $\mathrm{mod}(\gamma_g)$ can be also explained by means of the Rohlin property of finite group actions on C^*-algebras, which was defined in [I]. By Proposition 4.20, the canonical centralizers naturally induce an action α of G on the C^*-algebra $C^*(Y,\psi)$. Take a compact open subset U of X and consider the projection $p \in C^*(Y,\psi)$ corresponding to the characteristic function on $U \times G$. Because α_g fixes p, the restriction of α on $pC^*(X,\phi)p$ is well-defined. For every $v \in \widetilde{V}_n$ and $g \in G$, let $q_{v,g}$ be the minimum element in the set of finite paths from w_0 to the vertex in $\widetilde{W}_n$ corresponding with (v,g). Define

$$U_{n,g} = \bigcup_{v \in \widetilde{V}_n} \bigcup_{i=0}^{h_B(v)-1} \psi^i(U(q_{v,g}))$$

and denote the characteristic function on this clopen set by $e_g(n)$. Then, $\{(pe_g(n))_n\}_{g\in G}$ forms a family of Rohlin projections for $\alpha|pC^*(Y,\psi)p$. Moreover, by using the famous Evans-Kishimoto intertwining argument, we can classify this type of finite group actions up to conjugacy by the associated K-theoretical data.

In [M1, Theorem 3.3], we proved that kernel of the mod map cannot contain a finite group except for cyclic groups in the case of CM systems. However, we can construct LCCM systems whose kernel of the mod map contain a given finite group.

Theorem 6.9. *For a non-unital scaled simple dimension group (H, H^+, Σ) except for $\mathbb{Z}$ and a finite group G of order m, the following are equivalent.*

(i) *The abelian group H is uniquely m-divisible, that is, the homomorphism $h \mapsto mh$ is an automorphism of H.*

(ii) *There exists an LCCM system (Y,ψ) such that $T(\psi)$ contains a finite subgroup isomorphic to G and $(K^0(Y,\psi), K^0(Y,\psi)^+, \Sigma(Y,\psi))$ is isomorphic to (H, H^+, Σ) as a scaled dimension group.*

Proof. One readily verifies (ii)⇒(i) from Lemma 6.8. Let us show the opposite implication.

Let $B = (V,E)$ be an almost simple Bratteli diagram whose associated unital dimension group is isomorphic to the unitization of (H, H^+, Σ). Put $\infty_B = (e_n)_n$ and $r(e_n) = v_n$. We may assume that B satisfies the condition of Remark 2.5. Let us denote by p_n the unique finite path from the top vertex v_0 to v_n. We would like to define a partial order on each E_n and a G-valued cocycle c inductively. We will construct c in such a way that it takes a constant value on each $U(q)$ for a finite path q from v_0 to a vertex in $\widetilde{V}$, and write the constant value by $c(q)$, shortly, if it has been already defined.

Assume that the partial order has been defined on $E_1, E_2, \ldots, E_n$ and that the cocycle c has been determined on the compact open set $U(p_n)^c$. We would like to define a partial order on E_{n+1} and determine the cocycle c on $U(p_n) \setminus U(p_{n+1})$. By telescoping $B = (V,E)$, for every $v \in \widetilde{V}_n$ and $w \in \widetilde{V}_{n+1}$ one may assume that number of edges between v and w is a multiple of m and number of edges from v_n to w is greater than $m \times \#\widetilde{V}_n$. Take $w \in \widetilde{V}_{n+1}$ arbitrarily and put $l = h_B(w)$. We denote by q_v the maximum element in the set of finite paths from v_0 to $v \in V_n$. In order to get $\mathrm{mod}(\gamma_g) = id$ for all $g \in G$, we must require the following: if $(q_1, q_2, \ldots, q_l)$ is the ordered list of finite paths from v_0 to w, then for all $v \in \widetilde{V}_n$

$$\sum_{q_i \text{ is an extension of } q_v} c(q_{i+1})c(q_{i+2})\cdots c(q_l) \quad \in \mathbb{Z}[G]$$

is a scalar multiple of $\sum_{g\in G} g$. Since we can determine the value of the cocycle c freely on a finite path q_i which goes through v_n, the above condition can be achieved easily. We can do the same thing for all vertices in $\widetilde{V}_{n+1}$ and the induction is completed.

Now, the minimality of the skew product extension (Y, ψ) is clear from the above construction. $\square$

Next, we would like to consider skew product extensions by infinite countable discrete groups.

