# Classification of homomorphisms into simple $\mathcal{Z}$-stable $C^*$-algebras

Hiroki Matui
Graduate School of Science
Chiba University
Inage-ku, Chiba 263-8522, Japan

**Abstract**

We classify unital monomorphisms into certain simple $\mathcal{Z}$-stable $C^*$-algebras up to approximate unitary equivalence. The domain algebra $C$ is allowed to be any unital separable commutative $C^*$-algebra, or any unital simple separable nuclear $\mathcal{Z}$-stable $C^*$-algebra satisfying the UCT such that $C \otimes B$ is of tracial rank zero for a UHF algebra $B$. The target algebra $A$ is allowed to be any unital simple separable $\mathcal{Z}$-stable $C^*$-algebra such that $A \otimes B$ has tracial rank zero for a UHF algebra $B$, or any unital simple separable exact $\mathcal{Z}$-stable $C^*$-algebra whose projections separate traces and whose extremal traces are finitely many.

## 1 Introduction

Consider unital monomorphisms $\varphi, \psi : C \to A$ from a $C^*$-algebra $C$ to a simple $C^*$-algebra $A$. In this paper we study the problem to determine when $\varphi$ and $\psi$ are approximately unitarily equivalent, i.e. when there exists a sequence of unitaries $(u_n)_n$ in $A$ such that $\varphi(x) = \lim u_n \psi(x) u_n^*$ holds for any $x \in C$. This problem is known to be closely related to the classification problem for the simple $C^*$-algebra $A$. In the recent progress of Elliott's program to classify nuclear $C^*$-algebras via $K$-theoretic invariants (see [30] for an introduction to this subject), the Jiang-Su algebra plays a central role. The Jiang-Su algebra $\mathcal{Z}$, which was introduced by X. Jiang and H. Su in [12], is a unital, simple, separable, infinite dimensional, stably finite and nuclear $C^*$-algebra $KK$-equivalent to $\mathbb{C}$. A $C^*$-algebra $A$ is said to be $\mathcal{Z}$-stable if $A \otimes \mathcal{Z}$ is isomorphic to $A$. $\mathcal{Z}$-stability implies many nice properties from the point of view of classification theory. Among other things, if $A$ is a unital separable simple $\mathcal{Z}$-stable $C^*$-algebra, then $A$ is either purely infinite or stably finite. If, in addition, $A$ is stably finite, then $A$ must have stable rank one and weakly unperforated $K_0(A)$ (see [12, 7, 31]). All classes of unital simple infinite dimensional $C^*$-algebras for which Elliott's classification conjecture is confirmed consist of $\mathcal{Z}$-stable algebras. It is then natural to consider classification of unital monomorphisms from certain $C^*$-algebras into simple $\mathcal{Z}$-stable $C^*$-algebras which are not necessarily of real rank zero. In the present paper we give a positive solution for large classes of unital stably finite $C^*$-algebras (Theorem 6.6, Corollary 6.8, Theorem 7.1).

Classification of homomorphisms from $C(X)$ into a unital simple algebra has a long history. The earliest result for this subject is the classical Brown-Douglas-Fillmore theory

[3]. They showed that two unital monomorphisms $\varphi$ and $\psi$ from $C(X)$ to the Calkin algebra $B(H)/K(H)$ are unitarily equivalent if and only if $KK(\varphi) = KK(\psi)$. M. Dadarlat [4] showed that two monomorphisms from $C(X)$ to a unital simple purely infinite $C^*$-algebra are approximately unitarily equivalent if and only if they give the same element in $KL(C(X), A)$. In the case that the target algebra $A$ is stably finite, G. Gong and H. Lin [8] showed that for a unital simple separable $C^*$-algebra $A$ with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and a unique quasitrace $\tau$, two unital monomorphisms $\varphi, \psi : C(X) \to A$ are approximately unitarily equivalent if and only if $KL(\varphi) = KL(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$. H. Lin [17] obtained the same result for the case that the target algebra $A$ is of tracial rank zero. P. W. Ng and W. Winter [26] also obtained the same result for the case that $X$ is a path connected space and $A$ is a $\mathcal{Z}$-stable $C^*$-algebra of real rank zero. Similar classification up to approximate unitary equivalence is also known for more general domain algebras. G. A. Elliott [6] showed that two homomorphisms $\varphi$ and $\psi$ between AT algebras of real rank zero are approximately unitarily equivalent if and only if $K_i(\varphi) = K_i(\psi)$ for each $i = 0, 1$. K. E. Nielsen and K. Thomsen [25] obtained the analogous result for general AT algebras. H. Lin [17, 20] classified unital homomorphisms from AH algebras into simple separable $C^*$-algebras of tracial rank no more than one. Classification up to asymptotic unitary equivalence is also studied in [27, 13, 18].

It should be noted that all the target algebras in these results are assumed to have many non-trivial projections (and most of them are of real rank zero). Indeed almost nothing is known so far when the target algebra does not contain non-trivial projections. The present paper gives a first non-trivial general result for this subject. Our target algebras consist of two classes $\mathcal{C}$ and $\mathcal{C}'$. The class $\mathcal{C}$ is the family of all unital simple separable $\mathcal{Z}$-stable $C^*$-algebras $A$ such that $A \otimes Q$ has tracial rank zero, where $Q$ denotes the universal UHF algebra. The classification theorems in [35, 21] assert that any nuclear $C^*$-algebras $A, B \in \mathcal{C}$ satisfying the UCT are isomorphic if and only if their $K$-groups are isomorphic as graded ordered groups. The other class $\mathcal{C}'$ is the family of all unital simple separable stably finite $\mathcal{Z}$-stable exact $C^*$-algebras whose extremal traces are finitely many and whose projections separate traces. The Jiang-Su algebra $\mathcal{Z}$ itself is in $\mathcal{C} \cap \mathcal{C}'$ and any unital simple separable $\mathcal{Z}$-stable exact $C^*$-algebra with a unique trace is in $\mathcal{C}'$. In order to extend the target to $C^*$-algebras not necessarily of real rank zero, we need a new invariant $\Theta_{\varphi,\psi}$, which is a homomorphism from $K_1(C)$ to $\mathrm{Aff}(T(A))/\operatorname{Im} D_A$ (Lemma 3.1). Roughly speaking, if $A$ is of real rank zero, then the range of the dimension map $D_A$ is uniformly dense in $\mathrm{Aff}(T(A))$. Therefore this invariant trivially vanishes. When $A$ is not of real rank zero, it is not the case that $\operatorname{Im} D_A$ is dense in $\mathrm{Aff}(T(A))$, and so the homomorphism $\Theta_{\varphi,\psi}$ must be taken into account.

The paper is organized as follows. In Section 2 we collect preliminary material. The most important one is the notion of $\mathrm{Bott}(\varphi, u)$. In Section 3 we introduce the homomorphism $\Theta_{\varphi,\psi}$ for a pair of unital monomorphisms $\varphi, \psi : C \to A$. In Section 4 we give a classification theorem of unital monomorphisms from commutative $C^*$-algebras to certain unital simple $C^*$-algebras of real rank zero. The results in Section 4 (especially Theorem 4.5) partly overlap with those obtained in [17]. But the proof given in [17] is quite lengthy, and so we provide a simpler and self-contained proof for the reader's convenience. Section 5 is devoted to the proof of a version of the so called basic homotopy lemma (see [19]). In Section 6 we prove the classification theorem for the case that the domain algebra is commutative (Theorem 6.6) by combining the results obtained in Section 4 and 5. We also extend the classification theorem to the case that the domain is a unital AH algebra (Corollary 6.8). In Section 7 we prove the classification theorem for the case that the domain is a nuclear $C^*$-algebra in $\mathcal{C}$ satisfying the UCT (Theorem 7.1).

*Acknowledgement.* I would like to thank Huaxin Lin for valuable comments. I also like to thank the referee for a number of helpful comments.

# 2 Preliminaries

## 2.1 Notations

We let log be the standard branch defined on the complement of the negative real axis. For a Lipschitz continuous function $f$, we denote its Lipschitz constant by $\mathrm{Lip}(f)$. We denote by $\mathbb{K}$ the $C^*$-algebra of all compact operators on $\ell^2(\mathbb{Z})$. The normalized trace on $M_n$ is written by tr and the unnormalized trace on $M_n$ or $\mathbb{K}$ is written by Tr. The finite cyclic group of order $n$ is written by $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

Let $A$ be a $C^*$-algebra. For $a, b \in A$, we mean by $[a, b]$ the commutator $ab - ba$. We write $a \approx_\varepsilon b$ when $\|a - b\| < \varepsilon$. The set of tracial states on $A$ is denoted by $T(A)$ and the collection of all continuous bounded affine maps from $T(A)$ to $\mathbb{R}$ is denoted by $\mathrm{Aff}(T(A))$. We regard $\mathrm{Aff}(T(A))$ as a real Banach space with the sup norm. The dimension map $D_A : K_0(A) \to \mathrm{Aff}(T(A))$ is defined by $D_A([p])(\tau) = \tau(p)$. For a unital positive linear map $\varphi : A \to B$ between unital $C^*$-algebras, $T(\varphi) : T(B) \to T(A)$ denotes the affine continuous map induced by $\varphi$. We say that a $C^*$-algebra $A$ has strict comparison of projections if for projections $p, q \in A \otimes \mathbb{K}$, $(\tau \otimes \mathrm{Tr})(p) < (\tau \otimes \mathrm{Tr})(q)$ for any $\tau \in T(A)$ implies that $p$ is Murray-von Neumann equivalent to a subprojection of $q$. When $\varphi$ is a homomorphism between $C^*$-algebras, $K_0(\varphi)$ and $K_1(\varphi)$ mean the induced homomorphisms on $K$-groups.

A unital completely positive linear map is called a ucp map for short. A ucp map $\varphi : A \to B$ is said to be $(G, \delta)$-multiplicative if

$$\|\varphi(ab) - \varphi(a)\varphi(b)\| < \delta$$

holds for any $a, b \in G$, where $G$ is a subset of $A$. For two ucp maps $\varphi, \psi : A \to B$, we write $\varphi \sim_{G,\delta} \psi$, when there exists a unitary $u \in B$ such that

$$\|\varphi(a) - u\psi(a)u^*\| < \delta$$

holds for any $a \in G$.

## 2.2 The entire $K$-group

We recall the mod $p$ $K$-theory introduced by C. Schochet [33]. The $K_i$-group of a $C^*$-algebra $A$ with the coefficient module $\mathbb{Z}_n$ for $i = 0, 1$, $n \in \mathbb{N}$ is defined by

$$K_i(A; \mathbb{Z}_n) = K_i(A \otimes \mathcal{O}_{n+1}),$$

where $\mathcal{O}_{n+1}$ is the Cuntz algebra. For notational convenience, we set $K_i(A; \mathbb{Z}_0) = K_i(A)$. Although our definition looks different from the original one in [33], it gives an equivalent theory to the conventional one ([33, Theorem 6.4]). The entire $K$-group $\underline{K}(A)$ of $A$ is defined by

$$\underline{K}(A) = \bigoplus_{n=0}^{\infty} (K_0(A;\mathbb{Z}_n) \oplus K_1(A;\mathbb{Z}_n)).$$

For each $i = 0, 1$ and $n \in \mathbb{N}$, we have the Künneth exact sequence

$$0 \longrightarrow K_i(A) \otimes \mathbb{Z}_n \longrightarrow K_i(A;\mathbb{Z}_n) \longrightarrow \mathrm{Tor}(K_i(A), \mathbb{Z}_n) \longrightarrow 0.$$

It is known that this exact sequence splits unnaturally. For $C^*$-algebras $A, B$, we denote by $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ the set of all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ preserving the direct sum decomposition and commuting with natural coefficient transformations and the Bockstein operations (see [5, 14] for details). M. Dadarlat and T. A. Loring [5] proved the following universal multicoefficient theorem.

**Theorem 2.1.** *Let $A$ be a $C^*$-algebra satisfying the UCT and let $B$ be a $\sigma$-unital $C^*$-algebra. Then there exists a short exact sequence*

$$0 \to \bigoplus_{i=0,1} \mathrm{Pext}(K_i(A), K_{1-i}(B)) \to KK(A, B) \to \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \to 0,$$

*where $\mathrm{Pext}(K_i(A), K_{1-i}(B))$ is the subgroup of $\mathrm{Ext}(K_i(A), K_{1-i}(B))$ consisting of the pure extensions. The sequence is natural in each variable.*

Let $A$ and $B$ be $C^*$-algebras. Suppose that $A$ satisfies the UCT and $B$ is $\sigma$-unital. In [30, Section 2.4], the $KL$-group $KL(A, B)$ is defined as the quotient of $KK(A, B)$ by the image of $\mathrm{Pext}(K_*(A), K_{1-*}(B))$. Thus, by the theorem above, $KL(A, B)$ is identified with $\mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$. Throughout this paper we keep this identification. For a homomorphism $\varphi : A \to B$, we denote by $K_i(\varphi;\mathbb{Z}_n)$ the homomorphism from $K_i(A;\mathbb{Z}_n)$ to $K_i(B;\mathbb{Z}_n)$ induced by $\varphi$. We set

$$KL(\varphi) = (K_i(\varphi;\mathbb{Z}_n))_{i,n} \in \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)).$$

If $\varphi : A \to B$ and $\psi : A \to B$ are approximately unitarily equivalent, then $KL(\varphi) = KL(\psi)$ holds (see [30]). For $\kappa \in KL(A, B) = \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ and $i = 0, 1$, we denote its $K_i$-component by $K_i(\kappa) \in \mathrm{Hom}(K_i(A), K_i(B))$.

### 2.3 Almost multiplicative ucp maps

For a $C^*$-algebra $A$, we mean by $P(A)$ the set of all projections of $A$. When $A$ is unital, we mean by $U(A)$ the set of all unitaries of $A$. The connected component of the identity in $U(A)$ is denoted by $U(A)_0$. Let $U_\infty(A)$ be the union of $U(A \otimes M_n)$'s via the embedding

$$U(A \otimes M_n) \ni u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U(A \otimes M_{n+1}).$$

Likewise, we let $U_\infty(A)_0$ denote the union of $U(A \otimes M_n)_0$'s.

