

# Classification of uniformly outer actions of $\mathbb{Z}^2$ on UHF algebras

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## Abstract

We give a complete classification up to cocycle conjugacy of uniformly outer actions of  $\mathbb{Z}^2$  on UHF algebras. In particular, it is shown that any two uniformly outer actions of  $\mathbb{Z}^2$  on a UHF algebra of infinite type are cocycle conjugate. We also classify them up to outer conjugacy.

## 1 Introduction

In [N1], H. Nakamura introduced the Rohlin property for  $\mathbb{Z}^N$ -actions on unital  $C^*$ -algebras and obtained the Rohlin type theorem for  $\mathbb{Z}^2$ -actions on UHF algebras (see Theorem 2.4). He also classified product type actions of  $\mathbb{Z}^2$  up to cocycle conjugacy. In this paper, we extend this result and complete classification of uniformly outer actions of  $\mathbb{Z}^2$  on UHF algebras up to cocycle conjugacy.

In the theory of operator algebras, the problem of classifying group actions has a long history. A fundamental tool for the classification theory of actions is the (noncommutative) Rohlin property, which was first introduced by A. Connes for single automorphisms of finite von Neumann algebras ([C]). In the framework of  $C^*$ -algebras, A. Kishimoto established a non-commutative Rohlin type theorem for single automorphisms of UHF algebras and certain AT algebras, and classified them up to outer conjugacy ([K1],[K2],[K3]). Nakamura used the same technique for automorphisms of purely infinite simple nuclear  $C^*$ -algebras and classified them up to outer conjugacy ([N2]). Recently, the second named author showed that any two outer actions of  $\mathbb{Z}^N$  on the Cuntz algebra  $\mathcal{O}_2$  are cocycle conjugate ([M]). In the case of finite group actions, M. Izumi introduced a notion of the Rohlin

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property and classified a large class of actions ([I2],[I3]). The reader may consult the survey paper [I1] for the Rohlin property of automorphisms of  $C^*$ -algebras.

The aim of this paper is to extend these results to uniformly outer actions of  $\mathbb{Z}^2$  on UHF algebras. More precisely, we will introduce a  $K$ -theoretical invariant for  $\mathbb{Z}^2$ -actions on a UHF algebra and show that it is a complete invariant for cocycle conjugacy. When the UHF algebra  $A$  is of infinite type, that is,  $A$  is isomorphic to  $A \otimes A$ , the range of this invariant is zero, and so any two actions are cocycle conjugate. Note that, in [N1, Theorem 15], essentially the same invariant was used to distinguish infinitely many product type actions which are not cocycle conjugate to each other.

The content of this paper is as follows. In Section 2, we collect notations and basic facts relevant to this paper. Notions of the ultraproduct algebra  $A^\infty$  and the central sequence algebra  $A_\infty$  will help our analysis. For  $\mathbb{Z}^2$ -actions on unital  $C^*$ -algebras, we recall the definition of the Rohlin property from [N1]. In Section 3, we introduce the notion of admissible cocycles. For an almost cocycle of an approximately inner  $\mathbb{Z}^2$ -action on an AF algebra, we can associate an element of the  $K_0$ -group. An almost cocycle is said to be admissible, if the associated element in the  $K_0$ -group is zero. Section 4 is devoted to the cohomology vanishing theorem for  $\mathbb{Z}^2$ -actions on UHF algebras with the Rohlin property. One of the difficulties in the study of  $\mathbb{Z}^2$ -actions on  $C^*$ -algebras is that one has to formulate a two-dimensional version of Berg's technique in order to obtain the so-called cohomology vanishing theorem. In our situation, we need to construct a certain homotopy of unitaries, and admissibility of a cocycle is necessary for that. In Section 5, a  $K$ -theoretical invariant for  $\mathbb{Z}^2$ -actions is introduced. We also observe that the invariant is exhausted by uniformly outer actions. In Section 6, for given two uniformly outer  $\mathbb{Z}^2$ -actions  $\alpha, \beta$  with the same invariant, we show the existence of cocycles which transform  $\alpha$  to  $\beta$  approximately. Kishimoto's classification result of single automorphisms is used to obtain the admissibility of cocycles. Finally, the main theorems are given. D. E. Evans and A. Kishimoto introduced in [EK] an intertwining argument for automorphisms, which is an equivariant version of Elliott's intertwining argument for classification of  $C^*$ -algebras. By using the Evans-Kishimoto intertwining argument, we get the classification theorem up to cocycle conjugacy. The classification up to outer conjugacy is also obtained.

## 2 Preliminaries

Let  $A$  be a unital  $C^*$ -algebra. We denote by  $U(A)$  the group of unitaries in  $A$ . For  $u \in U(A)$ , we let  $\text{Ad } u(a) = uau^*$  for  $a \in A$  and call it an inner automorphism of  $A$ . For any  $a, b \in A$ , we write  $[a, b] = ab - ba$  and call it the commutator of  $a$  and  $b$ . We let  $\log$  be the standard branch defined on the complement of the negative real axis.

First, we would like to recall the classification result of single automorphisms of UHF algebras.

**Definition 2.1** ([K1, Definition 1.2]). An automorphism  $\alpha$  of a unital  $C^*$ -algebra  $A$  is said to be uniformly outer if for any  $a \in A$ , any non-zero projection  $p \in A$  and any  $\varepsilon > 0$ , there exist projections  $p_1, p_2, \dots, p_n$  in  $A$  such that  $p = \sum p_i$  and  $\|p_i a \alpha(p_i)\| < \varepsilon$  for all  $i = 1, 2, \dots, n$ .

We say that an action  $\alpha$  of a discrete group on  $A$  is uniformly outer if  $\alpha_n$  is uniformly

outer for every element  $n$  of the group other than the identity element. The following result can be found in [K1] (see also [BEK]).

**Theorem 2.2** ([K1, Theorem 1.3]). *Let  $A$  be a UHF algebra and let  $\alpha, \beta \in \text{Aut}(A)$  be automorphisms with which the associated actions of  $\mathbb{Z}$  are uniformly outer. Then for any  $\varepsilon > 0$ , there exist an automorphism  $\sigma$  and a unitary  $u \in A$  such that*

$$\|u - 1\| < \varepsilon \quad \text{and} \quad \text{Ad } u \circ \alpha = \sigma \circ \beta \circ \sigma^{-1}.$$

We remark that Kishimoto generalized the result above to AT algebras in [K2] and [K3]. In this paper, we would like to study a  $\mathbb{Z}^2$  version of this classification theorem on UHF algebras. A key point for the proof of the theorem above is the Rohlin property of automorphisms or  $\mathbb{Z}$ -actions. We would like to recall the definition of the Rohlin property for  $\mathbb{Z}^2$ -actions on unital  $C^*$ -algebras (see [N1, Section 2] or [I1, Section 3]).

For  $m = (m_1, m_2)$  and  $n = (n_1, n_2)$  in  $\mathbb{Z}^2$ ,  $m \leq n$  means  $m_1 \leq n_1$  and  $m_2 \leq n_2$ . For  $m = (m_1, m_2) \in \mathbb{N}^2$ , we let

$$m\mathbb{Z}^2 = \{(m_1 n_1, m_2 n_2) \in \mathbb{Z}^2 \mid (n_1, n_2) \in \mathbb{Z}^2\}.$$

For simplicity, we denote  $\mathbb{Z}^2/m\mathbb{Z}^2$  by  $\mathbb{Z}_m$ . Moreover, we may identify  $\mathbb{Z}_m = \mathbb{Z}^2/m\mathbb{Z}^2$  with

$$\{(n_1, n_2) \in \mathbb{Z}^2 \mid 0 \leq n_i \leq m_i - 1 \text{ for each } i = 1, 2\}.$$

The canonical basis of  $\mathbb{Z}^2$  as well as its image in  $\mathbb{Z}_m$  is denoted by  $\xi_1 = (1, 0)$  and  $\xi_2 = (0, 1)$ .

**Definition 2.3.** Let  $\alpha$  be an action of  $\mathbb{Z}^2$  on a unital  $C^*$ -algebra  $A$ . Then  $\alpha$  is said to have the Rohlin property, if for any  $m \in \mathbb{N}$  there exist  $R \in \mathbb{N}$  and  $m^{(1)}, m^{(2)}, \dots, m^{(R)} \in \mathbb{N}^2$  with  $m^{(1)}, \dots, m^{(R)} \geq (m, m)$  satisfying the following: For any finite subset  $\mathcal{F}$  of  $A$  and  $\varepsilon > 0$ , there exists a family of projections

$$e_g^{(r)} \quad (r = 1, 2, \dots, R, g \in \mathbb{Z}_{m^{(r)}})$$

in  $A$  such that

$$\sum_{r=1}^R \sum_{g \in \mathbb{Z}_{m^{(r)}}} e_g^{(r)} = 1, \quad \|[a, e_g^{(r)}]\| < \varepsilon, \quad \|\alpha_{\xi_i}(e_g^{(r)}) - e_{g+\xi_i}^{(r)}\| < \varepsilon$$

for any  $a \in \mathcal{F}$ ,  $r = 1, 2, \dots, R$ ,  $i = 1, 2$  and  $g \in \mathbb{Z}_{m^{(r)}}$ , where  $g + \xi_i$  is understood modulo  $m^{(r)}\mathbb{Z}^2$ .

Nakamura proved the following Rohlin type theorem for  $\mathbb{Z}^2$ -actions on UHF algebras in [N1].

**Theorem 2.4** ([N1, Theorem 3]). *Let  $\alpha$  be an action of  $\mathbb{Z}^2$  on a UHF algebra  $A$ . Then the following conditions are equivalent.*

- (1)  $\alpha$  has the Rohlin property.
- (2)  $\alpha$  is uniformly outer.

In this paper, we deal with central sequence algebras, which simplify the arguments a little. Let  $A$  be a separable  $C^*$ -algebra. We set

$$c_0(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n \rightarrow \infty} \|a_n\| = 0\},$$

$$A^\infty = \ell^\infty(\mathbb{N}, A) / c_0(A).$$

We identify  $A$  with the  $C^*$ -subalgebra of  $A^\infty$  consisting of equivalence classes of constant sequences. We let

$$A_\infty = A^\infty \cap A'.$$

When  $\alpha$  is an automorphism of  $A$  or an action of a discrete group on  $A$ , we can consider its natural extension on  $A^\infty$  and  $A_\infty$ . We denote it by the same symbol  $\alpha$ .

Using these notations, we can restate the definition of the Rohlin property of a  $\mathbb{Z}^2$ -action  $\alpha$  on a unital separable  $C^*$ -algebra  $A$  as follows. For any  $m \in \mathbb{N}$  there exist  $R \in \mathbb{N}$ ,  $m^{(1)}, m^{(2)}, \dots, m^{(R)} \in \mathbb{N}^2$  with  $m^{(1)}, \dots, m^{(R)} \geq (m, m)$  and a family of projections

$$e_g^{(r)} \quad (r = 1, 2, \dots, R, g \in \mathbb{Z}_{m^{(r)}})$$

in  $A_\infty = A^\infty \cap A'$  such that

$$\sum_{r=1}^R \sum_{g \in \mathbb{Z}_{m^{(r)}}} e_g^{(r)} = 1, \quad \alpha_{\xi_i}(e_g^{(r)}) = e_{g+\xi_i}^{(r)}$$

for any  $r = 1, 2, \dots, R$ ,  $i = 1, 2$  and  $g \in \mathbb{Z}_{m^{(r)}}$ , where  $g + \xi_i$  is understood modulo  $m^{(r)}\mathbb{Z}^2$ .