Proposition 6.10. *Let (X, ϕ) be a CM or LCCM system and G be a discrete group. Suppose the skew product extension (Y, ψ) of (X, ϕ) associated with a G-valued cocycle c is a minimal system. Then, G must be locally finite. In particular, a skew product extension of a CM system associated with a discrete group valued cocycle never be an LCCM system.*

Proof. The proof is by contradiction. Assume G is not locally finite. We can find a finite subset F which generates an infinite subgroup of G. Hence, there exists a sequence $\{g_i\}_{i\in\mathbb{N}} \subset G$ such that $g_i^{-1}g_{i+1} \in F$ for all $i \in \mathbb{N}$ and $g_i \neq g_j$ for all $i \neq j$.

Define a function $c : X \times \mathbb{Z} \to G$ so that $\psi^k(x, e) = (\phi^k(x), c(x, k))$ holds for every $x \in X$ and $k \in \mathbb{Z}$. Take a compact open subset U of X. Set

$$E_n = \{x \in U \ ; \ F \not\subset \{c(x, k); |k| \leq n, \phi^k(x) \in U\}\}$$

for each $n \in \mathbb{N}$. Evidently, E_n is a compact set and $E_{n+1} \subset E_n$. Let us show that E_n is not empty. Since (Y, ψ) is an LCCM system, there exists a point $x \in U$ such that the positive semi-orbit of (x, e) is dense in Y. Put

$$L_i = \{k \in \mathbb{N} \ ; \ \phi^k(x) \in U, \ c(x, k) = g_i\}$$

for every $i \in \mathbb{N}$. If a natural number k greater than n satisfies $k \in L_i$ and $\{k-n, k-n+1, \ldots, k+n\} \cap L_{i+1} = \emptyset$, then we get $\phi^k(x) \in E_n$. Therefore, we may assume that such a natural number does not exist, and so there exist a natural number i and an increasing sequence $\{k_j\}_{j\geq i}$ such that $c(x, k_j) = g_j$, $k_{j+1} - k_j \leq n$ and $\phi^{k_j}(x) \in U$ for all $j \geq i$. Because $\{c(y, m); y \in U, |m| \leq n\}$ is a finite set, this contradicts density of the positive semi-orbit of (x, e).

Since E_n is not empty for each $n \in \mathbb{N}$, their intersection $\bigcap E_n$ contains at least one point and we have a contradiction. $\square$

If G is a locally finite group, we can construct an LCCM system and a cocycle $c : X \to G$ so that the skew product extension becomes minimal. (Especially, there exists a surjective homomorphism between LCCM systems which is not proper.) Moreover, by using the same technique as in the proof of Theorem 6.9, we can construct the canonical centralizers in the kernel of the mod map.

At the last of this paper, we would like to point out that the results of [GPS3] and [M2] can be also translated to the locally compact case with a slight modification.

The following "dynamical realization" theorem can be proved by exactly the same way as [GPS3, Theorem 4.1].

Theorem 6.11. *Let (X, ϕ) be an LCCM system and (H, H^+, Σ) be a non-unital scaled simple dimension group. Suppose $\iota : K^0(X, \phi) \to H$ is a scale preserving order embedding. When $\iota(K^0(X, \phi))$ is order dense in H and the quotient group $H/\iota(K^0(X, \phi))$ is torsion free, there exist an LCCM system (Y, ψ) and a proper LCCM homomorphism $\pi : (Y, \psi) \to (X, \phi)$ such that the following are satisfied.*

(i) *The homomorphism π is almost one-to-one and at most two-to-one, that is, every point of X has at most two preimages in Y and there exists a point $y \in Y$ with $\pi^{-1}(\pi(y)) = \{y\}$.*

(ii) *There exists an order isomorphism $\alpha : H \to K^0(Y, \psi)$ satisfying $\alpha(\Sigma) = \Sigma(Y, \psi)$ and $\alpha \circ \iota = \pi^*$.*

In the above theorem, the assumption of torsion freeness of the quotient group is always needed because of Lemma 6.4. The dynamical realization problem without the assumption of order density is still open for CM and LCCM systems.

In [M2], we computed the dimension group of topological joinings of CM systems in a very special setting. The same computation can be done for LCCM systems.