For a unital $C^*$-algebra $A$, we set

$$\mathcal{K}_0(A) = P(A \otimes \mathbb{K}) \cup \bigcup_{n=1}^{\infty} P(A \otimes \mathcal{O}_{n+1}),$$

$$\mathcal{K}_1(A) = U_\infty(A) \cup \bigcup_{n=1}^{\infty} U(A \otimes \mathcal{O}_{n+1})$$

and $\mathcal{K}(A) = \mathcal{K}_0(A) \cup \mathcal{K}_1(A)$. Let $\varphi : A \to B$ be a $(G, \delta)$-multiplicative ucp map. For $p \in \mathcal{K}_0(A)$, if $G$ is sufficiently large and $\delta$ is sufficiently small, then $(\varphi \otimes \mathrm{id})(p)$ is close to a projection and one can consider its equivalence class in $K_0(B; \mathbb{Z}_n)$. We denote this class by $\varphi_\#(p)$. In a similar fashion, for $u \in \mathcal{K}_1(A)$, if $G$ is sufficiently large and $\delta$ is sufficiently small, then $(\varphi \otimes \mathrm{id})(u)$ is close to a unitary and one can consider its equivalence class in $K_1(B; \mathbb{Z}_n)$. We denote this class by $\varphi_\#(u)$. Thus, for any finite subset $L \subset \mathcal{K}(A)$, if $\varphi$ is a sufficiently multiplicative ucp map, then $\varphi_\# | L : L \to \underline{K}(B)$ is well-defined. In this paper, whenever we write $\varphi_\#(x)$ or $\varphi_\# | L$, the ucp map $\varphi$ is always assumed to be sufficiently multiplicative so that they are well-defined. When $\varphi$ is sufficiently multiplicative, we can verify the following easily: $\varphi_\#(p) = \varphi_\#(q)$ for Murray-von Neumann equivalent projections $p, q \in \mathcal{K}_0(A)$, $\varphi_\#(p+q) = \varphi_\#(p) + \varphi_\#(q)$ for orthogonal projections $p, q \in \mathcal{K}_0(A)$, $\varphi_\#(u) = 0$ for any $u \in U_\infty(A)_0 \cup U(A \otimes \mathcal{O}_{n+1})_0$ and $\varphi_\#(uv) = \varphi_\#(u) + \varphi_\#(v)$ for any $u, v \in \mathcal{K}_1(A)$. Therefore $\varphi_\#$ gives rise to a 'partial homomorphism' from $\underline{K}(A)$ to $\underline{K}(B)$.

Next, we would like to recall the notion of $\mathrm{Bott}(\varphi, w)$ introduced in [19]. Let $\varphi : A \to B$ be a unital homomorphism between unital $C^*$-algebras and let $w \in B$ be a unitary satisfying

$$\| [\varphi(a), w] \| < \delta$$

for every $a \in G$, where $G$ is a large finite subset of $A$ and $\delta > 0$ is a small positive real number. For a projection $p \in A \otimes C$, $(w \otimes 1)(\varphi \otimes \mathrm{id})(p) + (\varphi \otimes \mathrm{id})(1-p)$ in $B \otimes C$ is close to a unitary, where $C$ is $M_n$ or $\mathcal{O}_{n+1}$. We denote the equivalence class of this unitary by $\mathrm{Bott}(\varphi, w)(p) \in K_1(A \otimes C)$. Next, we would like to introduce $\mathrm{Bott}(\varphi, w)(u) \in K_0(A \otimes C)$ for a unitary $u \in A \otimes C$. To this end we need to recall the notion of Bott elements associated with almost commuting unitaries ([19, 2.11]). There exists a universal constant $\delta_0 > 0$ such that for any unitaries $v_1, v_2$ in a $C^*$-algebra $D$ satisfying $\| [v_1, v_2] \| < \delta_0$, the self-adjoint element

$$e(v_1, v_2) = \begin{pmatrix} f(v_1) & g(v_1) + h(v_1) v_2 \\ g(v_1) + v_2^* h(v_1) & 1 - f(v_1) \end{pmatrix} \in M_2(D)$$

has a spectral gap at $1/2$, where $f, g, h$ are certain universal real-valued continuous functions on $\mathbb{T}$ ([22, Section 3]). Then one can consider the $K_0$-class

$$[1_{[1/2, \infty)}(e(v_1, v_2))] - \left[ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \in K_0(D)$$

and call it the Bott element associated with $v_1, v_2$. In our setting, for a unitary $u \in A \otimes C$, we can consider the Bott element in $K_0(A \otimes C)$ corresponding to the almost commuting unitaries $(\varphi \otimes \mathrm{id})(u)$ and $w \otimes 1$. We denote it by $\mathrm{Bott}(\varphi, w)(u) \in K_0(A \otimes C)$. Thus, for a finite subset $L \subset \mathcal{K}(A)$, when $G$ is large enough and $\delta$ is small enough, then $\mathrm{Bott}(\varphi, w)|L : L \to \underline{K}(B)$ is well-defined. In this paper, whenever we write $\mathrm{Bott}(\varphi, w)|L$, $G$ and $\delta$ are always assumed to be chosen so that $\mathrm{Bott}(\varphi, w)|L$ is well-defined. In the same way as above, we can see that $\mathrm{Bott}(\varphi, w)$ gives rise to a 'partial homomorphism' from $K_i(A \otimes C)$ to $K_{1-i}(B \otimes C)$.

### 2.4 The target algebras

We denote the Jiang-Su algebra by $\mathcal{Z}$ ([12]). When a $C^*$-algebra $A$ satisfies $A \cong A \otimes \mathcal{Z}$, we say that $A$ is $\mathcal{Z}$-stable. We let $Q$ denote the universal UHF algebra, that is, $Q$ is the UHF algebra satisfying $K_0(Q) = \mathbb{Q}$.

We introduce four classes $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{C}$ and $\mathcal{C}'$ of unital simple separable stably finite $C^*$-algebras as follows.

**Definition 2.2.** We define $\mathcal{T}$ to be the class of all infinite dimensional unital simple separable $C^*$-algebras with tracial rank zero. Let $\mathcal{T}'$ be the class of infinite dimensional unital simple separable exact $C^*$-algebras $A$ with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and finitely many extremal tracial states. We let $\mathcal{C}$ be the class of unital simple separable $\mathcal{Z}$-stable $C^*$-algebras $A$ such that $A \otimes Q$ is in $\mathcal{T}$. Let $\mathcal{C}'$ be the class of unital simple separable stably finite $\mathcal{Z}$-stable exact $C^*$-algebras $A$ whose projections separate traces and whose extremal traces are finitely many.

**Remark 2.3.** (1) Any $A \in \mathcal{T}$ has real rank zero, stable rank one, weakly unperforated $K_0(A)$ and strict comparison of projections (see [14, Chapter 3]).

(2) Exactness of $A \in \mathcal{T}'$ is assumed only for the purpose of using the fact that any quasitrace on an exact $C^*$-algebra is a trace ([10]). By [1, Corollary 6.9.2], any $A \in \mathcal{T}'$ has strict comparison of projections.

(3) If $A \in \mathcal{C}$, then $A \otimes B$ has tracial rank zero for any UHF algebra $B$ by [24, Lemma 2.4], that is, $A \otimes B$ belongs to $\mathcal{T}$.

(4) Let $A \in \mathcal{C}'$ and let $B$ be a UHF algebra. Then $A \otimes B$ has real rank zero by [2, Theorem 1.4 (f)] and has stable rank one by [28, Corollary 6.6] (or [31]). By [29, Theorem 5.2] (or [7]), $K_0(A \otimes B)$ is weakly unperforated. It follows that $A \otimes B$ is in $\mathcal{T}'$.

(5) Of course, $\mathcal{Z}$ itself is in $\mathcal{C} \cap \mathcal{C}'$.

To continue, we fix a notation. Let $A$ and $B$ be unital stably finite $C^*$-algebras and let $\xi \in \mathrm{Hom}(K_0(A), K_0(B))$. We say that $\xi$ is unital when $\xi([1]) = [1]$. We say that $\xi$ is positive (resp. strictly positive) when $\xi(K_0(A)_+) \subset K_0(B)_+$ (resp. $\xi(K_0(A)_+ \setminus \{0\}) \subset K_0(B)_+ \setminus \{0\}$). Assume further that $A$ satisfies the UCT. We denote by $KL(A, B)_{+,1}$ the set of all $\kappa \in KL(A, B)$ such that $K_0(\kappa)$ is unital and strictly positive.

**Lemma 2.4.** *Let $X$ be a connected compact metrizable space and let $B$ be a unital stably finite $C^*$-algebra. For $\xi \in \mathrm{Hom}(K_0(C(X)), K_0(B))$ the following are equivalent.*

(1) *$\xi$ is unital and strictly positive.*

(2) *$\xi$ is unital and positive.*

*If $K_0(B)$ is simple and weakly unperforated, then the two conditions above are equivalent to the following condition.*

(3) *$\xi$ is unital and $\xi(\mathrm{Ker}\, D_{C(X)}) \subset \mathrm{Ker}\, D_B$.*

*Proof.* (1)⇒(2) is clear. To show (2)⇒(1), assume that $\xi$ is unital and positive. By [1, Corollary 6.3.6], $K_0(C(X))$ is a simple ordered group. Hence for any $x \in K_0(C(X))_+ \setminus \{0\}$ there exists $n \in \mathbb{N}$ such that $[1] \leq nx$. Then $[1] \leq n\xi(x)$ in $K_0(A)$, and so $\xi(x) \neq 0$.

Assume that $K_0(B)$ is simple and weakly unperforated. Let us show (2)⇒(3). Take $x \in \operatorname{Ker} D_{C(X)}$ and $\tau \in T(B)$. The composition of $\tau$ and $\xi$ is a state on $K_0(C(X))$. By [1, Corollary 6.10.3 (e)], any state on $K_0(C(X))$ comes from a trace. Therefore $\tau(\xi(x)) = 0$. It remains to show that (3) implies (1). Take $x \in K_0(C(X))_+ \setminus \{0\}$. Let $\rho$ be a state on $K_0(B)$. The composition of $\rho$ and $\xi$ is a state on $K_0(C(X))$. By [1, Corollary 6.10.3 (e)] it comes from a trace on $C(X)$, and so $\rho(\xi(x)) > 0$. By [1, Theorem 6.8.5], $K_0(B)$ has the strict ordering from its states. It follows that $\xi(x)$ is in $K_0(B) \setminus \{0\}$. □

We recall the following three theorems from [15, 26].

**Theorem 2.5** ([15, Corollary 4.6])**.** *Let $A$ be a unital simple separable $C^*$-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Then there exist a unital simple separable AH algebra $B$ with real rank zero and slow dimension growth and a unital homomorphism $\varphi : B \to A$ which induces a graded ordered isomorphism from $K_*(B)$ to $K_*(A)$.*

**Theorem 2.6** ([26, Theorem 0.1])**.** *Let $X$ be a path connected compact metrizable space and let $A$ be a unital simple separable exact $C^*$-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Let $\kappa \in KL(C(X), A)_{+,1}$ and let $\lambda : T(A) \to T(C(X))$ be an affine continuous map such that $\lambda(\tau)$ gives a strictly positive measure on $X$ for any $\tau \in T(A)$. Then there exists a unital monomorphism $\varphi : C(X) \to A$ such that $KL(\varphi) = \kappa$ and $T(\varphi) = \lambda$.*

**Theorem 2.7** ([26, Theorem 0.2])**.** *Let $X$ be a path connected compact metrizable space and let $A$ be a unital simple separable exact $\mathcal{Z}$-stable $C^*$-algebra with real rank zero. Let $\varphi, \psi : C(X) \to A$ be unital monomorphisms. Then $\varphi$ and $\psi$ are approximately unitarily equivalent if and only if $KL(\varphi) = KL(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(A)$.*

**Remark 2.8.** (1) The proof of [26, Theorem 0.1] uses [15, Corollary 4.6], and in the statement of [15, Corollary 4.6] $A$ is assumed to be nuclear. But the proof given there does not use nuclearity, and so we omit it. In the statement of [26, Theorem 0.2], $A$ is also assumed to be nuclear. But its proof needs only exactness of $A$.

(2) The condition (b) of [26, Theorem 0.1] automatically follows from other assumptions, because any traces on $C(X)$ induce the same state on $K_0(C(X))$ ([1, Corollary 6.10.3 (a)]) and $K_0(C(X))$ has no other states ([1, Corollary 6.10.3 (e)]).

We give a generalization of Theorem 2.6 for later use.

**Corollary 2.9.** *Let $C = \bigoplus_{i=1}^n p_i(C(X_i) \otimes M_{k_i})p_i$, where $X_i$ is a path connected compact metrizable space and $p_i \in C(X_i) \otimes M_{k_i}$ is a projection. Let $A$ be a unital simple separable exact $C^*$-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Let $\kappa \in KL(C, A)_{+,1}$ and let $\lambda : T(A) \to T(C)$ be an affine continuous map such that $\lambda(\tau)$ is a faithful trace for any $\tau \in T(A)$. Suppose that $\lambda(\tau)(p_i) = \tau(K_0(\kappa)([p_i]))$ holds for any $\tau \in T(A)$ and $i = 1, 2, \dots, n$. Then there exists a unital monomorphism $\varphi : C \to A$ such that $KL(\varphi) = \kappa$ and $T(\varphi) = \lambda$.*

*Proof.* It is clear that the case $C = C(X) \otimes M_k$ follows immediately from Theorem 2.6. Let us consider the case $C = p(C(X) \otimes M_k)p$, where $X$ is a path connected compact metrizable space. Let $m \in \mathbb{N}$ be the rank of $p$. There exist $l \in \mathbb{N}$ and a projection $q \in C \otimes M_l \subset C(X) \otimes M_{kl}$ such that $p \otimes e$ is a subprojection of $q$ and $q$ is Murray-von Neumann equivalent to $1_{C(X)} \otimes r$, where $e \in M_l$ is a minimal projection of $M_l$ and $r \in M_{kl}$ is a projection of rank $k$. We can find a projection $\tilde{q} \in A \otimes M_l$ such that $K_0(\kappa)([q]) = [\tilde{q}]$. Set $C_0 = q(C \otimes M_l)q \cong C(X) \otimes M_k$ and $A_0 = \tilde{q}(A \otimes M_l)\tilde{q}$. For any tracial state $\tau \in T(A)$, $mk^{-1}(\tau \otimes \mathrm{Tr})$ gives a tracial state on $A_0$, and this correspondence induces a homeomorphism between $T(A)$ and $T(A_0)$. Likewise there exists a natural homeomorphism between $T(C)$ and $T(C_0)$. The identifications

$$KL(C, A)_{+,1} \cong KL(C_0, A_0)_{+,1}, \quad T(A) \cong T(A_0) \quad \text{and} \quad T(C) \cong T(C_0)$$

allow us to regard $\kappa$ as an element of $KL(C_0, A_0)_{+,1}$ and $\lambda$ as an affine continuous map from $T(A_0)$ to $T(C_0)$. Therefore the previous case shows that there exists $\varphi : C_0 \to A_0$ realizing $\kappa$ and $\lambda$. From $[\varphi(p \otimes e)] = K_0(\kappa)([p \otimes e]) = [1_A \otimes e]$, there exists a unitary $u \in A \otimes M_l$ such that $u\varphi(p \otimes e)u^* = 1_A \otimes e$. The restriction of $\mathrm{Ad}\, u \circ \varphi$ to $(p \otimes e)C_0(p \otimes e)$ gives a desired unital monomorphism from $C$ to $A$.