Let  $\xi_1 = (1, 0)$  and  $\xi_2 = (0, 1)$  be the canonical basis of  $\mathbb{Z}^2$ . Let  $\alpha$  be an action of  $\mathbb{Z}^2$  on a unital  $C^*$ -algebra  $A$ . A family of unitaries  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A$  is called an  $\alpha$ -cocycle, if

$$u_n \alpha_n(u_m) = u_{n+m}$$

for all  $n, m \in \mathbb{Z}^2$ . If a pair of unitaries  $u_1, u_2 \in U(A)$  satisfies

$$u_1 \alpha_{\xi_1}(u_2) = u_2 \alpha_{\xi_2}(u_1),$$

then it determines uniquely an  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  such that  $u_{\xi_i} = u_i$ . We may also call the pair  $\{u_1, u_2\}$  an  $\alpha$ -cocycle. An  $\alpha$ -cocycle  $\{u_n\}_n$  in  $A$  is called a coboundary, if there exists  $v \in U(A)$  such that

$$u_n = v \alpha_n(v^*)$$

for all  $n \in \mathbb{Z}^2$ , or equivalently, if

$$u_{\xi_i} = v \alpha_{\xi_i}(v^*)$$

for each  $i = 1, 2$ .

When  $\{u_n\}_{n \in \mathbb{Z}^2}$  is an  $\alpha$ -cocycle, it turns out that a new action  $\tilde{\alpha}$  of  $\mathbb{Z}^2$  on  $A$  can be defined by

$$\tilde{\alpha}_n(x) = \text{Ad } u_n \circ \alpha_n(x) = u_n \alpha_n(x) u_n^*$$

for each  $x \in A$ . We call  $\tilde{\alpha}$  the perturbed action of  $\alpha$  by  $\{u_n\}_n$ .

Two actions  $\alpha$  and  $\beta$  of  $\mathbb{Z}^2$  on  $A$  are said to be cocycle conjugate, if there exists an  $\alpha$ -cocycle  $\{u_n\}_n$  in  $A$  such that the perturbed action of  $\alpha$  by  $\{u_n\}_n$  is conjugate to  $\beta$ . The purpose of this paper is to classify uniformly outer actions of  $\mathbb{Z}^2$  on UHF algebras up to cocycle conjugacy.

### 3 Admissible cocycles

Throughout this section, we let  $A$  be a unital AF algebra and let  $\alpha$  be an action of  $\mathbb{Z}^2$  on  $A$  such that  $\alpha_n$  is approximately inner for every  $n \in \mathbb{Z}^2$ . To simplify notation, we may denote  $\alpha_{\xi_i}$  by  $\alpha_i$  for each  $i = 1, 2$ .

A pair of unitaries  $\{u_1, u_2\}$  in  $A$  is called an almost  $\alpha$ -cocycle, when

$$\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < 1.$$

For such a pair  $u_1, u_2$ , we can construct a closed path of unitaries as follows. We put

$$a = \frac{1}{2\pi\sqrt{-1}} \log(u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*),$$

and set a path  $k : [0, 1] \rightarrow U(A)$  from  $u_2\alpha_2(u_1)$  to  $u_1\alpha_1(u_2)$  by

$$k(t) = e^{2\pi\sqrt{-1}ta}u_2\alpha_2(u_1).$$

Let  $h_i : [0, 1] \rightarrow U(A)$  be a path from 1 to  $u_i$  for each  $i = 1, 2$ . Then

$$\tilde{h}_1 : t \mapsto u_2\alpha_2(h_1(t))$$

is a path from  $u_2$  to  $u_2\alpha_2(u_1)$  and

$$\tilde{h}_2 : t \mapsto u_1\alpha_1(h_2(t))$$

is a path from  $u_1$  to  $u_1\alpha_1(u_2)$ . By connecting these paths, we obtain a closed path  $H : [0, 1] \rightarrow U(A)$ . More precisely,

$$H(t) = \begin{cases} h_1(5t) & \text{if } 0 \leq t \leq 1/5 \\ \tilde{h}_2(5t - 1) & \text{if } 1/5 \leq t \leq 2/5 \\ k(3 - 5t) & \text{if } 2/5 \leq t \leq 3/5 \\ \tilde{h}_1(4 - 5t) & \text{if } 3/5 \leq t \leq 4/5 \\ h_2(5 - 5t) & \text{if } 4/5 \leq t \leq 1. \end{cases}$$

Since  $\alpha$  is approximately inner, the  $K_1$ -class of this closed path  $H$  in  $K_1(SA) \cong K_0(A)$  does not depend on the choice of  $h_1$  and  $h_2$ . We denote this element in  $K_0(A)$  by

$$\kappa(u_1, u_2, \alpha_1, \alpha_2) \in K_0(A).$$

Clearly  $\kappa(u_1, u_2, \alpha_1, \alpha_2)$  is homotopy invariant in the following sense: if  $u_i : [0, 1] \rightarrow U(A)$  is a path of unitaries satisfying

$$\|u_1(t)\alpha_1(u_2(t)) - u_2(t)\alpha_2(u_1(t))\| < 1$$

for every  $t \in [0, 1]$ , then

$$\kappa(u_1(0), u_2(0), \alpha_1, \alpha_2) = \kappa(u_1(1), u_2(1), \alpha_1, \alpha_2).$$

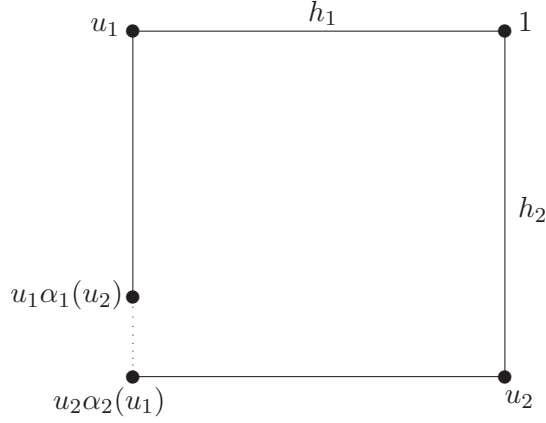


Figure 1: The definition of  $\kappa(u_1, u_2, \alpha_1, \alpha_2)$

**Lemma 3.1.** *Let  $\{u_1, u_2\}$  be an almost  $\alpha$ -cocycle with  $\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < \varepsilon < 1$ . Then,*

$$|\tau(\kappa(u_1, u_2, \alpha_1, \alpha_2))| < \frac{\arcsin \varepsilon}{2\pi}$$

for any tracial state  $\tau \in T(A)$ .

*Proof.* Let  $k, h_1, h_2, \tilde{h}_1, \tilde{h}_2$  and  $H$  be the paths of unitaries as above. We may assume that  $H$  is piecewise smooth. For any tracial state  $\tau \in T(A)$ , we have

$$\tau(\kappa(u_1, u_2, \alpha_1, \alpha_2)) = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{H}(t)H(t)^*) dt.$$

Since  $\alpha_2$  is approximately inner,  $\tau \circ \alpha_2$  is equal to  $\tau$ . Therefore the contribution from  $h_1$  and  $\tilde{h}_1$  cancels out. Similarly, the contribution from  $h_2$  and  $\tilde{h}_2$  also cancels out. Therefore we get  $\tau(\kappa(u_1, u_2, \alpha_1, \alpha_2)) = \tau(a)$ , where

$$a = \frac{1}{2\pi\sqrt{-1}} \log(u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*).$$

Since  $\|a\| < \pi^{-1} \arcsin(\varepsilon/2) < (2\pi)^{-1} \arcsin \varepsilon$ , the proof is completed.  $\square$

Suppose that a family of unitaries  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A^\infty$  is an  $\alpha$ -cocycle. Let  $(u_{\xi_i}(k))_{k \in \mathbb{N}}$  be a representing sequence in  $\ell^\infty(\mathbb{N}, A)$  of  $u_{\xi_i}$ . We may assume that  $u_{\xi_i}(k)$  is in  $U(A)$ . For sufficiently large  $k \in \mathbb{N}$ , the pair  $\{u_{\xi_1}(k), u_{\xi_2}(k)\}$  is an almost  $\alpha$ -cocycle in  $A$ . In addition, from the lemma above, we can see that

$$-1 \leq \kappa(u_{\xi_1}(k), u_{\xi_2}(k), \alpha_{\xi_1}, \alpha_{\xi_2}) \leq 1$$

for sufficiently large  $k \in \mathbb{N}$ . Hence

$$\kappa^\infty(u_{\xi_1}, u_{\xi_2}, \alpha_{\xi_1}, \alpha_{\xi_2}) = (\kappa(u_{\xi_1}(k), u_{\xi_2}(k), \alpha_{\xi_1}, \alpha_{\xi_2}))_{k \in \mathbb{N}} \in K_0(A^\infty)$$

is well-defined.

**Definition 3.2.** Let  $A$  be a unital AF algebra and let  $\alpha$  be an action of  $\mathbb{Z}^2$  on  $A$  such that  $\alpha_n$  is approximately inner for every  $n \in \mathbb{Z}^2$ .

- (1) An  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A$  is said to be admissible, if  $\kappa(u_{\xi_1}, u_{\xi_2}, \alpha_{\xi_1}, \alpha_{\xi_2})$  is zero in  $K_0(A)$ .
- (2) An  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A^\infty$  is said to be admissible, if  $\kappa^\infty(u_{\xi_1}, u_{\xi_2}, \alpha_{\xi_1}, \alpha_{\xi_2})$  is zero in  $K_0(A^\infty)$ .

Observe that if  $\alpha$ -cocycle is a coboundary, then it is admissible. When  $K_0(A)$  has no non-trivial infinitesimal elements, from Lemma 3.1, we can see that any  $\alpha$ -cocycle in  $A$  is admissible.

Let  $\{u_1, u_2\}$  be an almost  $\alpha$ -cocycle in  $A$ . For  $j \in \mathbb{N}$  and  $i = 1, 2$ , we define  $u_i^{(j)} \in U(A)$  by

$$u_i^{(1)} = u_i \text{ and } u_i^{(j+1)} = u_i^{(j)} \alpha_i^j(u_i).$$

Suppose that

$$\|u_1^{(j)} \alpha_1^j(u_2^{(k)}) - u_2^{(k)} \alpha_2^k(u_1^{(j)})\| < 1$$

for every  $j = 1, 2, \dots, m$  and  $k = 1, 2, \dots, n$ . Thus, the pair  $\{u_1^{(j)}, u_2^{(k)}\}$  is an almost cocycle for the  $\mathbb{Z}^2$ -action generated by  $\alpha_1^j$  and  $\alpha_2^k$ .

**Lemma 3.3.** *In the setting above, one has*

$$\kappa(u_1^{(m)}, u_2^{(n)}, \alpha_1^m, \alpha_2^n) = nm\kappa(u_1, u_2, \alpha_1, \alpha_2)$$

in  $K_0(A)$ .