Theorem 6.12. *Let $\pi_i : (Y_i, \psi_i) \to (X, \phi)$ be proper LCCM homomorphisms for $i = 0, 1$. Suppose*

$$\{x \in X \ ; \ \#\pi_0^{-1}(x) \neq 1\} \cap \{x \in X \ ; \ \#\pi_1^{-1}(x) \neq 1\} = \emptyset.$$

When we set $Z = \{(y_0, y_1); \pi_0(y_0) = \pi_1(y_1)\}$ and $\tau = \psi_0 \times \psi_1 | Z$, the system (Z, τ) is an LCCM system. The dimension group of (Z, τ) is isomorphic to the relative direct sum of $K^0(Y_i, \psi_i)$ with respect to $\pi_i^*(K^0(X, \phi))$ and the scale $\Sigma(Z, \tau)$ is given by

$$\{[g_0, g_1] \in K^0(Z, \tau) \ ; \ \exists \ h_0, h_1 \in K^0(X, \phi) \text{ s.t. } g_i \leq \pi_i^*(h_i) \text{ and } h_0 + h_1 \in \Sigma(X, \phi)\}.$$

Although the countable infinite product of the locally compact Cantor set is not locally compact, the projective limit of LCCM systems with proper connecting homomorphisms is a well-defined LCCM system. The scaled dimension group of the projective limit system is isomorphic to the inductive limit of the scaled dimension group of each LCCM system. Hence, [M2, Corollary 8] is valid for LCCM systems. Because we can prove [M2, Lemma 9] in the locally compact case, the next theorem is obtained.

Theorem 6.13. *Let (X, ϕ) be an LCCM system, $\gamma \in C(\phi)$ be a homeomorphism which has essentially infinite order. Then, there exists an LCCM system (Z, τ), a proper LCCM homomorphism $\Phi : (Z, \tau) \to (X, \phi)$ and a non-invertible centralizer $\tilde{\gamma} \in C(\tau)$ such that $\gamma \circ \Phi = \Phi \circ \tilde{\gamma}$ and Φ induces an isomorphism Φ^* between the scaled dimension groups $K^0(X, \phi)$ and $K^0(Z, \tau)$.*

Example 6.14. We would like to give an LCCM system which admits a non-invertible centralizer acting trivially on the dimension group. Set the Cantor set X by

$$X = \prod_{n=1}^{\infty} \{0, 1, 2, \ldots, 3^{2^n - 1}\}$$

and let (X, ϕ) be the odometer system (of type 3^∞). When we define the ϕ-homogeneous subset A by

$$A = \{(x_n)_n \in X \ ; \ x_n = 1, 2, \ldots, 2^n \text{ for all } n \in \mathbb{N}\},$$

the scaled dimension group of the LCCM system (Y, ψ) arising from the closed subset A is equal to $(\mathbb{Z}[1/2], \mathbb{Z}[1/2]^+, \mathbb{Z}[1/2]^+)$. Let (Z, τ) be the odometer system of type 2^∞. Consider the product system $(Y \times Z, \psi \times \tau)$. Since Z is isomorphic to the projective limit of $\mathbb{Z}/2^n\mathbb{Z}$ as a topological group, the product system $(Y \times Z, \psi \times \tau)$ can be written by a projective limit of LCCM systems. More precisely, when we denote by (Y_n, ψ_n) the product system of (Y, ψ) and the periodic system on $\mathbb{Z}/2^n\mathbb{Z}$, the projective limit of (Y_n, ψ_n)'s is isomorphic to $(Y \times Z, \psi \times \tau)$. By using the method developed in this section, one can check that (Y_n, ψ_n) is indeed an LCCM system and the canonical centralizers act on the dimension group trivially. Hence, $(Y \times Z, \psi \times \tau)$ is also an LCCM system and the centralizer $id \times \tau$ is contained in $T(\psi \times \tau)$. The natural projection π from $Y \times Z$ to Y gives a proper LCCM homomorphism and the scaled dimension groups of (Y, ψ) and $(Y \times Z, \psi \times \tau)$ are isomorphic via π^*. Because the centralizer $id \times \tau$ is of essentially infinite order, we obtain the desired LCCM system from the above theorem.

As observed in the example above, the product system of a CM system and an LCCM system may be minimal. We remark that, however, a product of two LCCM systems (X, ϕ) and (Y, ψ) never be minimal since any point of $X_+^c \times Y_-^c$ does not have dense orbits.

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