We now turn to the general case. Let $C = \bigoplus_{i=1}^n p_i(C(X_i) \otimes M_{k_i})p_i$, $\kappa$ and $\lambda$ be as in the statement. Let $\gamma_i : p_i(C(X_i) \otimes M_{k_i})p_i \to C$ be the canonical embedding. Choose projections $\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n \in A$ so that $K_0(\kappa)([p_i]) = [\tilde{p}_i]$ and $\tilde{p}_1 + \tilde{p}_2 + \dots + \tilde{p}_n = 1$. For each $i = 1, 2, \dots, n$, $\kappa \circ KL(\gamma_i)$ is regarded as an element of $KL(p_iC, \tilde{p}_iA\tilde{p}_i)_{+,1}$. For $\tau \in T(A)$, the restriction of $\tau/\tau(\tilde{p}_i)$ to $\tilde{p}_iA\tilde{p}_i$ is a tracial state. Similarly the restriction of $\lambda(\tau)/\tau(\tilde{p}_i)$ to $p_iC$ is also a tracial state, because $\lambda(\tau)(p_i) = \tau(K_0(\kappa)([p_i])) = \tau(\tilde{p}_i)$. It is not so hard to check that

$$\lambda_i : \tau/\tau(\tilde{p}_i) \mapsto \lambda(\tau)/\tau(\tilde{p}_i)$$

gives rise to an affine continuous map from $T(\tilde{p}_iA\tilde{p}_i)$ to $T(p_iC)$. We have already shown that $\kappa \circ KL(\gamma_i)$ and $\lambda_i$ are realized by a unital monomorphism $\varphi_i : p_iC \to \tilde{p}_iA\tilde{p}_i$. Then $\varphi = \varphi_1 + \varphi_2 + \dots + \varphi_n$ is a unital monomorphism from $C$ to $A$ satisfying $KL(\varphi) = \kappa$ and $T(\varphi) = \lambda$. $\square$

## 3 Determinants of unitaries

In this section, we would like to introduce a homomorphism

$$\Theta_{\varphi,\psi} : K_1(C) \to \mathrm{Aff}(T(A))/\operatorname{Im} D_A,$$

which plays an important role in the main theorems of this paper (Theorem 6.6, Corollary 6.8, Theorem 7.1).

Let $A$ be a unital $C^*$-algebra. For $\tau \in T(A)$, the de la Harpe-Skandalis determinant ([11]) associated with $\tau$ is written by

$$\Delta_\tau : U_\infty(A)_0 \to \mathbb{R}/D_A(K_0(A))(\tau).$$

It is well-known that $\Delta_A(u)(\tau) = \Delta_\tau(u)$ gives a homomorphism

$$\Delta_A : U_\infty(A)_0 \to \mathrm{Aff}(T(A))/\operatorname{Im} D_A.$$

Let $C, A$ be unital $C^*$-algebras and let $\varphi, \psi : C \to A$ be unital homomorphisms satisfying $K_1(\varphi) = K_1(\psi)$ and $T(\varphi) = T(\psi)$. In what follows, we use the same notation $\varphi, \psi$ for the homomorphisms from $C \otimes M_n$ to $A \otimes M_n$ induced by $\varphi, \psi$. For $u \in U_\infty(C)$, we can consider $\Delta_A(\varphi(u^*)\psi(u))$, as $\varphi(u^*)\psi(u)$ belongs to $U_\infty(A)_0$.

**Lemma 3.1.** *In the setting above, we have the following.*

(1) *There exists a homomorphism*

$$\Theta_{\varphi,\psi} : K_1(C) \to \operatorname{Aff}(T(A))/\operatorname{Im} D_A$$

*such that* $\Theta_{\varphi,\psi}([u]) = \Delta_A(\varphi(u^*)\psi(u))$ *for any* $u \in U_\infty(C)$.

(2) *For any* $w \in U(A)$, $\Theta_{\varphi,\operatorname{Ad} w \circ \psi} = \Theta_{\varphi,\psi}$.

(3) *If* $\varphi$ *and* $\psi$ *are approximately unitarily equivalent, then* $\operatorname{Im} \Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$.

(4) *If* $C$ *satisfies the UCT and* $KL(\varphi)$ *equals* $KL(\psi)$, *then the homomorphism* $\Theta_{\varphi,\psi}$ *factors through* $K_1(C)/\operatorname{Tor}(K_1(C))$.

*Proof.* (1) We first show that $\Delta_A(\varphi(u^*)\psi(u))$ equals $\Delta_A(\varphi(v^*)\psi(v))$ when $u, v \in U_\infty(C)$ satisfy $uv^* \in U_\infty(C)_0$. We can find $n \in \mathbb{N}$ and piecewise smooth paths of unitaries $x : [0,1] \to U(A \otimes M_n)$, $y : [0,1] \to U(A \otimes M_n)$ and $z : [0,1] \to U(C \otimes M_n)$ such that $x(0) = \varphi(u)$, $x(1) = \psi(u)$, $y(0) = \varphi(v)$, $y(1) = \psi(v)$ and $z(0) = u$, $z(1) = v$. Define $h : [0,1] \to U(A \otimes M_n)$ by

$$h(t) = \begin{cases} x(4t) & 1 \le t \le 1/4 \\ \psi(z(4t-1)) & 1/4 \le t \le 1/2 \\ y(3-4t) & 1/2 \le t \le 3/4 \\ \varphi(z(4-4t)) & 3/4 \le t \le 1. \end{cases}$$

Since $h$ is a closed path of unitaries, one has

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 (\tau \otimes \operatorname{Tr})(\dot{h}(t)h(t)^*)\, dt \in D_A(K_0(A))(\tau)$$

for any $\tau \in T(A)$. It is easy to see that the contribution from $t \mapsto \psi(z(4t-1))$ and $t \mapsto \varphi(z(4-4t))$ cancels out, because of $T(\varphi) = T(\psi)$. It follows that

$$\frac{1}{2\pi\sqrt{-1}} \left( \int_0^1 (\tau \otimes \operatorname{Tr})(\dot{x}(t)x(t)^*)\, dt - \int_0^1 (\tau \otimes \operatorname{Tr})(\dot{y}(t)y(t)^*)\, dt \right) \in D_A(K_0(A))(\tau)$$

for any $\tau \in T(A)$, which implies $\Delta_A(\varphi(u^*)\psi(u)) = \Delta_A(\varphi(v^*)\psi(v))$. It follows that $\Theta_{\varphi,\psi} : K_1(C) \to \operatorname{Aff}(T(A))/\operatorname{Im} D_A$ is well-defined as a map by $\Theta_{\varphi,\psi}([u]) = \Delta_A(\varphi(u^*)\psi(u))$ for any $u \in U_\infty(C)$.

For any $u, v \in U_\infty(C)$, $\operatorname{diag}(uv, 1)$ is homotopic to $\operatorname{diag}(u, v)$, and so

$$\begin{aligned} \Delta_A(\operatorname{diag}(\varphi(uv)^*\psi(uv), 1)) &= \Delta_A(\operatorname{diag}(\varphi(u)^*\psi(u), \varphi(v)^*\psi(v))) \\ &= \Delta_A(\varphi(u^*)\psi(u)) + \Delta_A(\varphi(v^*)\psi(v)). \end{aligned}$$

Hence we can conclude that $\Theta_{\varphi,\psi}$ is a homomorphism.

(2) can be shown in a similar fashion to the proof of (ii)⇒(i) of [13, Theorem 3.1]. We leave the details to the reader.

(3) follows from (1) and (2).

(4) Let

$$M_{\varphi,\psi} = \{f \in C([0,1], A) \mid f(0) = \varphi(c),\ f(1) = \psi(c) \quad \text{for some } c \in C\}$$

be the mapping torus of $\varphi, \psi : C \to A$. Since $K_i(\varphi) = K_i(\psi)$ for $i = 0, 1$, from the short exact sequence

$$0 \longrightarrow SA \longrightarrow M_{\varphi,\psi} \xrightarrow{\pi} C \longrightarrow 0$$

of $C^*$-algebras, we obtain the following short exact sequence of abelian groups:

$$0 \longrightarrow K_0(A) \longrightarrow K_1(M_{\varphi,\psi}) \xrightarrow{K_1(\pi)} K_1(C) \longrightarrow 0.$$

By $KL(\varphi) = KL(\psi)$, this exact sequence is pure (see Theorem 2.1). Thus, the quotient map $K_1(\pi)$ has a right inverse on any finitely generated subgroup of $K_1(C)$. Let $R_{\varphi,\psi} : K_1(M_{\varphi,\psi}) \to \operatorname{Aff}(T(A))$ be the rotation map introduced in [19, Section 2] (see also [13, Section 1]). It is easy to verify that

$$R_{\varphi,\psi}(x) + \operatorname{Im} D_A = \Theta_{\varphi,\psi}(K_1(\pi)(x))$$

holds for every $x \in K_1(M_{\varphi,\psi})$. Therefore $\Theta_{\varphi,\psi}$ kills torsion of $K_1(A)$, because $\operatorname{Aff}(T(A))$ is torsion free. In other words, $\Theta_{\varphi,\psi}$ factors through $K_1(C)/\operatorname{Tor}(K_1(C))$. $\square$

# 4 $C^*$-algebras of real rank zero

In this section we give a classification result of unital monomorphisms from $C(X)$ to a $C^*$-algebra in $\mathcal{T} \cup \mathcal{T}'$ (Theorem 4.8). We begin with the following lemma, which is a variant of [8, Lemma 2.2]. A similar argument is also found in [14, Lemma 6.2.7].

**Lemma 4.1.** *Let $X$ be a compact metrizable space. For any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exist a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. Let $\varphi$ and $\psi$ be $(G, \delta)$-multiplicative ucp maps from $C(X)$ to $M_n$ such that*

$$|\operatorname{tr}(\varphi(f)) - \operatorname{tr}(\psi(f))| < \delta \quad \forall f \in G.$$

*Then there exist a projection $p \in M_n$, $(F, \varepsilon)$-multiplicative ucp maps $\varphi', \psi' : C(X) \to pM_np$ and a unital homomorphism $\sigma : C(X) \to (1-p)M_n(1-p)$ such that $\varphi \sim_{F,\varepsilon} \varphi' \oplus \sigma$, $\psi \sim_{F,\varepsilon} \psi' \oplus \sigma$ and $\operatorname{tr}(p) < \varepsilon$.*

*Proof.* Suppose that we are given a finite subset $F \subset C(X)$ and $\varepsilon > 0$. We may assume that elements of $F$ are of norm one. The proof is by contradiction. If the lemma was false, then we would have a sequence of pairs of ucp maps $\varphi_n$ and $\psi_n$ from $C(X)$ to $M_{m_n}$ such that

$$\|\varphi_n(fg) - \varphi_n(f)\varphi_n(g)\| \to 0, \quad \|\psi_n(fg) - \psi_n(f)\psi_n(g)\| \to 0$$

and

$$|\operatorname{tr}(\varphi_n(f)) - \operatorname{tr}(\psi_n(f))| \to 0$$

as $n \to \infty$ for any $f, g \in C(X)$, and the conclusion of the lemma does not hold for any $\varphi_n, \psi_n$. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. Define

$$\bigoplus_{\omega} M_{m_n} = \left\{ (a_n)_n \in \prod M_{m_n} \mid \lim_{n \to \omega} \|a_n\| = 0 \right\}.$$

We set $A = \prod M_{m_n} / \bigoplus_\omega M_{m_n}$ and let $\pi : \prod M_{m_n} \to A$ be the quotient map. Define ucp maps $\tilde{\varphi}$ and $\tilde{\psi}$ from $C(X)$ to $\prod M_{m_n}$ by $\tilde{\varphi}(f) = (\varphi_n(f))_n$ and $\tilde{\psi}(f) = (\psi_n(f))_n$ for $f \in C(X)$. Clearly $\pi \circ \tilde{\varphi}$ and $\pi \circ \tilde{\psi}$ are unital homomorphisms from $C(X)$ to $A$. One can define a tracial state $\tau \in T(A)$ by

$$\tau(\pi((a_n)_n)) = \lim_{n \to \omega} \operatorname{tr}(a_n)$$

for $(a_n)_n \in \prod M_{m_n}$. Then we have $\tau \circ \pi \circ \tilde{\varphi} = \tau \circ \pi \circ \tilde{\psi}$. Let $\mu$ be the probability measure on $X$ corresponding to $\tau \circ \pi \circ \tilde{\varphi} = \tau \circ \pi \circ \tilde{\psi}$.

Any $x \in X$ has an open neighborhood $U_x$ such that $\mu(\overline{U_x} \setminus U_x) = 0$ and $|f(y) - f(y')| < \varepsilon/3$ for any $y, y' \in U_x$ and $f \in F$. (Such $U_x$ exists by the following reason. Let $d(\cdot, \cdot)$ be a metric compatible with the topology of $X$ and let $C_r = \{y \in X \mid d(x, y) = r\}$ for $r > 0$. There exist only countably many $r$ such that $\mu(C_r) > 0$, because $\mu$ is a probability measure. Hence it is easy to find $r > 0$ so that $\mu(C_r) = 0$ and $|f(x) - f(y)| < \varepsilon/6$ for any $y \in X$ with $d(x, y) < r$ and $f \in F$. Then $U_x = \{y \in X \mid d(x, y) < r\}$ meets the requirement.) Since $X$ is compact, we can find $x_1, x_2, \dots, x_k \in X$ such that $U_{x_1} \cup \dots \cup U_{x_k} = X$. Consider open subsets of the form $W = V_1 \cap V_2 \cap \dots \cap V_k$ satisfying $\mu(W) > 0$, where $V_i$ is either $U_{x_i}$ or $X \setminus \overline{U_{x_i}}$. Let $W_1, W_2, \dots, W_l$ be these open subsets. Evidently $W_i$'s are pairwise disjoint. Then we have

$$\mu(X \setminus (W_1 \cup W_2 \cup \dots \cup W_l)) = 0$$

and $|f(y) - f(y')| < \varepsilon/3$ for any $y, y' \in W_i$ and $f \in F$. Choose $z_i \in W_i$ for each $i = 1, 2, \dots, l$. The $C^*$-algebra $\prod M_{m_n}$ has real rank zero and so does $A$. Accordingly, the hereditary subalgebra of $A$ generated by $\pi(\tilde{\varphi}(C_0(W_i)))$ contains an approximate unit consisting of projections. It follows that there exists a projection

$$p_i \in \overline{\pi(\tilde{\varphi}(C_0(W_i))) A \pi(\tilde{\varphi}(C_0(W_i)))}$$

satisfying $\tau(p_i) > \mu(W_i) - \varepsilon/l$. It is easy to see that $\|\pi(\tilde{\varphi}(f))p_i - f(z_i)p_i\| < \varepsilon/3$ holds for any $f \in F$. Extend $\pi \circ \tilde{\varphi}$ to a unital homomorphism from $C(X)^{**}$ to $A^{**}$ and define $\bar{p}_i = \pi(\tilde{\varphi}(1_{W_i}))$. Then $\bar{p}_i$ commutes with $\pi(\tilde{\varphi}(C(X)))$ and $p_i \leq \bar{p}_i$. Similarly one can find projections $q_i$ in the hereditary subalgebra generated by $\pi(\tilde{\psi}(C_0(W_i)))$ and $\bar{q}_i$ in $A^{**} \cap \pi(\tilde{\psi}(C(X)))'$ satisfying analogous properties.