*Proof.* It suffices to show

$$\kappa(u_1^{(j+1)}, u_2^{(k)}, \alpha_1^{j+1}, \alpha_2^k) = \kappa(u_1^{(j)}, u_2^{(k)}, \alpha_1^j, \alpha_2^k) + \kappa(u_1, u_2^{(k)}, \alpha_1, \alpha_2^k).$$

Let  $h_1 : [0, 1] \rightarrow U(A)$  be a path from 1 to  $u_1$  and let  $h_2 : [0, 1] \rightarrow U(A)$  be a path from 1 to  $u_2^{(k)}$ . Let  $H$  be a closed path of unitaries obtained by connecting a short path from  $u_1^{(j)} \alpha_1^j(u_2^{(k)})$  to  $u_2^{(k)} \alpha_2^k(u_1^{(j)})$ , a short path from  $u_1^{(j+1)} \alpha_1^{j+1}(u_2^{(k)})$  to  $u_2^{(k)} \alpha_2^k(u_1^{(j+1)})$  and the following four paths:

$$\begin{aligned} t &\mapsto u_1^{(j)} \alpha_1^j(h_1(t)), & t &\mapsto u_1^{(j)} \alpha_1^j(h_2(t)) \\ t &\mapsto u_2^{(k)} \alpha_2^k(u_1^{(j)} \alpha_1^j(h_1(t))), & t &\mapsto u_1^{(j+1)} \alpha_1^{j+1}(h_2(t)). \end{aligned}$$

Then, the  $K_0$ -class of this closed path  $H$  in  $K_1(SA) \cong K_0(A)$  is equal to

$$\kappa(u_1^{(j+1)}, u_2^{(k)}, \alpha_1^{j+1}, \alpha_2^k) - \kappa(u_1^{(j)}, u_2^{(k)}, \alpha_1^j, \alpha_2^k).$$

Let  $H'$  be a closed path of unitaries obtained by connecting a short path from  $u_2^{(k)} \alpha_2^k(u_1)$  to  $u_1 \alpha_1(u_2^{(k)})$  and the following four paths:

$$t \mapsto h_1(t), \quad t \mapsto h_2(t)$$

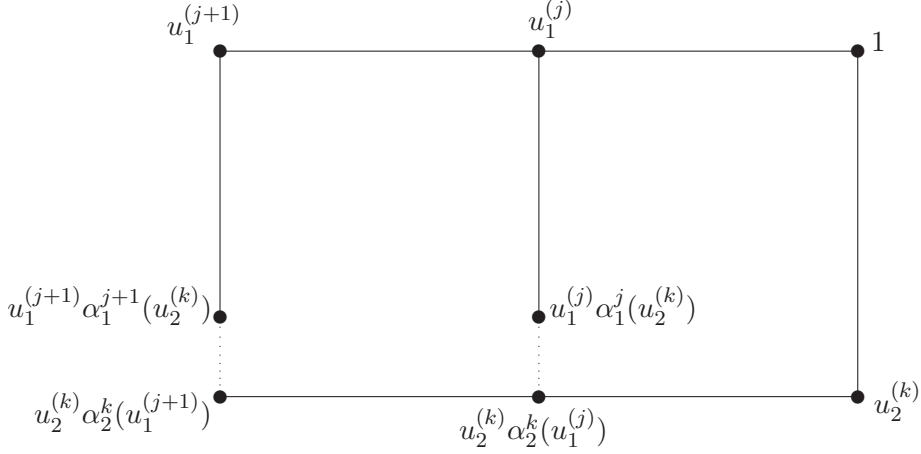


Figure 2:  $\kappa(u_1^{(j+1)}, u_2^{(k)}, \alpha_1^{j+1}, \alpha_2^k)$  and  $\kappa(u_1^{(j)}, u_2^{(k)}, \alpha_1^j, \alpha_2^k)$

$$t \mapsto u_2^{(k)} \alpha_2^k(h_1(t)), \quad t \mapsto u_1 \alpha_1(h_2(t)).$$

By definition, the  $K_0$ -class of  $H'$  equals  $\kappa(u_1, u_2^{(k)}, \alpha_1, \alpha_2^k)$ . It follows from

$$u_1^{(j)} \alpha_1^j(u_2^{(k)})(u_2^{(k)} \alpha_2^k(u_1^{(j)}))^* u_2^{(k)} \alpha_2^k(u_1^{(j+1)}) = u_1^{(j)} \alpha_1^j(u_2^{(k)} \alpha_2^k(u_1))$$

that  $H$  is homotopic to the closed path  $u_1^{(j)} \alpha_1^j(H'(\cdot))$ , which implies the desired equality.  $\square$

Let  $\{u_1, u_2\}$  be an almost  $\alpha$ -cocycle in  $A$  such that

$$\|u_1 \alpha_1(u_2) - u_2 \alpha_2(u_1)\| < \varepsilon.$$

Suppose that a unitary  $v$  in  $A$  satisfies

$$\|v - u_1 \alpha_1(v) u_1^*\| < \delta.$$

Then we have

$$\|u_1 \alpha_1(v u_2) - v u_2 \alpha_2(u_1)\| < \varepsilon + \delta.$$

Thus, if  $\delta$  is sufficiently small, then the pair  $\{u_1, v u_2\}$  is again an almost  $\alpha$ -cocycle.

**Lemma 3.4.** *In the setting above, we have*

$$\kappa(u_1, v u_2, \alpha_1, \alpha_2) = \kappa(u_1, u_2, \alpha_1, \alpha_2) + \kappa(1, v, \text{Ad } u_1 \circ \alpha_1, \text{id})$$

in  $K_0(A)$ .

*Proof.* One can prove this statement in a similar fashion to Lemma 3.3.  $\square$

The following two lemmas are key ingredients for the cohomology vanishing theorem in the next section. We denote the  $\ell^\infty$ -norm on  $\mathbb{R}^2$  by  $\|\cdot\|$ . We put

$$E = \{t \in \mathbb{R}^2 \mid \|t\| \leq 1\}$$



and

$$\partial E = \{t \in \mathbb{R}^2 \mid \|t\| = 1\}.$$

For a Lipschitz continuous function  $f$ , we denote its Lipschitz constant by  $\text{Lip}(f)$ .

**Lemma 3.5.** *Let  $\varepsilon > 0$  be a sufficiently small number. Suppose that a pair of unitaries  $u_1, u_2 \in A$  satisfies*

$$\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < \varepsilon$$

*and  $\kappa(u_1, u_2, \alpha_1, \alpha_2) = 0$ . Then there exists a continuous map  $z : \partial E \rightarrow U(A)$  such that the following hold.*

- (1)  $z(1, 1) = 1$ .
- (2)  $\|z(-1, t) - u_1\alpha_1(z(1, t))\| < \varepsilon$  and  $\|z(t, -1) - u_2\alpha_2(z(t, 1))\| < \varepsilon$  for all  $t \in [-1, 1]$ .
- (3)  $\text{Lip}(z) \leq 4$ .
- (4) *The  $K_1$ -class of  $z$  in  $K_1(C(\partial E) \otimes A)$  is zero.*

*Proof.* This is immediate from the definition of  $\kappa(u_1, u_2, \alpha_1, \alpha_2)$  and the fact that any unitary in an AF algebra can be connected to the identity by a path of unitaries with length less than  $\pi + \delta$ , where  $\delta$  is an arbitrary positive real number. We leave the details to the reader.  $\square$

**Lemma 3.6.** *Let  $A_0 \subset A$  be a unital finite dimensional  $C^*$ -subalgebra. For any  $\varepsilon > 0$ , there exist  $\delta > 0$  and a unital finite dimensional  $C^*$ -subalgebra  $A_1 \subset A$  such that the following holds. Suppose that a pair of unitaries  $u_1, u_2 \in A \cap A'_1$  satisfies*

$$\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < \delta$$

*and  $\kappa(u_1, u_2, \alpha_1, \alpha_2) = 0$ . Then there exists a continuous map  $z : \partial E \rightarrow U(A \cap A'_0)$  such that the following hold.*

- (1)  $z(1, 1) = 1$ .
- (2)  $\|z(-1, t) - u_1\alpha_1(z(1, t))\| < \varepsilon$  and  $\|z(t, -1) - u_2\alpha_2(z(t, 1))\| < \varepsilon$  for all  $t \in [-1, 1]$ .
- (3)  $\text{Lip}(z) \leq 4$ .
- (4) *The  $K_1$ -class of  $z$  in  $K_1(C(\partial E) \otimes A)$  is zero.*

*Proof.* Choose sufficiently small  $\delta > 0$ . Take a unital finite dimensional  $C^*$ -subalgebra  $A_1$  so that for any unitary  $w \in A_0$ , there exist  $a, b, c \in A_1$  such that

$$\|w - a\| < \delta, \quad \|\alpha_1^{-1}(w) - b\| < \delta \quad \text{and} \quad \|\alpha_2^{-1}(w) - c\| < \delta.$$

Suppose that a pair of unitaries  $u_1, u_2 \in A \cap A'_1$  satisfies

$$\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < \delta$$

and  $\kappa(u_1, u_2, \alpha_1, \alpha_2) = 0$ . Since  $A \cap A'_1$  is also an AF algebra, for each  $i = 1, 2$ , there exists a path of unitaries  $h_i$  in  $A \cap A'_1$  from 1 to  $u_i$  with length less than  $\pi + \delta$ . By using these paths, in the same way as the lemma above, we can find a continuous map  $z_0 : \partial E \rightarrow U(A)$  such that the following hold.

- $z_0(1, 1) = 1$ .
- For any  $w \in U(A_0)$  and  $(s, t) \in \partial E$ ,  $\|[w, z_0(s, t)]\| < 6\delta$ .
- $\|z_0(-1, t) - u_1\alpha_1(z_0(1, t))\| < \delta$  and  $\|z_0(t, -1) - u_2\alpha_2(z_0(t, 1))\| < \delta$  for all  $t \in [-1, 1]$ .
- $\text{Lip}(z_0) \leq \pi + 2\delta$ .
- The  $K_1$ -class of  $z_0$  in  $K_1(C(\partial E) \otimes A)$  is zero.

Using the Haar measure on the compact group  $U(A_0)$ , we define  $z_1 : \partial E \rightarrow A \cap A'_0$  by

$$z_1(s, t) = \int w z_0(s, t) w^* dw$$

for every  $(s, t) \in \partial E$ . Then one has  $\|z_1(s, t) - z_0(s, t)\| < 6\delta$  and  $\text{Lip}(z_1) \leq \pi + 2\delta$ . Let

$$z_1(s, t) = z(s, t)|z_1(s, t)|$$

be the polar decomposition of  $z_1(s, t)$  in  $A \cap A'_0$ . It is not so hard to see that  $z(s, t)$  is the desired map.  $\square$

## 4 Cohomology vanishing

We let  $M_n$  denote the  $C^*$ -algebra of all  $n$  by  $n$  matrices, and  $\mathbb{T}$  denote the unit circle of  $\mathbb{C}$ . For a Lipschitz continuous function  $f$ , its Lipschitz constant is denoted by  $\text{Lip}(f)$ .

**Lemma 4.1.** *Let  $u : [0, 1] \rightarrow M_n$  be a path of unitaries such that  $\text{Lip}(u) \leq C$ . Suppose that a continuous function  $\lambda : [0, 1] \rightarrow \mathbb{T}$  satisfies  $\lambda(t) \in \text{Sp}(u(t))$  for any  $t \in [0, 1]$ . Then, we have  $\text{Lip}(\lambda) \leq C$ .*

*Proof.* It suffices to show that for each  $t \in [0, 1]$ , there exists  $\delta > 0$  such that  $|\lambda(s) - \lambda(t)| \leq C|s - t|$  holds for all  $s \in (t - \delta, t + \delta)$ . Take  $t \in [0, 1]$ . There exists  $\delta > 0$  such that

$$|\lambda(t) - \mu| > 2C\delta$$

for any  $\mu \in \text{Sp}(u(t)) \setminus \{\lambda(t)\}$ . From  $\text{Lip}(u) \leq C$ , one has  $\|u(s) - u(t)\| \leq C|s - t|$ . Therefore, the distance from  $\lambda(s)$  to  $\text{Sp}(u(t))$  is not more than  $C|s - t|$ . Since  $\lambda$  is continuous and  $C|s - t| < C\delta$  for  $s \in (t - \delta, t + \delta)$ , we get  $|\lambda(s) - \lambda(t)| \leq C|s - t|$  for all  $s \in (t - \delta, t + \delta)$ .  $\square$

The following lemma is an immediate consequence of [ELP, Corollary 6.15] (and is also a special case of [L, Lemma 3.8]). When  $v$  and  $w$  are almost commuting unitaries in  $M_n$ , we denote the Bott element associated with  $v$  and  $w$  by  $\text{Bott}(v, w) \in K_0(M_n) \cong \mathbb{Z}$ . The Bott element  $\text{Bott}(v, w)$  can be calculated as follows. We put

$$a = \frac{1}{2\pi\sqrt{-1}} \log(vwv^*w^*).$$

Since  $\det(vwv^*w^*) = 1$ , we have  $\text{Tr}(a) \in \mathbb{Z}$ , where  $\text{Tr}$  is the unnormalized trace on  $M_n$ . Then  $\text{Bott}(v, w)$  is equal to  $\text{Tr}(a)$ .

**Lemma 4.2.** *For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that the following holds. For any  $n \in \mathbb{N}$  and unitaries  $v, w \in M_n$  satisfying*

$$\|[v, w]\| < \delta \text{ and } \text{Bott}(v, w) = 0,$$

*there exists a continuous path of unitaries  $w : [0, 1] \rightarrow M_n$  such that*

$$w(0) = 1, \quad w(1) = w, \quad \text{Lip}(w) \leq \pi + \varepsilon$$

*and*

$$\|[v, w(t)]\| < \varepsilon$$

*for all  $t \in [0, 1]$ .*

**Proposition 4.3.** *For any  $C > 0$  and  $\varepsilon > 0$ , there exists  $C' > 0$  such that the following holds. Let  $n \in \mathbb{N}$  and let  $u : [0, 1] \rightarrow M_n$  be a path of unitaries such that  $u(0) = u(1) = 1$  and  $\text{Lip}(u) \leq C$ . If the  $K_1$ -class of  $u$  in  $K_1(C_0(0, 1) \otimes M_n)$  is zero, then we can find a path of self-adjoint elements  $h : [0, 1] \rightarrow M_n$  such that the following are satisfied.*

- (1)  $\|u(t) - e^{2\pi\sqrt{-1}h(t)}\| < \varepsilon$  for all  $t \in [0, 1]$ .
- (2)  $h(0) = h(1) = 0$ .
- (3)  $\text{Lip}(h) \leq C'$ .

*Proof.* By applying Lemma 4.2 to  $\varepsilon/2$ , we get  $\delta > 0$ . We may assume that  $\delta$  is less than  $\varepsilon/2$ . Choose  $L \in \mathbb{N}$  so that  $2C/L < \delta$ . Put  $C' = 2CL/3 + C/6$ .

Suppose that  $u : [0, 1] \rightarrow M_n$  satisfies the conditions in the statement. We would like to construct a path of self-adjoint elements  $h : [0, 1] \rightarrow M_n$  satisfying (1), (2) and (3). One can find continuous functions  $\lambda_1, \lambda_2, \dots, \lambda_n$  from  $[0, 1]$  to  $\mathbb{T}$  such that  $u(t)$  is unitarily equivalent to  $\text{diag}(\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t))$  for each  $t \in [0, 1]$ . Since the  $K_1$ -class of  $u$  in  $K_1(C_0(0, 1) \otimes M_n)$  is zero, we may assume that the rotation number of  $\lambda_i$  is zero for every  $i = 1, 2, \dots, n$ . Thus, there exist continuous functions  $g_1, g_2, \dots, g_n$  from  $[0, 1]$  to  $\mathbb{R}$  such that  $g_i(0) = g_i(1) = 0$  and  $\lambda_i(t) = e^{2\pi\sqrt{-1}g_i(t)}$  for all  $i = 1, 2, \dots, n$  and  $t \in [0, 1]$ . By Lemma 4.1, we have  $\text{Lip}(\lambda_i) \leq C$ . It follows from  $6 < 2\pi$  that the Lipschitz constant of  $g_i$  is less than  $C/6$ . In particular,

$$\max\{|g_i(t)| \mid t \in [0, 1], i = 1, 2, \dots, n\} \leq \frac{C}{12}.$$

We divide  $[0, 1]$  to closed intervals  $I_k = [k/L, (k+1)/L]$  for  $k = 0, 1, \dots, L-1$  and construct the path  $h$  on each interval. For each  $k = 0, 1, \dots, L$ , we set  $u_k = u(k/L)$  and choose rank one projections  $p_{k,1}, p_{k,2}, \dots, p_{k,n}$  so that

$$\sum_{i=1}^n p_{k,i} = 1$$

and

$$u_k = \sum_{i=1}^n \lambda_i(k/L) p_{k,i}.$$

For each  $k = 0, 1, \dots, L-1$ , there exists a unitary  $w_k$  in  $M_n$  such that

$$w_k p_{k,i} w_k^* = p_{k+1,i}$$

for every  $i = 1, 2, \dots, n$ . From  $\text{Lip}(u) \leq C$  and  $\text{Lip}(\lambda_i) \leq C$ , we have

$$\|u_k - u_{k+1}\| \leq \frac{C}{L}$$

and

$$\|w_k u_k w_k^* - u_{k+1}\| \leq \frac{C}{L}.$$

It follows that

$$\|[u_k, w_k]\| \leq \frac{2C}{L} < \delta.$$

We claim that  $\text{Bott}(u_k, w_k)$  is zero. The two paths

$$I_k \ni t \mapsto \sum_{i=1}^n \lambda_i(t) p_{k+1,i} \in M_n$$

and  $u : I_k \rightarrow M_n$  give us a short path  $x$  from  $w_k u_k w_k^*$  to  $u_k$ . It is not so hard to see that the rotation number of  $\det(x)$  is zero. Hence, we obtain  $\text{Bott}(u_k, w_k) = 0$ . Therefore, from Lemma 4.2, we can find a path of unitaries  $z_k : I_k \rightarrow M_n$  such that

$$z_k(k/L) = 1, \quad z_k((k+1)/L) = w_k, \quad \text{Lip}(z_k) \leq 4L$$

and

$$\|[u_k, z_k(t)]\| < \varepsilon/2$$

for all  $t \in I_k$ .

We define a path of self-adjoint elements  $h_k : I_k \rightarrow M_n$  by

$$h_k(t) = z_k(t) \left( \sum_{i=1}^n g_i(t) p_{k,i} \right) z_k^*(t)$$

for  $k = 0, 1, \dots, L-1$  and  $t \in I_k$ . We have

$$\text{Lip}(h_k) \leq \frac{C}{12} \text{Lip}(z_k) + \frac{C}{6} + \frac{C}{12} \text{Lip}(z_k) \leq C',$$

and

$$\begin{aligned} & \|u(t) - e^{2\pi\sqrt{-1}h_k(t)}\| \\ & \leq \|u(t) - u_k\| + \|u_k - z_k(t)u_k z_k(t)^*\| + \|z_k(t)u_k z_k(t)^* - e^{2\pi\sqrt{-1}h_k(t)}\| \\ & \leq \frac{C}{L} + \frac{\varepsilon}{2} + \frac{C}{L} < \varepsilon \end{aligned}$$

for  $t \in I_k$ . By connecting  $h_0, h_1, \dots, h_{L-1}$ , we get a desired path  $h : [0, 1] \rightarrow M_n$ .  $\square$

The following is a well-known fact and one can find its proof in [HR, Lemma 4.8] for example.

**Lemma 4.4.** *Let  $h_1$  and  $h_2$  be self-adjoint elements in a unital  $C^*$ -algebra. Then*

$$\|e^{2\pi\sqrt{-1}h_1} - e^{2\pi\sqrt{-1}h_2}\| \leq 2\pi\|h_1 - h_2\|.$$

The following lemma is also well-known. We have been unable to find a suitable reference in the literature, so we include a proof for completeness.

**Lemma 4.5.** *Let  $A$  be a unital  $C^*$ -algebra and let  $u_1, u_2$  be unitaries in  $A$  such that  $\|u_i - 1\| < 1/2$  for  $i = 1, 2$ . Put  $2\pi\sqrt{-1}h_i = \log u_i$ . Then we have*

$$\|h_1 - h_2\| \leq \pi^{-1}\|u_1 - u_2\|.$$

*Proof.* Since  $\log z$  is written by the absolutely convergent power series

$$\log z = -\sum_{n=1}^{\infty} \frac{1}{n}(1-z)^n$$

on  $\{z \in \mathbb{C} \mid |z - 1| \leq 1/2\}$ , one has

$$\|h_1 - h_2\| = \frac{1}{2\pi} \|\log u_1 - \log u_2\| \leq \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \|(1-u_1)^n - (1-u_2)^n\|.$$

By an elementary estimate,

$$\|(1-u_1)^n - (1-u_2)^n\| \leq n2^{1-n}\|u_1 - u_2\|$$

is obtained, and so the proof is completed.  $\square$

The following proposition plays a crucial role in our cohomology vanishing theorem. We let  $E = \{t \in \mathbb{R}^2 \mid \|t\| \leq 1\}$  and  $\partial E = \{t \in \mathbb{R}^2 \mid \|t\| = 1\}$ , where  $\|\cdot\|$  is the  $\ell^\infty$ -norm on  $\mathbb{R}^2$ .

**Proposition 4.6.** *For any  $C > 0$ , there exists  $C' > 0$  such that the following holds. Let  $A$  be a unital AF algebra and let  $z : \partial E \rightarrow U(A)$  be a continuous map such that  $z(1, 1) = 1$  and  $\text{Lip}(z) \leq C$ . If the  $K_1$ -class of  $z$  in  $K_1(C(\partial E) \otimes A)$  is zero, then we can find a continuous map  $\tilde{z}$  from  $E$  to unitaries of  $A$  such that the following are satisfied.*

- (1)  $z(t) = \tilde{z}(t)$  for all  $t \in \partial E$ .
- (2)  $\text{Lip}(\tilde{z}) \leq C'$ .

*Proof.* By Proposition 4.3 and Lemma 4.4, there exists  $z_0 : E \rightarrow U(A)$  satisfying the following.

- $\|z(t) - z_0(t)\| < 1/2$  for all  $t \in \partial E$ .
- $\text{Lip}(z_0) \leq C_0$ , where  $C_0 > 0$  is a constant depending only on  $C > 0$ .