It is not so hard to find projections $p_i' \leq p_i$, $q_i' \leq q_i$ and a unitary $u \in A$ such that $p_i' = u q_i' u^*$ and

$$\tau(p_i') = \tau(q_i') = \min\{\tau(p_i), \tau(q_i)\}$$

for any $i = 1, 2, \dots, l$. Set $p = 1 - (p_1' + p_2' + \dots + p_l')$ and $q = 1 - (q_1' + q_2' + \dots + q_l')$. We have

$$\tau(p) = 1 - \sum_{i=1}^{l} \tau(p_i') < 1 - \sum_{i=1}^{l} (\mu(W_i) - \varepsilon/l) = \varepsilon.$$

Moreover,

$$
\begin{aligned}
(1-p)\pi(\tilde{\varphi}(f)) = \sum_{i=1}^{l} p_i'\pi(\tilde{\varphi}(f)) = \sum_{i=1}^{l} p_i'\bar{p}_i\pi(\tilde{\varphi}(f)) = \sum_{i=1}^{l} p_i'\pi(\tilde{\varphi}(f))\bar{p}_i \\
\approx_{\varepsilon/3} \sum_{i=1}^{l} f(z_i)p_i'\bar{p}_i = \sum_{i=1}^{l} f(z_i)p_i'
\end{aligned}
$$

holds for any $f \in F$. Similarly one has $\|(1-q)\pi(\tilde{\psi}(f)) - \sum_{i=1}^{l} f(z_i)q_i'\| < \varepsilon/3$ for any $f \in F$. It is well-known that the projections $p_i'$, $p$ in $A$ lift to projections $(p_{i,n}')_n$, $(p_n)_n$ in $\prod M_{m_n}$ satisfying $p_n + p_{1,n}' + \cdots + p_{l,n}' = 1$. Similarly the unitary $u \in A$ lifts to a unitary $(u_n)_n \in \prod M_{m_n}$. Define ucp maps $\varphi_n', \psi_n' : C(X) \to p_n M_{m_n} p_n$ and a unital homomorphism $\sigma_n : C(X) \to (1-p_n)M_{m_n}(1-p_n)$ by

$$
\varphi_n'(f) = p_n\varphi_n(f)p_n, \quad \psi_n'(f) = p_n u_n \psi_n(f) u_n^* p_n \quad \text{and} \quad \sigma_n(f) = \sum_{i=1}^{l} f(z_i)p_{i,n}'.
$$

It follows that there exists $n \in \mathbb{N}$ such that $\varphi_n'$ and $\psi_n'$ are $(F, 2\varepsilon/3)$-multiplicative, $\varphi_n \sim_{F,\varepsilon} \varphi_n' \oplus \sigma_n$ and $\psi_n \sim_{F,\varepsilon} \psi_n' \oplus \sigma_n$. This contradicts the assumption, and so the proof is completed. $\square$

We can prove the following lemma in the same way as above.

**Lemma 4.2.** *Let $X$ be a compact metrizable space. For any finite subset $F \subset C(X)$, $\varepsilon > 0$ and $m \in \mathbb{N}$, there exist a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. Let $A \in \mathcal{T}'$ be a $C^*$-algebra with at most $m$ extremal tracial states. Let $\varphi$ and $\psi$ be $(G, \delta)$-multiplicative ucp maps from $C(X)$ to $A$ such that*

$$
|\tau(\varphi(f)) - \tau(\psi(f))| < \delta \quad \forall f \in G, \ \tau \in T(A).
$$

*Then there exist a projection $p \in A$, $(F, \varepsilon)$-multiplicative ucp maps $\varphi', \psi' : C(X) \to pAp$ and a unital homomorphism $\sigma : C(X) \to (1-p)A(1-p)$ with finite dimensional range such that $\varphi \sim_{F,\varepsilon} \varphi' \oplus \sigma$, $\psi \sim_{F,\varepsilon} \psi' \oplus \sigma$ and $\tau(p) < \varepsilon$ for any $\tau \in T(A)$.*

*Proof.* Suppose that we are given a finite subset $F \subset C(X)$, $\varepsilon > 0$ and $m \in \mathbb{N}$. We may assume that elements of $F$ are of norm one. The proof is by contradiction. If the lemma was false, then we would have a sequence of $C^*$-algebras $(A_n)_n$ in $\mathcal{T}'$ and a sequence of pairs of ucp maps $\varphi_n$ and $\psi_n$ from $C(X)$ to $A_n$ as in the proof of Lemma 4.1. Define $\tilde{\varphi}$, $\tilde{\psi}$, $B = \prod A_n / \bigoplus_\omega A_n$ and $\pi : \prod A_n \to B$ in the same way. For each $n$, choose extremal tracial states $\tau_{1,n}, \tau_{2,n}, \ldots, \tau_{m,n} \in T(A_n)$ so that $\{\tau_{1,n}, \tau_{2,n}, \ldots, \tau_{m,n}\}$ exhausts all the extremal traces on $A_n$. For each $j = 1, 2, \ldots, m$, one can define $\tau_{j,\omega} \in T(B)$ by

$$
\tau_{j,\omega}(\pi((a_n)_n)) = \lim_{n\to\omega} \tau_{j,n}(a_n).
$$

We obtain a probability measure $\mu_j$ on $X$ corresponding to $\tau_{j,\omega} \circ \pi \circ \tilde{\varphi} = \tau_{j,\omega} \circ \pi \circ \tilde{\psi}$. In the same way as in Lemma 4.1, we can find pairwise disjoint open subsets $W_1, W_2, \ldots, W_l$ of $X$ such that

$$
\max_j \mu_j(W_i) > 0 \quad \forall i = 1, 2, \ldots, l, \quad \max_j \mu_j(X \setminus (W_1 \cup W_2 \cup \cdots \cup W_l)) = 0
$$

and $|f(y) - f(y')| < \varepsilon/3$ for any $y, y' \in W_i$ and $f \in F$. Choose $z_i \in W_i$ for each $i = 1, 2, \ldots, l$. In the same way as in Lemma 4.1, we also get a family of mutually orthogonal non-zero projections $p_1, p_2, \ldots, p_l$ in $B$ such that $\tau_{j,\omega}(p_i) > \mu_j(W_i) - \varepsilon/2l$ and $\|\pi(\tilde{\varphi}(f))p_i - f(z_i)p_i\| < \varepsilon/3$ for all $f \in F$. Similarly one can find mutually orthogonal non-zero projections $q_1, q_2, \ldots, q_l$ in $B$ for $\tilde{\psi}$. It is well-known that the projections $p_i$ (resp. $q_i$) lift to mutually orthogonal projections $(p_{i,n})_n$ (resp. $(q_{i,n})_n$) in $\prod A_n$. Then there exists $N \in \omega$ such that

$$p_{i,n} \neq 0, \quad q_{i,n} \neq 0, \quad \tau_{j,n}(p_{i,n}) > \mu_j(W_i) - \varepsilon/2l, \quad \tau_{j,n}(q_{i,n}) > \mu_j(W_i) - \varepsilon/2l$$

holds for every $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, m$ and $n \in N$. For each $n \in \mathbb{N}$, the image of $D_{A_n}$ is dense in $\mathrm{Aff}(T(A_n))$ by [1, Theorem 6.9.3]. It follows that for each $n \in N$ and $i = 1, 2, \ldots, l$ there exist projections $r_{i,n} \in A_n$ such that

$$\mu_j(W_i) - \varepsilon/2l < \tau_{j,n}(r_{i,n}) < \min\{\tau_{j,n}(p_{i,n}), \tau_{j,n}(q_{i,n})\} \quad \forall j = 1, 2, \ldots, m.$$

Besides $A_n$ satisfies strict comparison of projections (see Remark 2.3 (2)). Therefore, for $n \in N$, we can find projections $p'_{i,n} \leq p_{i,n}$, $q'_{i,n} \leq q_{i,n}$ and a unitary $u_n \in A_n$ such that $\tau_{j,n}(p'_{i,n}) > \mu_j(W_i) - \varepsilon/2l$ and $p'_{i,n} = u_n q'_{i,n} u_n^*$. For $n \notin N$, set $p'_{i,n} = q'_{i,n} = 0$ and $u_n = 1$. Let $p'_i, q'_i, u \in B$ be the image of $(p'_{i,n})_n$, $(q'_{i,n})_n$ and $(u_n)_n$ by $\pi$. Then we have $p'_i \leq p_i$, $q'_i \leq q_i$, $p'_i = u q'_i u^*$ and $\tau_{j,\omega}(p'_i) > \mu_j(W_i) - \varepsilon/l$. The rest of the proof is exactly the same as Lemma 4.1. □

We can show the same statement for the case that the target algebra is of tracial rank zero.

**Lemma 4.3.** *Let $X$ be a compact metrizable space. For any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exist a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. Let $A \in \mathcal{T}$ and let $\varphi$ and $\psi$ be $(G, \delta)$-multiplicative ucp maps from $C(X)$ to $A$ such that*

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta \quad \forall f \in G,\ \tau \in T(A).$$

*Then there exist a projection $p \in A$, $(F, \varepsilon)$-multiplicative ucp maps $\varphi', \psi' : C(X) \to pAp$ and a unital homomorphism $\sigma : C(X) \to (1-p)A(1-p)$ with finite dimensional range such that $\varphi \sim_{F,\varepsilon} \varphi' \oplus \sigma$, $\psi \sim_{F,\varepsilon} \psi' \oplus \sigma$ and $\tau(p) < \varepsilon$ for any $\tau \in T(A)$.*

*Proof.* Suppose that we are given a finite subset $F \subset C(X)$ and $\varepsilon > 0$. Applying Lemma 4.1 for $F$ and $\varepsilon$, we obtain $G \subset C(X)$ and $\delta > 0$. Let $A$ be a unital simple separable $C^*$-algebra with tracial rank zero and let $\varphi$ and $\psi$ be $(G, \delta)$-multiplicative ucp maps from $C(X)$ to $A$ such that

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta$$

for any $f \in G$ and $\tau \in T(A)$. Since $A$ has tracial rank zero, there exist a sequence of projections $e_n \in A$, a sequence of finite dimensional subalgebras $B_n$ of $A$ with $1_{B_n} = e_n$ and a sequence of ucp maps $\pi_n : A \to B_n$ such that the following hold.

- $\|[a, e_n]\| \to 0$ as $n \to \infty$ for any $a \in A$.
- $\|\pi_n(a) - e_n a e_n\| \to 0$ as $n \to \infty$ for any $a \in A$.

- $\tau(1-e_n) \to 0$ as $n \to \infty$ uniformly on $T(A)$.

It is easy to see that $\pi_n \circ \varphi$ and $\pi_n \circ \psi$ are $(G, \delta)$-multiplicative for sufficiently large $n \in \mathbb{N}$. We would like to show that

$$|\tau(\pi_n(\varphi(f))) - \tau(\pi_n(\psi(f)))| < \delta$$

holds for every $f \in G$, $\tau \in T(B_n)$ and sufficiently large $n \in \mathbb{N}$. To this end, we assume that there exist $\tau_n \in T(B_n)$ such that

$$\max_{f \in G} |\tau_n(\pi_n(\varphi(f))) - \tau_n(\pi_n(\psi(f)))| \geq \delta.$$

Let $\tau \in A^*$ be an accumulation point of $\tau_n \circ \pi_n$. Clearly $\tau$ is a tracial state of $A$ and $|\tau(\varphi(f)) - \tau(\psi(f))| \geq \delta$ for some $f \in G$, which is a contradiction.

Hence, Lemma 4.1 implies that, for sufficiently large $n \in \mathbb{N}$, there exist a projection $p_n \in B_n$, $(F, \varepsilon)$-multiplicative ucp maps $\varphi'_n, \psi'_n : C(X) \to p_n B_n p_n$ and a unital homomorphism $\sigma_n : C(X) \to (e_n - p_n) B_n (e_n - p_n)$ such that

$$\pi_n \circ \varphi \sim_{F,\varepsilon} \varphi'_n \oplus \sigma_n, \quad \pi_n \circ \psi \sim_{F,\varepsilon} \psi'_n \oplus \sigma_n$$

and $\tau(p_n) < \varepsilon$ for any $\tau \in T(B_n)$. Therefore the proof is completed. $\square$

The following is taken from [9, Theorem 3.1]. We remark that its origin is found in [4].

**Theorem 4.4** ([9, Theorem 3.1]). *Let $X$ be a compact metrizable space. For any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exist a finite subset $G \subset C(X)$, $\delta > 0$, $l \in \mathbb{N}$ and a finite subset $L \subset \mathcal{K}(C(X))$ satisfying the following: For any unital $C^*$-algebra $A$ with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and any $(G, \delta)$-multiplicative ucp maps $\varphi, \psi : C(X) \to A$ satisfying $\varphi_\#|L = \psi_\#|L$, there exist a unitary $u \in M_{l+1}(A)$ and $\{x_1, x_2, \ldots, x_l\} \subset X$ such that*

$$\|u \operatorname{diag}(\varphi(f), f(x_1), f(x_2), \ldots, f(x_l)) u^* - \operatorname{diag}(\psi(f), f(x_1), f(x_2), \ldots, f(x_l))\| < \varepsilon$$

*for any $f \in F$.*

The following theorem is a variant of [17, Theorem 4.6].