Define  $k : \partial E \rightarrow A$  and  $z_1 : [0, 1] \times \partial E \rightarrow U(A)$  by

$$k(t) = \frac{1}{2\pi\sqrt{-1}} \log(z(t)^* z_0(t))$$

and

$$z_1(s, t) = z(t) e^{2\pi\sqrt{-1}sk(t)}$$

for  $t \in \partial E$  and  $s \in [0, 1]$ . Clearly  $z_1(0, t) = z(t)$  and  $z_1(1, t) = z_0(t)$ . It follows from Lemma 4.5 and Lemma 4.4 that  $z_1(s, t)$  is Lipschitz continuous with some universal Lipschitz constant depending only on  $C > 0$ . Define

$$\tilde{z}(t) = \begin{cases} z_1(2 - 2\|t\|, t/\|t\|) & \text{if } \|t\| \geq 1/2 \\ z_0(2t) & \text{if } \|t\| \leq 1/2 \end{cases}$$

for  $t \in E$ . Then the assertion easily follows.  $\square$

Now we would like to prove the cohomology vanishing theorem.

**Theorem 4.7.** *Let  $A$  be a unital AF algebra and let  $\alpha$  be an action of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. Suppose that  $\alpha_n$  is approximately inner for any  $n \in \mathbb{Z}^2$ . Then, for any admissible  $\alpha$ -cocycle  $\{u_n\}_n$  in  $A^\infty$ , there exists a unitary  $v \in A^\infty$  such that*

$$u_n = v\alpha_n(v)^*$$

for any  $n \in \mathbb{Z}^2$ .

*Proof.* It suffices to show that, for any  $\varepsilon > 0$ , there exists a unitary  $v \in A^\infty$  such that

$$\|u_{\xi_i} - v\alpha_{\xi_i}(v)^*\| < \varepsilon$$

for each  $i = 1, 2$ . By applying Proposition 4.6 to  $C = 4$ , we obtain a constant  $C' > 0$ . Choose a natural number  $M$  so that  $2C'/M$  is less than  $\varepsilon$ . Since  $\alpha$  has the Rohlin property, there exist  $R \in \mathbb{N}$  and  $m^{(1)}, m^{(2)}, \dots, m^{(R)} \in \mathbb{Z}^2$  with  $m^{(1)}, \dots, m^{(R)} \geq (M, M)$  and which satisfies the requirement in Definition 2.3.

For each  $n \in \mathbb{Z}^2$ , let  $(u_n(k))_{k \in \mathbb{N}} \in \ell^\infty(\mathbb{N}, A)$  be a representing sequence of  $u_n$ . We may assume that  $u_n(k)$  is a unitary. Since  $\{u_n\}_n$  is admissible,  $\kappa(u_{\xi_1}(k), u_{\xi_2}(k), \alpha_{\xi_1}, \alpha_{\xi_2})$  is zero in  $K_0(A)$  for sufficiently large  $k \in \mathbb{N}$ .

For each  $r = 1, 2, \dots, R$  and  $i = 1, 2$ , let  $m_i^{(r)}$  be the  $i$ -th summand of  $m^{(r)}$ . We put  $\eta_{r,i} = m_i^{(r)} \xi_i \in \mathbb{Z}^2$ . For sufficiently large  $k \in \mathbb{N}$ , the pair of unitaries  $\{u_{\eta_{r,1}}(k), u_{\eta_{r,2}}(k)\}$  is an almost cocycle for  $\alpha_{\eta_{r,1}}, \alpha_{\eta_{r,2}}$ . Furthermore, by Lemma 3.3,  $\kappa(u_{\eta_{r,1}}(k), u_{\eta_{r,2}}(k), \alpha_{\eta_{r,1}}, \alpha_{\eta_{r,2}})$  is zero in  $K_0(A)$ . It follows from Lemma 3.5 that for sufficiently large  $k \in \mathbb{N}$ , there exists a continuous map  $z_k^{(r)} : \partial E \rightarrow U(A)$  such that the following are satisfied.

- $z_k^{(r)}(1, 1) = 1$ .
- For all  $t \in [-1, 1]$ ,

$$\lim_{k \rightarrow \infty} \|z_k^{(r)}(-1, t) - u_{\eta_{r,1}}(k) \alpha_{\eta_{r,1}}(z_k^{(r)}(1, t))\| = 0$$

and

$$\lim_{k \rightarrow \infty} \|z_k^{(r)}(t, -1) - u_{\eta_{r,2}}(k) \alpha_{\eta_{r,2}}(z_k^{(r)}(t, 1))\| = 0.$$

- $\text{Lip}(z_k^{(r)}) \leq 4$ .
- The  $K_1$ -class of  $z_k^{(r)}$  in  $K_1(C(\partial E) \otimes A)$  is zero.

From Proposition 4.6, for sufficiently large  $k \in \mathbb{N}$ , we obtain  $\tilde{z}_k^{(r)} : E \rightarrow U(A)$  such that the following are satisfied.

- $z_k^{(r)}(t) = \tilde{z}_k^{(r)}(t)$  for all  $t \in \partial E$ .
- $\text{Lip}(\tilde{z}_k^{(r)}) \leq C'$ .

For each  $r = 1, 2, \dots, R$ ,  $g = (g_1, g_2) \in \mathbb{Z}_{m^{(r)}}$  and sufficiently large  $k \in \mathbb{N}$ , we define  $w_g^{(r)}(k)$  in  $U(A)$  by

$$w_g^{(r)}(k) = \tilde{z}_k^{(r)} \left( \frac{2g_1}{m_1^{(r)}} - 1, \frac{2g_2}{m_2^{(r)}} - 1 \right).$$

Let  $w_g^{(r)}$  be the image of  $(w_g^{(r)}(k))_{k \in \mathbb{N}}$  in  $A^\infty$ . It is easily seen that one has the following for any  $r = 1, 2, \dots, R$ ,  $i = 1, 2$  and  $g = (g_1, g_2) \in \mathbb{Z}_{m^{(r)}}$ .

- If  $g_i \neq 0$ , then  $\|w_g^{(r)} - w_{g-\xi_i}^{(r)}\|$  is less than  $\varepsilon$ .
- If  $g_i = 0$ , then the distance from  $w_g^{(r)}$  to  $u_{\eta_{r,i}} \alpha_{\eta_{r,i}}(w_{g+\eta_{r,i}-\xi_i}^{(r)})$  is less than  $\varepsilon$ .

We can take a family of projections  $\{e_g^{(r)} \mid r = 1, 2, \dots, R, g \in \mathbb{Z}_{m^{(r)}}\}$  in  $A_\infty$  such that

$$\sum_{r=1}^R \sum_{g \in \mathbb{Z}_{m^{(r)}}} e_g^{(r)} = 1, \quad \alpha_{\xi_i}(e_g^{(r)}) = e_{g+\xi_i}^{(r)}$$

for any  $r = 1, 2, \dots, R$ ,  $i = 1, 2$  and  $g \in \mathbb{Z}_{m^{(r)}}$ , where  $g + \xi_i$  is understood modulo  $m^{(r)}\mathbb{Z}^2$ . Moreover, we may also assume that  $e_g^{(r)}$  commutes with  $u_g$  and  $\alpha_g(w_g^{(r)})$ . Define  $v \in \mathcal{U}(A^\infty)$  by

$$v = \sum_{r=1}^R \sum_{g \in \mathbb{Z}_{m^{(r)}}} u_g \alpha_g(w_g^{(r)}) e_g^{(r)}.$$

It is now routinely checked that  $\|u_{\xi_i} - v \alpha_{\xi_i}(v^*)\|$  is less than  $\varepsilon$  for each  $i = 1, 2$ .  $\square$

When  $A$  is a UHF algebra, we can show the following in a similar fashion to the theorem above.

**Theorem 4.8.** *Let  $A$  be a UHF algebra and let  $\alpha$  be an action of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. Then, for any admissible  $\alpha$ -cocycle  $\{u_n\}_n$  in  $A_\infty$ , there exists a unitary  $v \in A_\infty$  such that*

$$u_n = v \alpha_n(v)^*$$

for any  $n \in \mathbb{Z}^2$ .

*Proof.* It suffices to show that, for any  $\varepsilon > 0$  and any unital full matrix subalgebra  $A_0 \subset A$ , there exists a unitary  $v \in A^\infty \cap A'_0$  such that

$$\|u_{\xi_i} - v\alpha_{\xi_i}(v)^*\| < \varepsilon$$

for each  $i = 1, 2$ .

The proof is almost the same as that of Theorem 4.7. Choose  $C' > 0$ ,  $M \in \mathbb{N}$  and  $m^{(1)}, m^{(2)}, \dots, m^{(R)} \in \mathbb{Z}^2$  in the same way. By using Lemma 3.6 instead of Lemma 3.5, we may assume that the range of the map  $z_k^{(r)}$  is  $U(A \cap A'_0)$  for all  $r = 1, 2, \dots, R$  and sufficiently large  $k \in \mathbb{N}$ . By Lemma 3.6, the  $K_1$ -class of  $z_k^{(r)}$  in  $K_1(C(\partial E) \otimes A)$  is zero. It follows that the  $K_1$ -class of  $z_k^{(r)}$  in  $K_1(C(\partial E) \otimes (A \cap A'_0))$  is also zero, because  $A_0$  is a full matrix subalgebra. We apply Proposition 4.6 to  $A \cap A'_0$  and  $z_k^{(r)}$ , and obtain a continuous map  $\tilde{z}_k^{(r)} : E \rightarrow U(A \cap A'_0)$ . The rest of the proof is exactly the same as Theorem 4.7.  $\square$

The following corollary is an immediate consequence of the theorem above and we omit the proof. Note that if  $A$  is a UHF algebra, then any  $\alpha$ -cocycle in  $A$  is admissible.

**Corollary 4.9.** *Let  $A$  be a UHF algebra and let  $\alpha$  be an action of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. For any  $\varepsilon > 0$  and a finite subset  $\mathcal{F}$  of  $A$ , there exist  $\delta > 0$  and a finite subset  $\mathcal{G}$  of  $A$  such that the following holds: If a family of unitaries  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A$  is an  $\alpha$ -cocycle satisfying*

$$\|[u_{\xi_i}, a]\| < \delta$$

for each  $i = 1, 2$  and  $a \in \mathcal{G}$ , then we can find a unitary  $v \in U(A)$  satisfying

$$\|u_{\xi_i} - v\alpha_{\xi_i}(v^*)\| < \varepsilon$$

and

$$\|[v, a]\| < \varepsilon$$

for each  $i = 1, 2$  and  $a \in \mathcal{F}$ .

## 5 Invariants

In this section, we would like to introduce an invariant  $[\alpha]$  for a  $\mathbb{Z}^2$ -action  $\alpha$  on a UHF algebra. Let  $A$  be a UHF algebra and let  $\tau$  be the unique tracial state on  $A$ . Throughout this section, we identify  $K_0(A)$  and  $\tau(K_0(A)) \subset \mathbb{R}$ . We need the generalized determinant introduced by P. de la Harpe and G. Skandalis (see [HS] for details). For any  $u \in U(A)$ , there exists a piecewise smooth path  $h : [0, 1] \rightarrow U(A)$  such that  $h(0) = 1$  and  $h(1) = u$ . It is not so hard to see that

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt$$

is real. The determinant  $\Delta_\tau$  associated with the tracial state  $\tau$  is the map from  $U(A)$  to  $\mathbb{R}/\tau(K_0(A))$  defined by

$$\Delta_\tau(u) = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt + \tau(K_0(A)).$$



A crucial fact here is that  $\Delta_\tau$  is a group homomorphism. We also notice that  $\Delta_\tau$  is  $\alpha$ -invariant for any  $\alpha \in \text{Aut}(A)$ , that is,  $\Delta_\tau \circ \alpha = \Delta_\tau$ .