**Theorem 4.5.** *Let $X$ be a compact metrizable space, let $F \subset C(X)$ be a finite subset and let $\varepsilon > 0$. Then there exist a finite subset $L \subset \mathcal{K}(C(X))$ and a family of mutually orthogonal positive elements $h_1, h_2, \ldots, h_k \in C(X)$ of norm one such that the following holds. For any $\nu > 0$, one can find a finite subset $G \subset C(X)$ and $\delta > 0$ satisfying the following. For any $A \in \mathcal{T}$ and any $(G, \delta)$-multiplicative ucp maps $\varphi, \psi : C(X) \to A$ such that $\varphi_\#|L = \psi_\#|L$,*

$$\tau(\varphi(h_i)) \geq \nu \quad \forall \tau \in T(A),\ i{=}1, 2, \ldots, k$$

*and*

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta \quad \forall \tau \in T(A),\ f \in G,$$

*there exists a unitary $u \in A$ such that*

$$\|u \varphi(f) u^* - \psi(f)\| < \varepsilon$$

*holds for any $f \in F$.*

*Proof.* We say that a subset $Y \subset X$ is $(F,\varepsilon)$-dense if for any $x \in X$ there exists $y \in Y$ such that $|f(x) - f(y)| < \varepsilon$ for every $f \in F$. Choose an $(F, \varepsilon/7)$-dense finite subset $\{y_1, y_2, \dots, y_k\} \subset X$. For each $i = 1, 2, \dots, k$, choose an open neighborhood $U_i$ of $y_i$ so that $x \in U_i$ implies $|f(x) - f(y_i)| < \varepsilon/7$ for any $f \in F$ and that $U_1, U_2, \dots, U_k$ are mutually disjoint. Choose a positive function $h_i \in C_0(U_i)$ of norm one. By applying Theorem 4.4 to $F$ and $\varepsilon/7$, we obtain a finite subset $G_1 \subset C(X)$, $\delta_1 > 0$, $l \in \mathbb{N}$ and a finite subset $L \subset \mathcal{K}(C(X))$. There exist a finite subset $G_2 \subset C(X)$ and $\delta_2 > 0$ such that the following holds: For any unital $C^*$-algebra $A$ and any $(G_2, \delta_2)$-multiplicative ucp maps $\varphi, \psi : C(X) \to A$, if $\|\varphi(f) - \psi(f)\| < \delta_2$ for every $f \in G_2$, then $\varphi_\#|L = \psi_\#|L$. Suppose that $\nu > 0$ is given. Let

$$G_3 = F \cup G_1 \cup G_2 \cup \{h_1, h_2, \dots, h_k\}$$

and

$$\delta_3 = \min\{\varepsilon/7, \delta_1, \delta_2, \nu/(l+2)\}.$$

By applying Lemma 4.3 to $G_3$ and $\delta_3$, we obtain a finite subset $G \subset C(X)$ and $\delta > 0$.

Suppose that $A$ is a unital simple separable $C^*$-algebra $A$ with tracial rank zero and that $\varphi, \psi : C(X) \to A$ are $(G, \delta)$-multiplicative ucp maps satisfying $\varphi_\#|L = \psi_\#|L$,

$$\tau(\varphi(h_i)) \geq \nu \quad \forall \tau \in T(A),\ i{=}1, 2, \dots, k$$

and

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta \quad \forall \tau \in T(A),\ f \in G.$$

By lemma 4.3, there exist a projection $p \in A$, $(G_3, \delta_3)$-multiplicative ucp maps $\varphi', \psi' : C(X) \to pAp$, a unital homomorphism $\sigma : C(X) \to (1{-}p)A(1{-}p)$ with finite dimensional range such that $\varphi \sim_{G_3,\delta_3} \varphi' \oplus \sigma$, $\psi \sim_{G_3,\delta_3} \psi' \oplus \sigma$ and $\tau(p) < \delta_3$ for any $\tau \in T(A)$. Since $G_2$ is contained in $G_3$ and $\delta_2$ is not greater than $\delta_3$, by the choice of $G_2$ and $\delta_2$, we obtain $(\varphi' \oplus \sigma)_\#|L = (\psi' \oplus \sigma)_\#|L$, and hence $\varphi'_\#|L = \psi'_\#|L$. Besides, $\varphi'$ and $\psi'$ are $(G_1, \delta_1)$-multiplicative, because $G_1$ is contained in $G_3$ and $\delta_1$ is not greater than $\delta_3$. By Theorem 4.4, there exist a unitary $u \in M_{l+1}(pAp)$ and $\{x_1, x_2, \dots, x_l\} \subset X$ such that

$$\|u \operatorname{diag}(\varphi'(f), f(x_1), f(x_2), \dots, f(x_l))u^* - \operatorname{diag}(\psi'(f), f(x_1), f(x_2), \dots, f(x_l))\| < \varepsilon/7$$

for any $f \in F$. In what follows, for a positive linear functional $\rho$ on $C(X)$, we let $\mu_\rho$ denote the corresponding measure on $X$. For any $\tau \in T(A)$ and $i = 1, 2, \dots, k$, one has

$$\mu_{\tau\circ\sigma}(U_i) \geq \tau(\sigma(h_i)) > \tau(\varphi(h_i)) - \tau(\varphi'(h_i)) - \delta_3 > \nu - 2\delta_3 \geq l\delta_3.$$

It follows that there exists a unital homomorphism $\sigma' : C(X) \to (1{-}p)A(1{-}p)$ with finite dimensional range such that

$$\|\sigma(f) - \sigma'(f)\| < \varepsilon/7$$

for any $f \in F$ and $\mu_{\tau\circ\sigma'}(\{y_i\}) > l\delta_3$ for any $\tau \in T(A)$ and $i = 1, 2, \dots, k$. Since $\{y_1, y_2, \dots, y_k\}$ is $(F, \varepsilon/7)$-dense, we can find a unital homomorphism $\sigma'' : C(X) \to (1{-}p)A(1{-}p)$ with finite dimensional range such that

$$\|\sigma'(f) - \sigma''(f)\| < \varepsilon/7$$

for any $f \in F$ and $\mu_{\tau\circ\sigma''}(\{x_j\}) > \delta_3$ for any $\tau \in T(A)$ and $j = 1, 2, \ldots, l$. Then it is not so hard to see $\varphi' \oplus \sigma'' \sim_{F,\varepsilon/7} \psi' \oplus \sigma''$. Consequently we have

$$\begin{aligned}\varphi &\sim_{F,\varepsilon/7} \varphi' \oplus \sigma \sim_{F,\varepsilon/7} \varphi' \oplus \sigma' \sim_{F,\varepsilon/7} \varphi' \oplus \sigma'' \\ &\sim_{F,\varepsilon/7} \psi' \oplus \sigma'' \sim_{F,\varepsilon/7} \psi' \oplus \sigma' \sim_{F,\varepsilon/7} \psi' \oplus \sigma \sim_{F,\varepsilon/7} \psi.\end{aligned}$$

□

In the same fashion as above, one can prove the following by using Lemma 4.2 instead of Lemma 4.3 (see also [8, Corollary 2.17]).

**Theorem 4.6.** *Let $X$ be a compact metrizable space, let $F \subset C(X)$ be a finite subset and let $\varepsilon > 0$, $m \in \mathbb{N}$. Then there exist a finite subset $L \subset \mathcal{K}(C(X))$ and a family of mutually orthogonal positive elements $h_1, h_2, \ldots, h_k \in C(X)$ of norm one such that the following holds. For any $\nu > 0$, one can find a finite subset $G \subset C(X)$ and $\delta > 0$ satisfying the following. Let $A \in \mathcal{T}'$ be a $C^*$-algebra with at most $m$ extremal tracial states. For any $(G, \delta)$-multiplicative ucp maps $\varphi, \psi : C(X) \to A$ such that $\varphi_\#|L = \psi_\#|L$,*

$$\tau(\varphi(h_i)) \geq \nu \quad \forall \tau \in T(A),\ i{=}1, 2, \ldots, k$$

*and*

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta \quad \forall \tau \in T(A),\ f \in G,$$

*there exists a unitary $u \in A$ such that*

$$\|u\varphi(f)u^* - \psi(f)\| < \varepsilon$$

*holds for any $f \in F$.*

By using the theorems above, we obtain the following generalization of [17, Theorem 3.3].

**Theorem 4.7.** *Let $X$ be a compact metrizable space and let $A \in \mathcal{T} \cup \mathcal{T}'$. Let $\varphi : C(X) \to A$ be a unital monomorphism. Then for any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exist a finite subset $L \subset \mathcal{K}(C(X))$, a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. If $\psi : C(X) \to A$ is a $(G, \delta)$-multiplicative ucp map satisfying $\varphi_\#|L = \psi_\#|L$ and*

$$|\tau(\varphi(f)) - \tau(\psi(f))| < \delta$$

*for any $\tau \in T(A)$ and $f \in G$, then there exists a unitary $u \in A$ such that*

$$\|u\varphi(f)u^* - \psi(f)\| < \varepsilon$$

*holds for any $f \in F$.*

*Proof.* Applying Theorem 4.5 or Theorem 4.6, we obtain a finite subset $L \subset \mathcal{K}(C(X))$ and positive elements $h_1, h_2, \ldots, h_k \in C(X)$ of norm one. Since $A$ is simple and $\varphi$ is injective,

$$\nu = \min\{\tau(\varphi(h_i)) \mid \tau \in T(A),\ i = 1, 2, \ldots, k\}$$

is positive. Using Theorem 4.5 or Theorem 4.6 for $\nu$, we find a finite subset $G \subset C(X)$ and $\delta > 0$. It is clear that $G$ and $\delta$ meet the requirement. □

The following is an immediate consequence of the theorem above.

**Theorem 4.8.** *Let $X$ be a compact metrizable space and let $A \in \mathcal{T} \cup \mathcal{T}'$. Let $\varphi, \psi : C(X) \to A$ be unital monomorphisms. Then $\varphi$ and $\psi$ are approximately unitarily equivalent if and only if $KL(\varphi) = KL(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(A)$.*

**Corollary 4.9.** *Let $C$ be a unital AH algebra and let $A \in \mathcal{T} \cup \mathcal{T}'$. Let $\varphi, \psi : C \to A$ be unital monomorphisms. Then $\varphi$ and $\psi$ are approximately unitarily equivalent if and only if $KL(\varphi) = KL(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$ for all $\tau \in T(A)$.*

*Proof.* Although the proof is essentially the same as [17, Corollary 4.8], we present it for completeness. Without loss of generality, we may assume $C = p(C(X) \otimes M_k)p$, where $X$ is a compact metrizable space and $p \in C(X) \otimes M_k$ is a non-zero projection. We may further assume that the rank of $p(x) \in M_k$ is strictly positive for every $x \in X$.

We first consider the case $p = 1 \in C(X) \otimes M_k$. It is easy to see that there exists a unitary $u \in A$ such that $\varphi(1 \otimes a) = u\psi(1 \otimes a)u^*$ holds for any $a \in M_k$. Let $e$ be a minimal projection of $M_k$. Then $\varphi'(f) = \varphi(f \otimes e)$ and $\psi'(f) = u\psi(f \otimes e)u^*$ are unital monomorphisms from $C(X)$ to $\varphi(e)A\varphi(e)$. By Theorem 4.8, they are approximately unitarily equivalent. Hence $\varphi$ and $\psi$ are approximately unitarily equivalent.

Let us consider the general case. There exist $l \in \mathbb{N}$ and a projection $q \in C \otimes M_l \subset C(X) \otimes M_{kl}$ such that $p \otimes e$ is a subprojection of $q$ and $q$ is Murray-von Neumann equivalent to $1_{C(X)} \otimes r$, where $e \in M_l$ is a minimal projection of $M_l$ and $r \in M_{kl}$ is a projection of rank $k$. By the argument above, the restrictions of $\varphi \otimes \mathrm{id}_{M_l}$ and $\psi \otimes \mathrm{id}_{M_l}$ to $q(C \otimes M_l)q$ are approximately unitarily equivalent. It follows that their restrictions to $(p \otimes e)(C \otimes M_l)(p \otimes e) = C$ are also approximately unitarily equivalent, which completes the proof. □

In Section6 we will generalize the results above to the case that the target algebra $A$ belongs to $\mathcal{C} \cup \mathcal{C}'$.

# 5 Homotopy of unitaries

In this section, we prove the so called basic homotopy lemma for $A$ in $\mathcal{T} \cup \mathcal{T}'$ (Theorem 5.3 and Theorem 5.4). The basic idea of the proof is similar to that of [19, Theorem 8.1], but there are two main differences. One is the use of Theorem 2.6, which claims the existence of a unital monomorphism $\varphi : C(X) \to A$ realizing the given $\kappa \in KL(C(X), A)_{+,1}$ and $\lambda : T(A) \to T(C(X))$. The other point is that we allow $G \subset C(X)$ and $\delta > 0$ in Theorem 5.3 to depend on the given homomorphism $\varphi : C(X) \to A$. Although, as shown in [19], it is possible to state the theorem in a more general form, we do not pursue this here because the actual application discussed in Section 6 does not need that general form. These two points enable us to simplify the proof given in [19].

We let $\mathbb{T}$ denote the unit circle in the complex plane and let $z \in C(\mathbb{T})$ be the identity function $z(\exp(\pi\sqrt{-1}t)) = \exp(\pi\sqrt{-1}t)$. The following is a variant of [19, Lemma 6.4].

**Lemma 5.1.** *Let $X$ be a compact metrizable space and let $A \in \mathcal{T} \cup \mathcal{T}'$. For any finite subsets $F \subset C(X)$, $\widetilde{F} \subset C(X \times \mathbb{T})$ and $\varepsilon > 0$, there exist a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. For any $k \in \mathbb{N}$, any unital monomorphism $\varphi : C(X) \to A$ and a unitary $u \in A$ satisfying $\|[\varphi(f), u]\| < \delta$ for any $f \in G$, there exist a* *path of unitaries $w : [0,1] \to A$ and an $(\widetilde{F}, \varepsilon)$-multiplicative ucp map $\psi : C(X \times \mathbb{T}) \to A$ such that*

$$\|w(0) - u\| < \varepsilon, \quad \|w(1) - \psi(1 \otimes z)\| < \varepsilon, \quad \mathrm{Lip}(w) \le \pi,$$

$$\|[\varphi(f), w(t)]\| < \varepsilon, \quad \|\psi(f \otimes 1) - \varphi(f)\| < \varepsilon$$

*hold for any $f \in F$ and $t \in [0,1]$, and*

$$|\tau(\psi(f \otimes z^j))| < \varepsilon \|f\|$$

*holds for any $\tau \in T(A)$, $f \in C(X)$ and $j \in \mathbb{Z}$ with $1 \le |j| < k$.*

*Proof.* Without loss of generality, we may assume that all the elements of $F$ are of norm one. Applying Lemma 4.2 or Lemma 4.3 to

$$G_1 = \widetilde{F} \cup \{f \otimes 1 \mid f \in F\} \cup \{1 \otimes z\} \subset C(X \times \mathbb{T})$$

and $\delta_1 = \min\{\varepsilon/8, \varepsilon^2\}$, we obtain a finite subset $G_2 \subset C(X \times \mathbb{T})$ and $\delta_2 > 0$. We may assume that $G_2$ contains $G_1$ and that $\delta_2$ is less than $\delta_1$. Clearly there exist a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following holds: If $\varphi : C(X) \to A$ is a unital monomorphism and $u \in A$ is a unitary satisfying $\|[\varphi(f), u]\| < \delta$ for any $f \in G$, then one can find a $(G_2, \delta_2)$-multiplicative ucp map $\varphi_0 : C(X \times \mathbb{T}) \to A$ such that

$$\|\varphi_0(1 \otimes z) - u\| < \delta_2, \quad \|\varphi_0(f \otimes 1) - \varphi(f)\| < \delta_2$$

for every $f \in G_2$.

Suppose that we are given $k \in \mathbb{N}$, a unital monomorphism $\varphi : C(X) \to A$ and a unitary $u \in A$ satisfying $\|[\varphi(f), u]\| < \delta$ for every $f \in G$. We find $\varphi_0 : C(X \times \mathbb{T}) \to A$ as above. By using Lemma 4.2 or Lemma 4.3, there exist a projection $p \in A$, a $(G_1, \delta_1)$-multiplicative ucp map $\varphi_0' : C(X \times \mathbb{T}) \to pAp$ and a unital homomorphism $\sigma : C(X \times \mathbb{T}) \to (1-p)A(1-p)$ with finite dimensional range such that

$$\|\varphi_0(f) - (\varphi_0' \oplus \sigma)(f)\| < \delta_1 \quad \forall f \in G_1$$

and $\tau(p) < \delta_1$ for any $\tau \in T(A)$. We may further assume that there exists a unitary $u' \in pAp$ such that $\|u' - \varphi_0'(1 \otimes z)\| < \delta_1$. Since $\sigma$ has finite dimensional range, one can find $x_1, x_2, \dots, x_l \in X$, $y_1, y_2, \dots, y_l \in \mathbb{T}$ and projections $p_1, p_2, \dots, p_l \in A$ such that

$$\sum_{i=1}^{l} p_i = 1-p, \quad \sigma(f \otimes g) = \sum_{i=1}^{l} f(x_i) g(y_i) p_i$$

holds for any $f \in C(X)$ and $g \in C(\mathbb{T})$. By replacing each $p_i$ with its subprojection if necessary, we may assume that $D_A([p_i])$ belongs to $k \,\mathrm{Im}(D_A)$, because $\mathrm{Im}\, D_A$ is dense in $\mathrm{Aff}(T(A))$. Choose projections $q_{i,j}$ for $i = 1, 2, \dots, l$ and $j = 1, 2, \dots, k$ so that

$$\sum_{j=1}^{k} q_{i,j} = p_i, \quad kD_A([q_{i,j}]) = D_A([p_i]) \quad \forall j = 1, 2, \dots, k.$$

Define a homomorphism $\sigma' : C(X \times \mathbb{T}) \to (1-p)A(1-p)$ with finite dimensional range by

$$\sigma'(f \otimes g) = \sum_{i=1}^{l} \sum_{j=1}^{k} f(x_i) g(\zeta^j) q_{i,j},$$

where $\zeta = \exp(2\pi\sqrt{-1}/k)$. Define a ucp map $\psi : C(X \times \mathbb{T}) \to A$ by $\psi = \varphi_0' \oplus \sigma'$. It is clear that $\psi$ is $(G_1, \delta_1)$-multiplicative, and hence is $(\widetilde{F}, \varepsilon)$-multiplicative. Moreover, one has

$$\psi(f \otimes 1) = \varphi_0'(f \otimes 1) \oplus \sigma'(f \otimes 1) = \varphi_0'(f \otimes 1) \oplus \sigma(f \otimes 1) \approx_{\delta_1} \varphi_0(f \otimes 1) \approx_{\delta_2} \varphi(f)$$

for any $f \in F$. For any $\tau \in T(A)$, $f \in C(X)$ and $j \in \mathbb{Z}$ with $1 \leq |j| < k$, it is easy to see

$$\begin{aligned} |\tau(\psi(f \otimes z^j))| &\leq |\tau(\varphi_0'(f \otimes z^j))| + |\tau(\sigma'(f \otimes z^j))| \\ &= |\tau(\varphi_0'(f \otimes z^j))| \\ &\leq \|f\| \tau(p)^{1/2} < \varepsilon \|f\|. \end{aligned}$$

We construct a path of unitaries $w : [0,1] \to A$. By the definition of $\sigma'$, we can find a path of unitaries $v : [0,1] \to (1-p)A(1-p)$ such that

$$v(0) = \sigma(1 \otimes z), \quad v(1) = \sigma'(1 \otimes z), \quad \mathrm{Lip}(v) \leq \pi$$

and $[q_{i,j}, v(t)] = 0$ for any $i, j$ and $t \in [0,1]$. Define $w : [0,1] \to U(A)$ by $w(t) = u' \oplus v(t)$. Evidently we have

$$w(0) = u' \oplus \sigma(1 \otimes z) \approx_{\delta_1} \varphi_0'(1 \otimes z) \oplus \sigma(1 \otimes z) \approx_{\delta_1} \varphi_0(1 \otimes z) \approx_{\delta_2} u,$$
$$w(1) = u' \oplus \sigma'(1 \otimes z) \approx_{\delta_1} \varphi_0'(1 \otimes z) \oplus \sigma'(1 \otimes z) = \psi(1 \otimes z).$$

and $\mathrm{Lip}(w) \leq \pi$. Besides, for any $f \in F$ and $t \in [0,1]$, one can verify

$$\begin{aligned} [\varphi(f), w(t)] &\approx_{2\delta_2} [\varphi_0(f \otimes 1), w(t)] \approx_{2\delta_1} [(\varphi_0' \oplus \sigma)(f \otimes 1), u' \oplus v(t)] \\ &= [\varphi_0'(f \otimes 1), u'] \approx_{2\delta_1} [\varphi_0'(f \otimes 1), \varphi_0'(1 \otimes z)] \approx_{2\delta_1} 0, \end{aligned}$$

thereby completing the proof. □

**Remark 5.2.** In the lemma above, for $A$ in $\mathcal{T}$, one can see that $G \subset C(X)$ and $\delta$ depend only on $F \subset C(X)$ and $\varepsilon$. For $A$ in $\mathcal{T}'$, $G \subset C(X)$ and $\delta$ depend only on $F \subset C(X)$, $\varepsilon$ and the cardinality of extremal tracial states on $A$.

The following is a generalization of [19, Corollary 8.4].

**Theorem 5.3.** *Let $X$ be a path connected compact metrizable space and let $A \in \mathcal{T} \cup \mathcal{T}'$. Let $\varphi : C(X) \to A$ be a unital monomorphism. For any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exist a finite subset $L \subset \mathcal{K}(C(X))$, a finite subset $G \subset C(X)$ and $\delta > 0$ such that the following hold. If $u \in A$ is a unitary satisfying*

$$\|[\varphi(f), u]\| < \delta \quad \forall f \in G \quad \textit{and} \quad \mathrm{Bott}(\varphi, u)(x) = 0 \quad \forall x \in L,$$

*then there exists a path of unitaries $w : [0,1] \to A$ such that*

$$w(0) = u, \quad w(1) = 1, \quad \mathrm{Lip}(w) < 2\pi + \varepsilon$$

*and*

$$\|[\varphi(f), w(t)]\| < \varepsilon \quad \forall f \in F, \ t \in [0,1].$$

*Proof.* Let $p : C([-1,1]) \to \mathbb{C}$ be the point evaluation at $1 \in [-1,1]$ and let $q : C(\mathbb{T}) \to C([-1,1])$ be the unital monomorphism defined by $q(f)(t) = f(\exp(\pi\sqrt{-1}t))$.

By Lemma 2.4, $KL(\varphi) \circ KL(\mathrm{id} \otimes p)$ belongs to $KL(C(X \times [-1,1]), A)_{+,1}$. Define $\tau_0 \in T(C([-1,1]))$ by

$$\tau_0(f) = \frac{1}{2} \int_{-1}^{1} f(t)\, d\mu(t),$$

where $\mu$ is the Lebesgue measure on $\mathbb{R}$. Let $\lambda : T(C(X)) \to T(C(X \times [-1,1]))$ be the affine continuous map defined by $\lambda(\tau) = \tau \otimes \tau_0$. By applying Theorem 2.6 to $KL(\varphi) \circ KL(\mathrm{id} \otimes p)$ and $\lambda \circ T(\varphi)$, we get a unital monomorphism $\sigma : C(X \times [-1,1]) \to A$ such that $KL(\sigma) = KL(\varphi) \circ KL(\mathrm{id} \otimes p)$ and $T(\sigma) = \lambda \circ T(\varphi)$. Then $\sigma' = \sigma \circ (\mathrm{id} \otimes q)$ is a unital monomorphism from $C(X \times \mathbb{T})$ to $A$ such that

$$KL(\sigma') = KL(\sigma \circ (\mathrm{id} \otimes q)) = KL(\varphi) \circ KL(\mathrm{id} \otimes (p \circ q))$$

and $T(\sigma') = T(\mathrm{id} \otimes q) \circ \lambda \circ T(\varphi)$. Under the canonical isomorphism

$$\underline{K}(C(X \times \mathbb{T})) \cong \underline{K}(C(X)) \oplus \underline{K}(C_0(X \times (\mathbb{T} \setminus \{-1\}))),$$

$KL(\sigma') \in \mathrm{Hom}_\Lambda(\underline{K}(C(X \times \mathbb{T})), \underline{K}(A))$ corresponds to $KL(\varphi) \oplus 0$. It is also easy to see that $T(\sigma')(\tau) = (\tau \circ \varphi) \otimes \tau_0'$ for any $\tau \in T(A)$, where $\tau_0' \in T(C(\mathbb{T}))$ is the tracial state corresponding to the Haar measure on $\mathbb{T}$. From the construction, there exists a path of unitaries $w_1 : [0,1] \to A$ such that

$$w_1(0) = 1, \quad w_1(1) = \sigma'(1 \otimes z), \quad \mathrm{Lip}(w_1) = \pi$$

and $[\sigma'(f \otimes 1), w_1(t)] = 0$ for any $f \in C(X)$ and $t \in [0,1]$.

By applying Theorem 4.7 to $\sigma' : C(X \times \mathbb{T}) \to A$, $\{f \otimes 1 \mid f \in F\} \cup \{1 \otimes z\}$ and $\varepsilon/4$, we obtain a finite subset $L \subset \mathcal{K}(C(X \times \mathbb{T}))$, a finite subset $G_1 \subset C(X \times \mathbb{T})$ and $\delta_1 > 0$. Choose a sufficiently large finite subset $L_0 \subset \mathcal{K}(C(X))$, a sufficiently large finite subset $G_2 \subset C(X)$ and a sufficiently small real number $\delta_2 > 0$. By applying Lemma 5.1 to $G_2 \subset C(X)$, $G_1 \subset C(X \times \mathbb{T})$ and $\delta_2 > 0$, we obtain a finite subset $G \subset C(X)$ and $\delta > 0$.

Suppose that we are given a unitary $u \in A$ satisfying

$$\|[\varphi(f), u]\| < \delta \quad \forall f \in G \quad \text{and} \quad \mathrm{Bott}(\varphi, u)(x) = 0 \quad \forall x \in L_0.$$

Let $k \in \mathbb{N}$ be a sufficiently large natural number. By Lemma 5.1, one can find a path of unitaries $w_0 : [0,1] \to A$ and a $(G_1, \delta_2)$-multiplicative ucp map $\psi : C(X \times \mathbb{T}) \to A$ such that

$$\|w_0(0) - u\| < \delta_2, \quad \|w_0(1) - \psi(1 \otimes z)\| < \delta_2, \quad \mathrm{Lip}(w_0) \le \pi,$$

$$\|[\varphi(f), w_0(t)]\| < \delta_2, \quad \|\psi(f \otimes 1) - \varphi(f)\| < \delta_2$$

hold for any $f \in G_2$ and $t \in [0,1]$, and

$$|\tau(\psi(f \otimes z^j))| < \delta_2 \|f\|$$

holds for any $\tau \in T(A)$, $f \in C(X)$ and $j \in \mathbb{Z}$ with $1 \le |j| < k$. Hence, if $L_0 \subset \mathcal{K}(C(X))$ is large enough, $G_2 \subset C(X)$ is large enough and $\delta_2 > 0$ is small enough, then one can conclude $\psi_\#|L = \sigma'_\#|L$. In addition, if $k \in \mathbb{N}$ is chosen to be large enough, then we may assume

$$|\tau(\psi(f)) - \tau(\sigma'(f))| < \delta_1 \quad \forall \tau \in T(A),\ f \in G_1.$$

It follows from Theorem 4.7 that there exists a unitary $v \in A$ such that

$$\|v\sigma'(1 \otimes z)v^* - \psi(1 \otimes z)\| < \varepsilon/4, \quad \|v\sigma'(f \otimes 1)v^* - \psi(f \otimes 1)\| < \varepsilon/4 \quad \forall f \in F.$$

We define $w : [0,1] \to U(A)$ by $w(t) = w_0(t)vw_1(t)^*v^*$. Clearly one has $\mathrm{Lip}(w) \leq 2\pi$,

$$\|w(0) - u\| < \delta_2, \quad \|w(1) - 1\| < \delta_2 + \varepsilon/4$$

and

$$\|[\varphi(f), w(t)]\| < 3\delta_2 + \varepsilon/2 \quad \forall f \in F,\ t \in [0,1].$$

It is easy to perturb the path $w : [0,1] \to A$ a little bit so that $w(0) = u$ and $w(1) = 1$. $\square$

The following is an easy generalization of the theorem above.

**Theorem 5.4.** *Let $C$ be a unital $C^*$-algebra of the form $\bigoplus_{i=1}^n p_i M_{k_i}(C(X_i))p_i$, where $X_i$ is a path connected compact metrizable space and $p_i$ is a non-zero projection of $M_{k_i}(C(X_i))$. Let $A \in \mathcal{T} \cup \mathcal{T}'$. Let $\varphi : C \to A$ be a unital monomorphism. For any finite subset $F \subset C$ and $\varepsilon > 0$, there exist a finite subset $L \subset \mathcal{K}(C)$, a finite subset $G \subset C$ and $\delta > 0$ such that the following hold. If $u \in A$ is a unitary satisfying*

$$\|[\varphi(f), u]\| < \delta \quad \forall f \in G \quad \text{and} \quad \mathrm{Bott}(\varphi, u)(x) = 0 \quad \forall x \in L,$$

*then there exists a path of unitaries $w : [0,1] \to A$ such that*

$$w(0) = u, \quad w(1) = 1, \quad \mathrm{Lip}(w) < 2\pi + \varepsilon$$

*and*

$$\|[\varphi(f), w(t)]\| < \varepsilon \quad \forall f \in F,\ t \in [0,1].$$

*Proof.* We can prove this in a similar fashion to [19, Lemma 17.5] by using the theorem above. We omit the detail. It is worth noting that if $A$ is in $\mathcal{T} \cup \mathcal{T}'$ and $e \in A$ is a non-zero projection, then $eAe$ is also in $\mathcal{T} \cup \mathcal{T}'$. See also the proof of Corollary 4.9. $\square$

**Remark 5.5.** In the theorems above, if the target algebra $A$ satisfies $A \cong A \otimes Q$ (i.e. $A$ is $Q$-stable), then $K_i(A; \mathbb{Z}_n) = 0$ for any $i = 0, 1$ and $n \geq 1$ because $K_i(A)$ is torsion free and divisible. Therefore the entire $K$-group $\underline{K}(A)$ is canonically isomorphic to $K_0(A) \oplus K_1(A)$. Consequently we may assume that the finite subset $L \subset \mathcal{K}(C(X))$ in the statement is actually a finite subset of $P(C(X) \otimes \mathbb{K}) \cup U_\infty(C(X))$.