Let  $A$  be a UHF algebra. For every prime number  $p$ , we define

$$\zeta(p) = \sup\{k \in \mathbb{N} \cup \{0\} \mid [1] \text{ is divisible by } p^k \text{ in } K_0(A)\} \in \{0, 1, 2, \dots, \infty\},$$

and let  $\mathcal{P}(A)$  be the set of prime numbers  $p$  such that  $1 \leq \zeta(p) < \infty$ . Let  $p \in \mathcal{P}(A)$ . We put  $\theta(p) = p^{\zeta(p)}$ . There exists a unital full matrix subalgebra  $A_0 \subset A$  which is isomorphic to  $M_{\theta(p)}$ . It is clear that the canonical map from  $K_0(A \cap A'_0)$  to  $K_0(A)$  is injective and  $K_0(A)/K_0(A \cap A'_0)$  is isomorphic to  $\mathbb{Z}/\theta(p)\mathbb{Z}$ . We also set

$$\mathcal{Q}(A) = \{\theta(p) \mid p \in \mathcal{P}(A)\} \cup \{p^n \mid p \text{ with } \zeta(p) = \infty \text{ and } n \in \mathbb{N}\},$$

so that  $A$  is isomorphic to  $\bigotimes_{q \in \mathcal{Q}(A)} M_q$ .

Let  $\alpha, \beta$  be actions of  $\mathbb{Z}^2$  on  $A$ . To simplify notation, we denote  $\alpha_{\xi_i}, \beta_{\xi_i}$  by  $\alpha_i, \beta_i$  for each  $i = 1, 2$ . We would like to introduce an invariant  $[\beta, \alpha]$ . Choose a positive real number  $\delta_0 > 0$  so that, for any unital finite dimensional  $C^*$ -subalgebra  $A_0 \subset A$  and a unitary  $w \in A$  satisfying  $\|[w, a]\| < \delta_0$  for all  $a \in U(A_0)$ , there exists a piecewise smooth path of unitaries  $h : [0, 1] \rightarrow U(A)$  such that  $h(0) = 1$ ,  $h(1) = w$  and  $\|[h(t), a]\| < 1/8$  for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . We may assume  $\delta_0 < 1/8$ .

Take  $p \in \mathcal{P}(A)$  arbitrarily. Let  $A_0 \subset A$  be a unital full matrix subalgebra which is isomorphic to  $M_{\theta(p)}$ . Take a finite dimensional  $C^*$ -subalgebra  $A_1 \subset A$  so that for any unitary  $w \in U(A_0)$ , there exist unitaries  $a, b, c \in U(A_1)$  such that

$$\|w - a\| < \delta_0/7, \quad \|\beta_1^{-1}(w) - b\| < \delta_0/7 \quad \text{and} \quad \|\beta_2^{-1}(w) - c\| < \delta_0/7.$$

We can find unitaries  $u_1$  and  $u_2$  in  $A$  such that

$$\|\beta_i(a) - \text{Ad } u_i \circ \alpha_i(a)\| < \delta_0/7$$

for all  $a$  in  $\beta_1^{-1}(U(A_1)) \cup \beta_2^{-1}(U(A_1)) \cup \beta_1^{-1}\beta_2^{-1}(U(A_1))$ . Define  $x \in U(A)$  by

$$x = u_1 \alpha_1(u_2)(u_2 \alpha_2(u_1))^*.$$

It is easy to see that  $\|[x, a]\|$  is less than  $\delta_0$  for every  $a \in U(A_0)$ . There exists a piecewise smooth path of unitaries  $h : [0, 1] \rightarrow U(A)$  such that  $h(0) = 1$ ,  $h(1) = x$  and  $\|[h(t), a]\| < 1/2$  for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . Since  $\Delta_\tau$  is a group homomorphism and  $\alpha_i$ -invariant for each  $i = 1, 2$ , one has  $\Delta_\tau(x) = 0$ . Hence,

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt \in K_0(A).$$

We denote the image of this value in  $K_0(A)/K_0(A \cap A'_0) \cong \mathbb{Z}/\theta(p)\mathbb{Z}$  by  $[\beta, \alpha](p)$ . Thus,

$$\begin{aligned} [\beta, \alpha](p) &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt + K_0(A \cap A'_0) \\ &\in K_0(A)/K_0(A \cap A'_0) \cong \mathbb{Z}/\theta(p)\mathbb{Z}. \end{aligned}$$

**Lemma 5.1.** *In the setting above,  $[\beta, \alpha](p)$  does not depend on the choice of  $A_0, A_1, u_1, u_2$  and  $h$ .*

*Proof.* Let  $k : [0, 1] \rightarrow U(A)$  be another piecewise smooth path such that  $k(0) = 1, k(1) = x$  and  $\| [k(t), a] \| < 1/2$  for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . By connecting  $h$  and  $k$ , we get a closed path  $H$  which almost commutes with elements in  $U(A_0)$ . Clearly  $H$  is homotopic to a closed path in  $U(A \cap A'_0)$ , and so the difference between

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt$$

and

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{k}(t)k(t)^*) dt$$

is in  $K_0(A \cap A'_0)$ . Therefore,  $[\beta, \alpha](p) \in K_0(A)/K_0(A \cap A'_0)$  does not depend on the choice of the path  $h$ .

Next, we would like to verify that  $[\beta, \alpha](p)$  does not depend on the choice of  $u_1$ . Suppose that  $u'_1 \in U(A)$  also satisfies

$$\| \beta_1(a) - \text{Ad } u'_1 \circ \alpha_1(a) \| < \delta_0/7$$

for all  $a$  in  $\beta_1^{-1}(U(A_1)) \cup \beta_2^{-1}(U(A_1)) \cup \beta_1^{-1}\beta_2^{-1}(U(A_1))$ . Put  $v = u'_1 u_1^*$ . Then we have  $\| [a, v] \| < 2\delta_0/7$  for all  $a \in U(A_1)$ . Hence we can find a piecewise smooth path of unitaries  $k : [0, 1] \rightarrow U(A)$  such that  $k(0) = 1, k(1) = v$  and  $\| [k(t), a] \| < 1/8$  for all  $t \in [0, 1]$  and  $a \in U(A_1)$ . It follows that

$$\| [k(t), a] \| < \frac{2\delta_0}{7} + \frac{1}{8} \quad \text{and} \quad \| [k(t), \beta_2^{-1}(a)] \| < \frac{2\delta_0}{7} + \frac{1}{8}$$

for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . For the unitary  $x = u_1 \alpha_1(u_2)(u_2 \alpha_2(u_1))^*$ , there exists a piecewise smooth path of unitaries  $h : [0, 1] \rightarrow U(A)$  such that  $h(0) = 1, h(1) = x$  and  $\| [h(t), a] \| < 1/8$  for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . Define  $H : [0, 1] \rightarrow U(A)$  by

$$H(t) = k(t)h(t)u_2\alpha_2(k(t)^*)u_2^*.$$

We can see that

$$H(0) = 1, \quad H(1) = u'_1 \alpha_1(u_2)(u_2 \alpha_2(u'_1))^*$$

and

$$\| [H(t), a] \| < \frac{2\delta_0}{7} + \frac{1}{8} + \frac{1}{8} + \frac{5\delta_0}{7} + \frac{1}{8} = \delta_0 + 3/8 < 1/2$$

for all  $t \in [0, 1]$  and  $a \in U(A_0)$ . In addition, by [HS, Lemma 1],

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{H}(t)H(t)^*) dt = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt$$

Therefore,  $[\beta, \alpha](p)$  does not depend on the choice of  $u_1$ . Likewise, we can show that  $[\beta, \alpha](p)$  does not depend on the choice of  $u_2$ .

In order to show that  $[\beta, \alpha](p)$  does not depend on the choice of  $A_1$ , we take another finite dimensional  $C^*$ -subalgebra  $A_2$  so that, for any unitary  $w \in U(A_0)$ , there exist unitaries  $a, b, c \in U(A_2)$  such that

$$\|w - a\| < \delta_0/7, \quad \|\beta_1^{-1}(w) - b\| < \delta_0/7 \quad \text{and} \quad \|\beta_2^{-1}(w) - c\| < \delta_0/7.$$

Since we can find unitaries  $u_1, u_2$  in  $A$  such that

$$\|\beta_i(a) - \text{Ad } u_i \circ \alpha_i(a)\| < \delta_0/7$$

for all  $a$  in

$$\beta_1^{-1}(U(A_1) \cup U(A_2)) \cup \beta_2^{-1}(U(A_1) \cup U(A_2)) \cup \beta_1^{-1}\beta_2^{-1}(U(A_1) \cup U(A_2)),$$

the assertion is clear.

Finally, we observe that  $[\beta, \alpha](p)$  does not depend on the choice of  $A_0 \cong M_{\theta(p)}$ . Suppose that  $B_0$  is also a full matrix subalgebra of  $A$  which is isomorphic to  $M_{\theta(p)}$ . By taking the finite dimensional  $C^*$ -subalgebra  $A_1$  so large, we may assume that  $x = u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*$  can be connected to the identity by a piecewise smooth path of unitaries which almost commute with both  $A_0$  and  $B_0$ . Hence,  $[\beta, \alpha](p)$  does not depend on the choice of  $A_0$ .  $\square$

We put  $[\beta, \alpha] \in \prod_{p \in \mathcal{P}(A)} \mathbb{Z}/\theta(p)\mathbb{Z}$  by

$$[\beta, \alpha] = ([\beta, \alpha](p))_{p \in \mathcal{P}(A)}.$$

Furthermore, we set  $[\alpha] = [\text{id}, \alpha]$ . We will show that  $[\alpha]$  is the complete invariant for cocycle conjugacy in the next section.