# 6 $\mathcal{Z}$-stable $C^*$-algebras

In this section we prove Theorem 6.6 and Corollary 6.8. When $X$ is a finite CW complex, it is well-known that $K_*(C(X))$ is finitely generated.

**Lemma 6.1.** *Let $C$ be a $C^*$-algebra of the form $p(C(X)\otimes M_k)p$, where $X$ is a finite CW complex and $p\in C(X)\otimes M_k$ is a projection. Let $A\in\mathcal{T}\cup\mathcal{T}'$. Let $L\subset U_\infty(C)$ be a finite subset which generates $K_1(C)$ and let $\varphi: C\to A$ be a unital monomorphism. For any finite subset $F\subset C$ and $\varepsilon>0$, there exists $\delta>0$ such that the following holds. If $\xi: K_1(C)\to K_0(A)$ satisfies $\|D_A(\xi([w]))\|<\delta$ for any $w\in L$, then there exists a unitary $u\in A$ such that*

$$\|[\varphi(f),u]\|<\varepsilon$$

*for every $f\in F$ and*

$$\mathrm{Bott}(\varphi,u)(w)=\xi([w])$$

*for every $w\in L$.*

*Proof.* When $A$ is in $\mathcal{T}$, this lemma is contained in [18, Lemma 6.11]. Assume $A\in\mathcal{T}'$. By Theorem 2.5, there exist a unital simple AH algebra $B$ with real rank zero and slow dimension growth and a unital homomorphism $\psi: B\to A$ such that $K_*(\psi)$ gives a graded ordered isomorphism. The tracial simplexes $T(A)$ and $T(B)$ are naturally isomorphic to the state spaces of $K_0(A)$ and $K_0(B)$, respectively. Hence $T(\psi)$ induces an affine isomorphism from $T(A)$ to $T(B)$. It follows from Corollary 2.9 that there exists a unital homomorphism $\varphi_0: C\to B$ such that $KL(\varphi_0)=KL(\psi)^{-1}\circ KL(\varphi)$ and $T(\varphi_0)=T(\varphi)\circ T(\psi)^{-1}$. By Corollary 4.9, $\psi\circ\varphi_0$ and $\varphi$ are approximately unitarily equivalent. As $B$ is in $\mathcal{T}$, we have already known that the lemma holds for $\varphi_0: C\to B$. Therefore the lemma holds for $\psi\circ\varphi_0: C\to A$, and hence for $\varphi: C\to A$. $\square$

**Remark 6.2.** In the lemma above the finite subset $L\subset U_\infty(C)$ is allowed to be *any* finite subset which generates $K_1(C)$, though this point is not clearly mentioned in [18, Lemma 6.11]. This readily follows from the fact that (if $F$ is large enough and $\varepsilon$ is small enough, then) $\mathrm{Bott}(\varphi,u)$ gives rise to a 'partial homomorphism' from $K_1(C)$ to $K_0(A)$, as mentioned in Section 2.3.

In what follows, we frequently omit '$\otimes\,\mathrm{id}$', '$\otimes 1$' and '$\otimes\,\mathrm{Tr}$' to simplify notation. For example, $u\otimes 1\in A\otimes M_n$ is denoted by $u$.

**Lemma 6.3.** *Let $C$ be a $C^*$-algebra of the form $p(C(X)\otimes M_k)p$, where $X$ is a finite CW complex and $p\in C(X)\otimes M_k$ is a projection. Let $A\in\mathcal{T}\cup\mathcal{T}'$. Suppose that unital monomorphisms $\varphi,\psi: C\to A$ satisfy $KL(\varphi)=KL(\psi)$, $T(\varphi)=T(\psi)$. Let $L\subset U_\infty(C)$ be a finite subset which generates $K_1(C)$. For any finite subset $F\subset C$ and $\varepsilon>0$, there exists $\delta>0$ such that the following holds. If $\eta: K_1(C)\to\mathrm{Aff}(T(A))$ is a homomorphism satisfying*

$$\eta(x)+\mathrm{Im}\,D_A=\Theta_{\varphi,\psi}(x)\quad\forall x\in K_1(C)$$

*and*

$$\|\eta([w])\|<\delta\quad\forall w\in L,$$

*then there exists a unitary $u\in A$ such that*

$$\|\varphi(f)-u\psi(f)u^*\|<\varepsilon\quad\forall f\in F$$

*and*

$$\frac{1}{2\pi\sqrt{-1}}\tau(\log(\varphi(w)^*u\psi(w)u^*))=\eta([w])(\tau)\quad\forall\tau\in T(A),\ w\in L.$$

*Proof.* Applying Lemma 6.1 to $\psi$, $F$ and $\varepsilon/2$, we obtain $\delta > 0$. Suppose that $\eta \in \mathrm{Hom}(K_1(C), \mathrm{Aff}(T(A)))$ satisfies

$$\eta(x) + \mathrm{Im}\, D_A = \Theta_{\varphi,\psi}(x) \quad \forall x \in K_1(C)$$

and

$$\|\eta([w])\| < \delta/2 \quad \forall w \in L.$$

Choose a large finite subset $F_0 \subset C$ and a small real number $\varepsilon_0 > 0$. By virtue of Corollary 4.9, there exists a unitary $u_1 \in A$ such that

$$\|\varphi(f) - u_1\psi(f)u_1^*\| < \min\{\varepsilon_0, \varepsilon/2\} \quad \forall f \in F_0 \cup F.$$

Put $\psi' = \mathrm{Ad}\, u_1 \circ \psi$. For each $w \in U_\infty(C)$ satisfying $\|\varphi(w) - \psi'(w)\| < 2$, the function

$$z_w : \tau \mapsto \frac{1}{2\pi\sqrt{-1}}\tau(\log(\varphi(w)^*\psi'(w)))$$

gives an element of $\mathrm{Aff}(T(A))$. By [11, Lemma 1], we can see the following (see also the proof of Lemma 3.1).

- If $w_1, w_2 \in U_\infty(C)$ satisfy $\|\varphi(w_1) - \psi'(w_1)\| + \|\varphi(w_2) - \psi'(w_2)\| < 2$, then $z_{w_1w_2} = z_{w_1} + z_{w_2}$.
- If $w : [0,1] \to U(C \otimes M_n)$ is a path of unitaries satisfying $\|\varphi(w(t)) - \psi'(w(t))\| < 2$, then $z_{w(0)} = z_{w(1)}$.

Therefore, if $F_0$ is large enough and $\varepsilon_0$ is small enough, then there exists a homomorphism $\zeta : K_1(C) \to \mathrm{Aff}(T(A))$ such that

$$\zeta([w])(\tau) = z_w(\tau) = \frac{1}{2\pi\sqrt{-1}}\tau(\log(\varphi(w)^*\psi'(w))) \quad \forall \tau \in T(A),\ w \in L.$$

Clearly we may further assume that $\varphi(w)$ and $u_1\psi(w)u_1^*$ are close enough to imply $\|\zeta([w])\| < \delta/2$ for every $w \in L$. We also have $\eta([w]) - \zeta([w]) \in \mathrm{Im}\, D_A$ by Lemma 3.1 (2). Hence there exists $\xi \in \mathrm{Hom}(K_1(C), K_0(A))$ such that $D_A(\xi(x)) = \eta(x) - \zeta(x)$ for any $x \in K_1(C)$ and $\xi(x) = 0$ for any $x \in \mathrm{Tor}(K_1(C))$. Moreover one has $\|D_A(\xi([w]))\| < \delta/2 + \delta/2 = \delta$. It follows from Lemma 6.1 that there exists a unitary $u_2 \in A$ such that

$$\|[\psi(f), u_2]\| < \varepsilon/2 \quad \forall f \in F$$

and

$$\mathrm{Bott}(\psi, u_2)(w) = \xi([w]) \quad \forall w \in L.$$

Set $u = u_1u_2$. It is straightforward to check that

$$\|\varphi(f) - u\psi(f)u^*\| < \varepsilon$$

holds for any $f \in F$. Besides, for any $\tau \in T(A)$ and $w \in L$,

$$\begin{aligned}\tau(\log(\varphi(w)^* u\psi(w)u^*)) &= \tau(\log(\varphi(w)^* u_1u_2\psi(w)u_2^*u_1^*)) \\ &= \tau(\log(\varphi(w)^* u_1\psi(w)u_1^*u_1\psi(w)^*u_2\psi(w)u_2^*u_1^*)) \\ &= \tau(\log(\varphi(w)^* u_1\psi(w)u_1^*)) + \tau(\log(\psi(w)^*u_2\psi(w)u_2^*)) \\ &= 2\pi\sqrt{-1}(\zeta([w])(\tau) + D_A(\mathrm{Bott}(\psi, u_2)(w))(\tau)) \\ &= 2\pi\sqrt{-1}(\zeta([w])(\tau) + D_A(\xi([w]))(\tau)) \\ &= 2\pi\sqrt{-1}\eta([w])(\tau),\end{aligned}$$

where we have used [18, Theorem 3.6]. $\square$

The following lemma is an easy exercise and we leave it to the reader.

**Lemma 6.4.** *Let $L$ be a finitely generated abelian group and let $M$ be an abelian group. Let $N_0$ and $N_1$ be subgroups of $\mathbb{Q}$ and let $N \subset \mathbb{Q}$ be the subgroup generated by $N_0, N_1$. Then for any $\xi \in \mathrm{Hom}(L, M \otimes N)$, there exist $\xi_j \in \mathrm{Hom}(L, M \otimes N_j)$ such that $\xi = \xi_1 - \xi_0$.*

For each infinite supernatural number $\mathfrak{p}$ we let $M_{\mathfrak{p}}$ denote the UHF algebra of type $\mathfrak{p}$. Let $\mathfrak{p}, \mathfrak{q}$ be relatively prime infinite supernatural numbers such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} \cong Q$. As in [32], define a $C^*$-algebra $Z$ by

$$Z = \{f \in C([0,1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) \mid f(0) \in M_{\mathfrak{p}} \otimes \mathbb{C},\ f(1) \in \mathbb{C} \otimes M_{\mathfrak{q}}\}.$$

The following proposition is the main part of the proof of Theorem 6.6.

**Proposition 6.5.** *Let $X$ be a connected finite CW complex and let $A \in \mathcal{C} \cup \mathcal{C}'$. Suppose that two unital monomorphisms $\varphi, \psi : C(X) \to A$ satisfy $KL(\varphi) = KL(\psi)$, $T(\varphi) = T(\psi)$ and $\mathrm{Im}\,\Theta_{\varphi,\psi} \subset \overline{\mathrm{Im}\, D_A}$. Then for any finite subset $F \subset C(X)$ and $\varepsilon > 0$, there exists a unitary $u \in A \otimes Z$ such that*

$$\|\varphi(f) \otimes 1 - u(\psi(f) \otimes 1)u^*\| < \varepsilon$$

*holds for any $f \in F$.*

*Proof.* We write $Q = M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$, $B_0 = M_{\mathfrak{p}} \otimes \mathbb{C}$ and $B_1 = \mathbb{C} \otimes M_{\mathfrak{q}}$. By Remark 2.3, $A \otimes Q$, $A \otimes B_0$ and $A \otimes B_1$ are in $\mathcal{T} \cup \mathcal{T}'$. Set $\bar{\varphi}(f) = \varphi(f) \otimes 1$ and $\bar{\psi}(f) = \psi(f) \otimes 1$. We regard $\bar{\varphi}$ and $\bar{\psi}$ as homomorphisms from $C(X)$ into $A \otimes Q$ or $A \otimes B_j$. We identify $T(A \otimes Q)$, $T(A \otimes B_j)$ with $T(A)$. In the same way as Lemma 6.3, to simplify notation, for $u \in A$ we denote $u \otimes 1 \in A \otimes M_n$ by $u$. Similarly, for $\tau \in T(A)$, $\tau \otimes \mathrm{Tr}$ on $A \otimes M_n$ is written by $\tau$ for short.

Applying Theorem 5.3 to $\bar{\psi} : C(X) \to A \otimes Q$, $F$ and $\varepsilon/2$, we obtain a finite subset $L \subset \mathcal{K}(C(X))$, a finite subset $G_1 \subset C(X)$ and $\delta_1 > 0$. By Remark 5.5, we may and do assume that $L$ is written as $L = L_0 \cup L_1$, where $L_0$ is a finite subset of $P(C(X) \otimes \mathbb{K})$ and $L_1$ is a finite subset of $U_\infty(C(X))$. We may further assume that $L_1$ generates $K_1(C(X))$. Since $K_i(C(X))$ is finitely generated, one can find a finite subset $G_2 \subset C(X)$ and $\delta_2 > 0$ such that the following holds: For any unitary $w \in A \otimes Q$ satisfying $\|[\bar{\psi}(f), w]\| < \delta_2$ for any $f \in G_2$, there exist $\xi_i \in \mathrm{Hom}(K_i(C(X)), K_{1-i}(A \otimes Q))$ such that $\xi_i([s]) = \mathrm{Bott}(\bar{\psi}, w)(s)$ for any $s \in L_i$ and $i = 0, 1$ ([19, Section 2]). We may assume that $G_2$ contains $F \cup G_1$ and that $\delta_2$ is less than $\min\{\varepsilon/2, \delta_1/2\}$. By applying Lemma 6.3 to $\bar{\varphi}, \bar{\psi} : C(X) \to A \otimes B_j$, $G_2 \subset C(X)$ and $\delta_2/2$, we get $\delta_{3,j} > 0$ for each $j = 0, 1$.