**Lemma 5.2.** *Let  $A$  be a UHF algebra and let  $\alpha, \beta, \gamma$  be actions of  $\mathbb{Z}^2$  on  $A$ . Then one has  $[\gamma, \alpha] = [\gamma, \beta] + [\beta, \alpha]$ .*

*Proof.* Take  $p \in \mathcal{P}(A)$  arbitrarily. We will prove  $[\gamma, \alpha](p) = [\gamma, \beta](p) + [\beta, \alpha](p)$ . To simplify notation, we denote  $\alpha_{\xi_i}, \beta_{\xi_i}, \gamma_{\xi_i}$  by  $\alpha_i, \beta_i, \gamma_i$  for each  $i = 1, 2$ . Let  $A_0 \subset A$  be a unital full matrix subalgebra which is isomorphic to  $M_{\theta(p)}$ . Choose a sufficiently large finite subset  $\mathcal{F} \subset A$  and a sufficiently small positive real number  $\delta > 0$ . There exist unitaries  $v_1, v_2$  such that

$$\|\gamma_i(a) - \text{Ad } v_i \circ \beta_i(a)\| < \delta$$

for all  $a \in \mathcal{F}$  and  $i = 1, 2$ . We may also find piecewise smooth paths of unitaries  $u_1, u_2 : [1, \infty) \rightarrow U(A)$  such that

$$\beta_i(a) = \lim_{s \rightarrow \infty} \text{Ad } u_i(s) \circ \alpha_i(a)$$

for all  $a \in A$ . We may further assume

$$\|\beta_i(a) - \text{Ad } u_i(s) \circ \alpha_i(a)\| < \delta$$

for all  $a \in \mathcal{F} \cup \{v_1, v_2\}$ ,  $s \in [1, \infty)$  and  $i = 1, 2$ . We write  $u_1 = u_1(1)$  and  $u_2 = u_2(1)$ . Define  $x, y \in U(A)$  by

$$x = u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*$$

and

$$y = v_1\beta_1(v_2)(v_2\beta_2(v_1))^*.$$

There exist piecewise smooth paths of unitaries  $h, k : [0, 1] \rightarrow U(A)$  such that

$$h(0) = k(0) = 1, \quad h(1) = x, \quad k(1) = y$$

and  $h(t), k(t)$  almost commute with unitaries in  $A_0$ . We have

$$\begin{aligned} & [\gamma, \beta](p) + [\beta, \alpha](p) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt + \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{k}(t)k(t)^*) dt + K_0(A \cap A'_0) \end{aligned}$$

Since

$$\|\gamma_i(a) - \text{Ad } v_i u_i \circ \alpha_i(a)\| < 2\delta$$

for all  $a \in \mathcal{F}$  and  $i = 1, 2$ ,

$$z = v_1 u_1 \alpha_1(v_2 u_2)(v_2 u_2 \alpha_2(v_1 u_1))^*$$

almost commutes with unitaries in  $A_0$ . We may also assume that

$$z(t) = (v_1 u_1 \alpha_1(v_2) u_1^*) h(t) (v_2 u_2 \alpha_2(v_1) u_2^*)^*$$

almost commutes with unitaries in  $A_0$  for every  $t \in [0, 1]$ . It is straightforward to see  $z(1) = z$ . Let  $w : [0, 1] \rightarrow U(A)$  be a path of unitaries such that  $w(0) = 0$ ,  $w(1) = z(0)$  and  $w(t)$  almost commutes with unitaries in  $A_0$ . Then, one has

$$\begin{aligned} & [\gamma, \alpha](p) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{w}(t)w(t)^*) dt + \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{z}(t)z(t)^*) dt + K_0(A \cap A'_0) \\ &= \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{w}(t)w(t)^*) dt + \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{h}(t)h(t)^*) dt + K_0(A \cap A'_0). \end{aligned}$$

Therefore, it suffices to show

$$\frac{1}{2\pi\sqrt{-1}} \left( \int_0^1 \tau(\dot{k}(t)k(t)^*) dt - \int_0^1 \tau(\dot{w}(t)w(t)^*) dt \right) \in K_0(A \cap A'_0).$$

By connecting  $k$ ,  $w$  and a short path from  $y$  to  $z(0)$ , we obtain a closed path  $H$  which is almost commuting with unitaries in  $A_0$ . Let  $\lambda \in K_0(A \cap A'_0) \subset \mathbb{R}$  be the  $K_0$ -class determined by the closed path  $H$ . We would like to show

$$\lambda = \frac{1}{2\pi\sqrt{-1}} \left( \int_0^1 \tau(\dot{k}(t)k(t)^*) dt - \int_0^1 \tau(\dot{w}(t)w(t)^*) dt \right).$$

For  $s \in [1, \infty)$ , we let

$$f(s) = (v_1 u_1(s) \alpha_1(v_2) u_1(s)^*) (v_2 u_2(s) \alpha_2(v_1) u_2(s)^*)^*$$

Thus,  $f$  is a path from  $z(0)$  to  $y$ . For any  $\varepsilon > 0$ , there exists  $s \in [1, \infty)$  such that

$$\|\beta_1(v_2) - \text{Ad } u_1(s) \circ \alpha_1(v_2)\| < \varepsilon$$

and

$$\|\beta_2(v_1) - \text{Ad } u_2(s) \circ \alpha_2(v_1)\| < \varepsilon.$$

We define

$$a = \frac{1}{2\pi\sqrt{-1}} \log(f(s)y^*).$$

Then  $\|a\|$  is less than  $\arcsin 2\varepsilon$ . By connecting paths  $k, w, f : [1, s] \rightarrow U(A)$  and

$$g : t \mapsto e^{2\pi\sqrt{-1}ta}y,$$

we obtain a closed path  $H'$ . Clearly  $H$  is homotopic to  $H'$ , because they are close to each other. Moreover, we can see

$$\frac{1}{2\pi\sqrt{-1}} \int_1^s \tau(\dot{f}(t)f(t)^*) dt = 0$$

and

$$\left| \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\dot{g}(t)g(t)^*) dt \right| = |\tau(a)| < \arcsin 2\varepsilon.$$

It follows that

$$\left| \lambda - \frac{1}{2\pi\sqrt{-1}} \left( \int_0^1 \tau(\dot{k}(t)k(t)^*) dt - \int_0^1 \tau(\dot{w}(t)w(t)^*) dt \right) \right| < \arcsin 2\varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we get the conclusion.  $\square$

We need to recall the notion of outer conjugacy. Two actions  $\alpha$  and  $\beta$  of  $\mathbb{Z}^2$  on  $A$  is said to be outer conjugate, if there exist an automorphism  $\sigma \in \text{Aut}(A)$  and unitaries  $\{u_n\}_{n \in \mathbb{Z}^2} \subset U(A)$  such that

$$\text{Ad } u_n \circ \alpha_n(a) = \sigma \circ \beta_n \circ \sigma^{-1}(a)$$

for any  $n \in \mathbb{Z}^2$  and  $a \in A$ . Notice that  $\{u_n\}_n$  is not necessarily an  $\alpha$ -cocycle.

The following lemma is essentially contained in [N1].

**Lemma 5.3.** *Let  $A$  be a UHF algebra. For any  $f \in \prod_{p \in \mathcal{P}(A)} \mathbb{Z}/\theta(p)\mathbb{Z}$ , we can construct a uniformly outer  $\mathbb{Z}^2$ -action  $\gamma^f$  on  $A$  such that  $[\gamma^f] = f$ . Moreover, if  $f, g \in \prod \mathbb{Z}/\theta(p)\mathbb{Z}$  satisfy  $f(p) = g(p)$  for all but finitely many  $p \in \mathcal{P}(A)$ , then  $\gamma^f$  is outer conjugate to  $\gamma^g$ .*

*Proof.* Let

$$A = \bigotimes_{q \in \mathcal{Q}(A)} M_q$$

be a tensor product decomposition. For  $q \in \mathcal{Q}(A)$ , we define unitaries  $u_q, v_q \in M_q$  by

$$u_q = \begin{bmatrix} 1 & & & & & \\ & \omega & & & & \\ & & \omega^2 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \omega^{q-1} \end{bmatrix}, \quad v_q = \begin{bmatrix} 0 & \cdot & \cdots & \cdot & 0 & 1 \\ 1 & 0 & \cdots & & \cdot & 0 \\ 0 & 1 & \cdot & & & \cdot \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \cdot & & 0 & 1 & 0 & 0 \\ 0 & \cdot & \cdots & 0 & 1 & 0 \end{bmatrix},$$

where  $\omega = e^{2\pi\sqrt{-1}/q}$ . Take a decomposition  $\mathcal{Q}(A) = L_1 \cup L_2$  such that both  $L_1$  and  $L_2$  are infinite sets.

Take  $f \in \prod_{p \in \mathcal{P}(A)} \mathbb{Z}/\theta(p)\mathbb{Z}$ . For each  $p \in \mathcal{P}(A)$ , we regard  $f(p)$  as an element of  $\{0, 1, \dots, \theta(p) - 1\}$ . Define a  $\mathbb{Z}^2$ -action  $\gamma^f$  on  $A$  by

$$\gamma_{\xi_1}^f = \bigotimes_{q=\theta(p) \in L_f} \text{Ad } u_q^{f(p)} \otimes \bigotimes_{q \in L_1 \setminus L_f} \text{id} \otimes \bigotimes_{q \in L_2 \setminus L_f} \text{Ad } u_q.$$

and

$$\gamma_{\xi_2}^f = \bigotimes_{q \in L_f} \text{Ad } v_q \otimes \bigotimes_{q \in L_1 \setminus L_f} \text{Ad } v_q \otimes \bigotimes_{q \in L_2 \setminus L_f} \text{id}$$

where  $L_f = \{\theta(p) \in \mathcal{Q}(A) \mid f(p) \neq 0\}$ . We can verify that  $\gamma^f$  has the Rohlin property and  $[\gamma^f] = f$  (see the proof of [N1, Theorem 15 (1)]). The latter assertion is easy to see.  $\square$

## 6 Classification

Let  $A$  be a UHF algebra and let  $\tau$  be the unique tracial state on  $A$ . Throughout this section, we identify  $K_0(A)$  and  $\tau(K_0(A)) \subset \mathbb{R}$ . We begin with the following lemma.

**Lemma 6.1.** *Let  $A$  be a UHF algebra and let  $\alpha \in \text{Aut}(A)$  be an automorphism with the Rohlin property. Then, for any unitary  $u \in A_\infty$ , there exists a unitary  $v \in A_\infty$  such that*

$$u = v\alpha(v^*).$$

*Proof.* This is the cohomology vanishing for single automorphisms. The reader should see [HO] and [I1, Lemma 2.9].  $\square$

By means of the lemma above, one obtains the following.

**Lemma 6.2.** *Let  $A$  be a UHF algebra and let  $\alpha, \beta$  be actions of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. Suppose that there exists a unitary  $u_1 \in A$  such that  $\beta_{\xi_1}(a) = \text{Ad } u_1 \circ \alpha_{\xi_1}(a)$  for all  $a \in A$ . If  $[\alpha] = [\beta]$ , then for any finite subset  $\mathcal{F} \subset A$  and  $\varepsilon > 0$ , there exists  $u_2 \in U(A)$  satisfying the following.*

- (1)  $\|\beta_{\xi_2}(a) - \text{Ad } u_2 \circ \alpha_{\xi_2}(a)\|$  is less than  $\varepsilon$  for all  $a \in \mathcal{F}$ .
- (2)  $\|u_1 \alpha_{\xi_1}(u_2) - u_2 \alpha_{\xi_2}(u_1)\|$  is less than  $\varepsilon$ .
- (3)  $\kappa(u_1, u_2, \alpha_{\xi_1}, \alpha_{\xi_2})$  is zero.

*Proof.* To simplify notation, we denote  $\alpha_{\xi_i}, \beta_{\xi_i}$  by  $\alpha_i, \beta_i$  for each  $i = 1, 2$ . First, we claim the following: there exists a unitary  $u_2 \in A^\infty$  such that  $u_1 \alpha_1(u_2) = u_2 \alpha_2(u_1)$  and  $\beta_2(a) = \text{Ad } u_2 \circ \alpha_2(a)$  for all  $a \in A$ . Indeed, we can find  $x \in U(A^\infty)$  such that  $\beta_2(a) = \text{Ad } x \circ \alpha_2(a)$  for all  $a \in A$ . It is easy to see that  $y = u_1 \alpha_1(x)(x \alpha_2(u_1))^*$  is a unitary in  $A_\infty$ . Hence, from Lemma 6.1, one can find a unitary  $z \in A_\infty$  such that

$$y^* = z(\text{Ad } u_1 \circ \alpha_1)(z^*).$$

Then,  $u_2 = z^* x \in A^\infty$  meets the requirements.

Choose  $\delta > 0$  so that  $4\delta + \arcsin \delta < \varepsilon$ . Let

$$A = \bigotimes_{q \in \mathcal{Q}(A)} M_q$$

be a tensor product decomposition. Let  $\omega \in \text{Aut}(A)$  be a product type automorphism such that

$$\omega(e_{i,j}^{(q)}) = e_{i+1,j+1}^{(q)},$$

where  $(e_{i,j}^{(q)})_{i,j}$  is a system of matrix units in  $M_q$ . Then,  $\omega$  has the Rohlin property. Hence, by Theorem 2.2, there exists a unitary  $w$  and  $\sigma \in \text{Aut}(A)$  such that

$$\|w - 1\| < \delta \quad \text{and} \quad \text{Ad } wu_1 \circ \alpha_1 = \sigma \circ \omega \circ \sigma^{-1}.$$

Choose a full matrix subalgebra  $A_0 \cong M_r \subset A$  so that the following hold.

- $A_0$  is a tensor product of finitely many  $M_q$ 's above.
- For any  $a \in \mathcal{F}$ , there exists  $b \in A_0$  such that  $\|\sigma^{-1}(\beta_2(a)) - b\| < \delta$ .

Let  $\delta' > 0$  be a small positive number and let  $\mathcal{G} \subset A$  be a finite subset containing  $\mathcal{F}$ . From the claim above, we can find a unitary  $u_2 \in A$  such that

- $\|\beta_2(a) - \text{Ad } u_2 \circ \alpha_2(a)\|$  is less than  $\delta'$  for all  $a \in \mathcal{G}$ .
- $\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\|$  is less than  $\delta$ .

Put

$$a = \frac{1}{2\pi\sqrt{-1}} \log(u_1\alpha_1(u_2)(u_2\alpha_2(u_1))^*).$$

We let  $\kappa(u_1, u_2, \alpha_1, \alpha_2) = m/l$ , where  $m \in \mathbb{Z}$  and  $l \in \mathbb{N}$  are relatively prime. By Lemma 3.1,  $|m/l| = |\tau(a)|$  is less than  $(2\pi)^{-1} \arcsin \delta$ .

Let  $P = \{p \in \mathcal{P}(A) \mid p \text{ divides } r\}$ . For  $p \in P$ , let  $A_p \subset A_0$  be a unital full matrix subalgebra which is isomorphic to  $M_{\theta(p)}$ . By virtue of Lemma 5.2, we have  $[\beta, \alpha] = 0$ . It follows that, by taking  $\mathcal{G}$  so large and  $\delta' > 0$  so small, we may assume that

$$[\beta, \alpha](p) = \tau(a) + K_0(A \cap A'_p)$$

for all  $p \in P$ . Hence  $\tau(a)$  belongs to  $K_0(A \cap A'_p)$  for each  $p \in P$ . Thus,  $\tau(a) = m/l$  is in  $K_0(A \cap A'_0)$ , and so there exists a projection  $e$  in  $A \cap A'_0$  such that

$$e = \omega^l(e), \quad e + \omega(e) + \cdots + \omega^{l-1}(e) = 1.$$

Define a unitary  $v \in A$  by

$$v = \sum_{j=0}^{l-1} e^{2\pi\sqrt{-1}jm/l} \sigma(\omega^j(e)).$$

It is easily seen that

$$\|v - \text{Ad } wu_1 \circ \alpha_1(v)\| < 2\pi|m/l| < \arcsin \delta$$

and

$$\|[\beta_2(a), v]\| < 2\delta$$

for all  $a \in \mathcal{F}$ . Besides, it can be also checked that

$$\kappa(1, v, \text{Ad } wu_1 \circ \alpha_1, \text{id}) = -m/l.$$

We put  $u'_1 = wu_1$  and  $u'_2 = vu_2$ . Clearly we have

$$\|\beta_i(a) - u'_i \alpha_i(a) u_i'^*\| < 4\delta$$

for all  $a \in \mathcal{F}$  and  $i = 1, 2$ . In addition,

$$\|u'_1 \alpha_1(u'_2) - u'_2 \alpha_2(u'_1)\| < 4\delta + \arcsin \delta.$$

From Lemma 3.4, one also gets

$$\begin{aligned} \kappa(u'_1, u'_2, \alpha_1, \alpha_2) &= \kappa(wu_1, vu_2, \alpha_1, \alpha_2) \\ &= \kappa(wu_1, u_2, \alpha_1, \alpha_2) + \kappa(1, v, \text{Ad } wu_1 \circ \alpha_1, \text{id}) \\ &= \kappa(u_1, u_2, \alpha_1, \alpha_2) - m/l = 0, \end{aligned}$$

thereby completing the proof.  $\square$

**Lemma 6.3.** *Let  $A$  be a UHF algebra and let  $\alpha, \beta$  be actions of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. If  $[\alpha] = [\beta]$ , then there exists an admissible  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  in  $A^\infty$  such that*

$$\beta_n(a) = \text{Ad } u_n \circ \alpha_n(a)$$

for all  $n \in \mathbb{Z}^2$  and  $a \in A$ .

*Proof.* To simplify notation, we denote  $\alpha_{\xi_i}, \beta_{\xi_i}$  by  $\alpha_i, \beta_i$ . It suffices to show the following. For any  $\varepsilon > 0$  and a finite subset  $\mathcal{F}$  of  $A$ , there exist unitaries  $u_1, u_2$  in  $A$  such that the following are satisfied.

- $\|\beta_i(a) - \text{Ad } u_i \circ \alpha_i(a)\|$  is less than  $\varepsilon$  for all  $a \in \mathcal{F}$  and  $i = 1, 2$ .
- $\|u_1 \alpha_1(u_2) - u_2 \alpha_2(u_1)\|$  is less than  $\varepsilon$ .
- $\kappa(u_1, u_2, \alpha_1, \alpha_2)$  is zero.

By Theorem 2.2, there exists a unitary  $w_1 \in A$  and  $\sigma \in \text{Aut}(A)$  such that

$$\beta_1 = \text{Ad } w_1 \circ \sigma \circ \alpha_1 \circ \sigma^{-1}.$$

Let  $\alpha'_i = \sigma \circ \alpha_i \circ \sigma^{-1}$  for  $i = 1, 2$ . By Lemma 6.2, there exists  $w_2 \in U(A)$  such that

- $\|\beta_2(a) - \text{Ad } w_2 \circ \alpha'_2(a)\|$  is less than  $\varepsilon/2$  for all  $a \in \mathcal{F}$ .
- $\|w_1 \alpha'_1(w_2) - w_2 \alpha'_2(w_1)\|$  is less than  $\varepsilon/2$ .
- $\kappa(w_1, w_2, \alpha'_1, \alpha'_2)$  is zero.



We can find a unitary  $v \in U(A)$  such that

$$\|\sigma(a) - vav^*\| < \varepsilon/8$$

for all  $a$  in

$$\sigma^{-1}(\mathcal{G}) \cup \alpha_1(\sigma^{-1}(\mathcal{G})) \cup \alpha_2(\sigma^{-1}(\mathcal{G})),$$

where  $\mathcal{G} = \mathcal{F} \cup \{w_1, w_2\}$ . Then the unitaries  $u_1 = w_1v\alpha_1(v^*)$  and  $u_2 = w_2v\alpha_2(v^*)$  satisfy

$$\|\beta_i(a) - \text{Ad } u_i \circ \alpha_i(a)\| < \varepsilon$$

for all  $a \in \mathcal{F}$  and  $i = 1, 2$ , and

$$\|u_1\alpha_1(u_2) - u_2\alpha_2(u_1)\| < \varepsilon.$$

Besides, one can see

$$\begin{aligned} \kappa(u_1, u_2, \alpha_1, \alpha_2) &= \kappa(w_1v\alpha_1(v^*), w_2v\alpha_2(v^*), \alpha_1, \alpha_2) \\ &= \kappa(v^*w_1v, v^*w_2v, \alpha_1, \alpha_2) \\ &= \kappa(\sigma(v^*w_1v), \sigma(v^*w_2v), \alpha'_1, \alpha'_2) \\ &= \kappa(w_1, w_2, \alpha'_1, \alpha'_2) \\ &= 0. \end{aligned}$$

This completes the proof. □

Combining the lemma above and Theorem 4.7, we can prove the following.

**Proposition 6.4.** *Let  $A$  be a UHF algebra and let  $\alpha, \beta$  be actions of  $\mathbb{Z}^2$  on  $A$  with the Rohlin property. If  $[\alpha] = [\beta]$ , then for any  $\varepsilon > 0$  and a finite subset  $\mathcal{F}$  of  $A$ , there exists an  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  (in fact a coboundary) in  $A$  such that*

$$\|\beta_{\xi_i}(a) - \text{Ad } u_{\xi_i} \circ \alpha_{\xi_i}(a)\| < \varepsilon$$

for all  $a \in \mathcal{F}$  and  $i = 1, 2$ .

Now we are ready to prove our main results. The key ingredient of the proof is the Evans-Kishimoto intertwining argument [EK, Theorem 4.1]. See also [N2, Theorem 5], [I2, Theorem 3.5] and [M, Theorem 5.2] for this argument.

**Theorem 6.5.** *Let  $A$  be a UHF algebra and let  $\alpha$  and  $\beta$  be two uniformly outer actions of  $\mathbb{Z}^2$  on  $A$ . Then, the following conditions are equivalent.*

- (1)  $[\alpha] = [\beta]$ .
- (2)  $\alpha$  is cocycle conjugate to  $\beta$ .

*Proof.* The implication from (2) to (1) is clear. It suffices to show that (1) implies (2). Note that, by Theorem 2.4, both  $\alpha$  and  $\beta$  have the Rohlin property. The proof is carried out by the Evans-Kishimoto intertwining argument. In each step of this argument, we have to use Proposition 6.4 and Corollary 4.9 repeatedly. We omit the detail, because it is exactly the same as [M, Theorem 5.2]. □

A UHF algebra  $A$  is said to be of infinite type, if  $A$  is isomorphic to  $A \otimes A$ . This is equivalent to say that  $\mathcal{P}(A)$  is empty. As an immediate consequence, we get the following corollary.

**Corollary 6.6.** *Let  $A$  be a UHF algebra of infinite type. Then, any two uniformly outer actions of  $\mathbb{Z}^2$  on  $A$  are cocycle conjugate.*

**Remark 6.7.** From Theorem 6.5 and Theorem 4.7, one can actually show the following: Let  $\alpha$  and  $\beta$  be uniformly outer actions of  $\mathbb{Z}^2$  on a UHF algebra  $A$ . If  $[\alpha] = [\beta]$ , then for any  $\varepsilon > 0$ , there exist  $\sigma \in \text{Aut}(A)$  and an  $\alpha$ -cocycle  $\{u_n\}_{n \in \mathbb{Z}^2}$  such that

$$\text{Ad } u_n \circ \alpha_n(a) = \sigma \circ \beta_n \circ \sigma^{-1}(a),$$

$$\|u_{\xi_i} - 1\| < \varepsilon$$

for any  $n \in \mathbb{Z}^2$ ,  $a \in A$  and  $i = 1, 2$ .

Finally, we would like to give a classification result up to outer conjugacy.

**Theorem 6.8.** *Let  $A$  be a UHF algebra and let  $\alpha$  and  $\beta$  be two uniformly outer actions of  $\mathbb{Z}^2$  on  $A$ . Then, the following conditions are equivalent.*

- (1)  $[\alpha](p) = [\beta](p)$  for all but finitely many  $p \in \mathcal{P}(A)$ .
- (2)  $\alpha$  is outer conjugate to  $\beta$ .

*Proof.* This follows from Theorem 6.5 and Lemma 5.3. □

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