Since $K_1(C(X))$ is finitely generated and the homomorphism $\Theta_{\varphi,\psi}$ factors through $K_1(C(X))/\operatorname{Tor}(K_1(C(X)))$ by Lemma 3.1 (4), there exists $\eta \in \operatorname{Hom}(K_1(C(X)), \operatorname{Aff}(T(A)))$ such that

$$\eta(x) + \operatorname{Im} D_A = \Theta_{\varphi,\psi}(x) \quad \forall x \in K_1(C(X)).$$

Moreover, we may assume $\|\eta([w])\| < \min\{\delta_{3,0}, \delta_{3,1}\}$ for all $w \in L_1$ because $\operatorname{Im}\Theta_{\varphi,\psi}$ is contained in the closure of $\operatorname{Im} D_A$. It follows from Lemma 6.3 that there exists a unitary $u_j \in A \otimes B_j$ such that

$$\|\bar{\varphi}(f) - u_j \bar{\psi}(f) u_j^*\| < \delta_2/2 \quad \forall f \in G_2$$

and

$$\frac{1}{2\pi\sqrt{-1}} \tau(\log(\bar{\varphi}(w)^* u_j \bar{\psi}(w) u_j^*)) = \eta([w])(\tau) \quad \forall \tau \in T(A \otimes B_j),\ w \in L_1.$$

In particular one has $\|[\bar{\psi}(f), u_1^* u_0]\| < \delta_2$ for $f \in G_2$. From the choice of $G_2$ and $\delta_2$, we can find $\xi_i \in \operatorname{Hom}(K_i(C(X)), K_{1-i}(A \otimes Q))$ such that $\xi_i([s]) = \operatorname{Bott}(\bar{\psi}, u_1^* u_0)(s)$ holds for any $s \in L_i$ and $i = 0, 1$. By [18, Theorem 3.6],

$$\begin{aligned}
D_{A\otimes Q}(\xi_1([w]))(\tau) &= D_{A\otimes Q}(\operatorname{Bott}(\bar{\psi}, u_1^* u_0)(w))(\tau) \\
&= \frac{1}{2\pi\sqrt{-1}} \tau(\log(u_1^* u_0 \bar{\psi}(w) u_0^* u_1 \bar{\psi}(w)^*)) \\
&= \frac{1}{2\pi\sqrt{-1}} \big(\tau(\log(\bar{\varphi}(w)^* u_0 \bar{\psi}(w) u_0^*)) - \tau(\log(\bar{\varphi}(w)^* u_1 \bar{\psi}(w) u_1^*))\big) \\
&= \eta([w])(\tau) - \eta([w])(\tau) = 0
\end{aligned}$$

for any $\tau \in T(A \otimes Q)$. Thus $\operatorname{Im}\xi_1$ is contained in $\operatorname{Ker} D_{A\otimes Q}$. By Lemma 6.4, we can find $\xi_{1,j} : K_1(C(X)) \to \operatorname{Ker} D_{A\otimes B_j}$ such that $\xi_1 = \xi_{1,1} - \xi_{1,0}$, where $\operatorname{Ker} D_{A\otimes C}$ is naturally identified with $(\operatorname{Ker} D_A) \otimes K_0(C)$ for $C = Q, B_0, B_1$. In the same way, one obtains $\xi_{0,j} : K_0(C(X)) \to K_1(A \otimes B_j)$ such that $\xi_0 = \xi_{0,1} - \xi_{0,0}$.

We consider the following exact sequence of $C^*$-algebras:

$$0 \longrightarrow C_0(X \times (\mathbb{T} \setminus \{-1\})) \xrightarrow{\iota} C(X \times \mathbb{T}) \xrightarrow{\pi} C(X) \longrightarrow 0,$$

where $\pi$ is the evaluation at $-1 \in \mathbb{T}$. We write $S = C_0(\mathbb{T} \setminus \{-1\})$ for short. Let $\rho : C(X) \to C(X \times \mathbb{T})$ be the homomorphism defined by $\rho(f) = f \otimes 1$. Then $\pi \circ \rho$ is the identity on $C(X)$. This split exact sequence induces the isomorphism

$$(a, b) \mapsto KL(\rho)(a) + KL(\iota)(b)$$

from $\underline{K}(C(X)) \oplus \underline{K}(C(X) \otimes S)$ to $\underline{K}(C(X \times \mathbb{T}))$. Let $\omega_i : K_i(C(X) \otimes S) \to K_{1-i}(C(X))$ be the canonical isomorphism for each $i = 0, 1$. For each $j = 0, 1$, choose $\kappa_j \in KL(C(X) \otimes S, A \otimes B_j)$ such that $K_i(\kappa_j) = \xi_{1-i,j} \circ \omega_i$. Define $\tilde{\kappa}_j \in KL(C(X \times \mathbb{T}), A \otimes B_j)$ by

$$\tilde{\kappa}_j \circ KL(\rho) = KL(\bar{\psi}) \quad \text{and} \quad \tilde{\kappa}_j \circ KL(\iota) = \kappa_j.$$

Clearly $K_0(\tilde{\kappa}_j)$ is unital. Also, $K_0(\tilde{\kappa}_j) \circ K_0(\rho) = K_0(\bar{\psi})$ is (strictly) positive and the image of

$$K_0(\tilde{\kappa}_j) \circ K_0(\iota) = K_0(\kappa_j) = \xi_{1,j} \circ \omega_0$$

is contained in $\operatorname{Ker} D_{A\otimes B_j}$. It follows from Lemma 2.4 that $K_0(\tilde{\kappa}_j)$ is unital and (strictly) positive. Thus, $\tilde{\kappa}_j$ is in $KL(C(X\times\mathbb{T}), A\otimes B_j)_{+,1}$. Let $\tau_0 \in T(C(\mathbb{T}))$ be the tracial state corresponding to the Haar measure on $\mathbb{T}$ and define the affine continuous map $\lambda : T(A\otimes B_j) \to T(C(X\times\mathbb{T}))$ by $\lambda(\tau) = T(\bar{\psi})(\tau)\otimes\tau_0$. Thanks to Theorem 2.6, there exists a unital monomorphism $\sigma_j : C(X\times\mathbb{T}) \to A\otimes B_j$ such that $KL(\sigma_j) = \tilde{\kappa}_j$ and $T(\sigma_j) = \lambda$. Since $KL(\sigma_j\circ\rho) = KL(\bar{\psi})$ and $T(\sigma_j\circ\rho) = T(\bar{\psi})$, $\bar{\psi}$ and $\sigma_j\circ\rho$ are approximately unitarily equivalent by Theorem 4.8. Hence there exists a unitary $v_j \in A\otimes B_j$ such that

$$\|[\bar{\psi}(f), v_j]\| < \delta_2/2 \quad \forall f \in G_2$$

and

$$\begin{aligned}\operatorname{Bott}(\bar{\psi}, v_j)(s) &= (K_{1-i}(\sigma_j)\circ K_{1-i}(\iota)\circ\omega_{1-i}^{-1})([s]) \\ &= (K_{1-i}(\kappa_j)\circ\omega_{1-i}^{-1})([s]) \\ &= (\xi_{i,j}\circ\omega_{1-i}\circ\omega_{1-i}^{-1})([s]) \\ &= \xi_{i,j}([s])\end{aligned}$$

for any $s \in L_i$ and $i = 0, 1$.

It is easy to see that

$$\|\bar{\varphi}(f) - u_j v_j \bar{\psi}(f) v_j^* u_j^*\| < \delta_2/2 + \delta_2/2 = \delta_2$$

holds for any $f \in G_2$. In particular one has

$$\|[\bar{\psi}(f), v_1^* u_1^* u_0 v_0]\| < 2\delta_2 < \delta_1 \quad \forall f \in G_2.$$

Besides, when $G_2$ is sufficiently large and $\delta_2$ is sufficiently small, we get

$$\begin{aligned}\operatorname{Bott}(\bar{\psi}, v_1^* u_1^* u_0 v_0)([s]) &= \operatorname{Bott}(\bar{\psi}, v_1^*)([s]) + \operatorname{Bott}(\bar{\psi}, u_1^* u_0)([s]) + \operatorname{Bott}(\bar{\psi}, v_0)([s]) \\ &= -\xi_{i,1}([s]) + \xi_i([s]) + \xi_{i,0}([s]) = 0\end{aligned}$$

for any $s \in L_i$ and $i = 0, 1$, where we have used [18, (e2.6)]. Therefore, by Theorem 5.3, we can find a path of unitaries $w : [0,1] \to A\otimes Q$ such that $w(0) = v_1^* u_1^* u_0 v_0$, $w(1) = 1$ and

$$\|[\bar{\psi}(f), w(t)]\| < \varepsilon/2 \quad \forall f \in F,\ t \in [0,1].$$

Define a unitary $U \in Z$ by $U(t) = u_1 v_1 w(t)$. It is easy to see that

$$\|\varphi(f)\otimes 1 - U(\psi(f)\otimes 1)U^*\| < \varepsilon/2 + \delta_2 < \varepsilon$$

holds for any $f \in F$. $\square$

**Theorem 6.6.** *Let $X$ be a compact metrizable space and let $A \in \mathcal{C}\cup\mathcal{C}'$. For unital monomorphisms $\varphi, \psi : C(X) \to A$, the following two conditions are equivalent.*

1. *$\varphi$ and $\psi$ are approximately unitarily equivalent.*
2. *$KL(\varphi) = KL(\psi)$, $\tau\circ\varphi = \tau\circ\psi$ for any $\tau \in T(A)$ and $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$.*

*Proof.* It is straightforward to check that (1) implies (2). Indeed, $KL(\varphi) = KL(\psi)$ and $T(\varphi) = T(\psi)$ are clear. By Lemma 3.1, $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$ also follows.

We would like to show the other implication. We first consider the case that $X$ is a connected finite CW complex. Let $\varphi, \psi : C(X) \to A$ be unital monomorphisms satisfying $KL(\varphi) = KL(\psi)$, $T(\varphi) = T(\psi)$ and $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$. We may replace the target algebra $A$ with $A \otimes \mathcal{Z}$, because $A$ is $\mathcal{Z}$-absorbing. Since $\mathcal{Z}$ is strongly self-absorbing ([34]), there exists an approximately inner endomorphism $\pi : A \otimes \mathcal{Z} \to A \otimes \mathcal{Z}$ such that $\pi(A \otimes \mathcal{Z}) = A \otimes \mathbb{C}$. Hence $\varphi$ and $\pi \circ \varphi$ (resp. $\psi$ and $\pi \circ \psi$) are approximately unitarily equivalent. By [32, Proposition 3.3], the $C^*$-algebra $Z$ embeds unitally into $\mathcal{Z}$. It follows from Proposition 6.5 that $\pi \circ \varphi$ and $\pi \circ \psi$ are approximately unitarily equivalent. Therefore $\varphi$ and $\psi$ are approximately unitarily equivalent.

A general finite CW complex is a finite union of pairwise disjoint connected finite CW complexes. Since $A$ has cancellation by [31, Theorem 6.7] and $eAe$ is in $\mathcal{C} \cup \mathcal{C}'$ for any non-zero projection $e \in A$, the conclusion follows from the previous case.

Let $X$ be a compact metrizable space. Let $\{f_1, f_2, \dots, f_n\}$ be a finite subset of $C(X)$ and let $\varepsilon > 0$. By [23, Lemma 1], there exist a finite CW complex (actually a finite simplicial complex) $Y$, a finite subset $\{g_1, g_2, \dots, g_n\}$ of $C(Y)$ and a unital monomorphism $\sigma : C(Y) \to C(X)$ such that $\|f_i - \sigma(g_i)\| < \varepsilon/3$ for any $i = 1, 2, \dots, n$. Clearly $KL(\varphi \circ \sigma) = KL(\psi \circ \sigma)$, $T(\varphi \circ \sigma) = T(\psi \circ \sigma)$ and $\operatorname{Im}\Theta_{\varphi\circ\sigma,\psi\circ\sigma}$ is contained in the closure of $\operatorname{Im} D_A$. It follows from the argument above that $\varphi \circ \sigma$ and $\psi \circ \sigma$ are approximately unitarily equivalent. Hence there exists a unitary $u \in A$ such that $\|\varphi(\sigma(g_i)) - u\psi(\sigma(g_i))u^*\| < \varepsilon/3$, which implies $\|\varphi(f_i) - u\psi(f_i)u^*\| < \varepsilon$. Thus, $\varphi$ and $\psi$ are approximately unitarily equivalent. $\square$

**Remark 6.7.** In the theorem above, if $A$ has real rank zero, then the image of $D_A$ is dense in $\operatorname{Aff}(T(A))$. Hence the condition $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$ is trivially satisfied.

**Corollary 6.8.** *Let $C$ be a unital AH algebra and let $A \in \mathcal{C} \cup \mathcal{C}'$. For unital monomorphisms $\varphi, \psi : C \to A$, the following two conditions are equivalent.*

(1) *$\varphi$ and $\psi$ are approximately unitarily equivalent.*

(2) *$KL(\varphi) = KL(\psi)$, $\tau \circ \varphi = \tau \circ \psi$ for any $\tau \in T(A)$ and $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$.*

*Proof.* We can prove this in the same way as Corollary 4.9. $\square$

## 7 Homomorphisms between simple $\mathcal{Z}$-stable $C^*$-algebras

In this section we prove Theorem 7.1. The main idea is almost the same as Proposition 6.5 and Theorem 6.6. The proof is, however, somewhat lengthy because we must work with finitely generated subgroups of $K_*(C)$ so as to use Lemma 6.4.

**Theorem 7.1.** *Let $C$ be a nuclear $C^*$-algebra in $\mathcal{C}$ satisfying the UCT and let $A \in \mathcal{C} \cup \mathcal{C}'$. For any unital homomorphisms $\varphi, \psi : C \to A$, the following are equivalent.*

(1) *$\varphi$ and $\psi$ are approximately unitarily equivalent.*

(2) *$KL(\varphi) = KL(\psi)$ and $\operatorname{Im}\Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$.*

*Proof.* The implication (1)⇒(2) is trivial. We would like to show the other implication. Note that $KL(\varphi) = KL(\psi)$ implies $T(\varphi) = T(\psi)$, because projections of $C$ separate traces. Since $\mathcal{Z}$ is strongly self-absorbing ([34]) and $C$ (resp. $A$) is $\mathcal{Z}$-stable by assumption, there exists an isomorphisms $\pi_C : C \to C \otimes \mathcal{Z}$ (resp. $\pi_A : A \to A \otimes \mathcal{Z}$) which is approximately unitarily equivalent to the unital monomorphism $c \mapsto c \otimes 1$. It is not so hard to see that any unital homomorphism $\varphi : C \to A$ is approximately unitarily equivalent to $\pi_A^{-1} \circ (\varphi \otimes \mathrm{id}) \circ \pi_C$. Hence, for given homomorphisms $\varphi, \psi$ satisfying $KL(\varphi) = KL(\psi)$ and $\operatorname{Im} \Theta_{\varphi,\psi} \subset \overline{\operatorname{Im} D_A}$, it suffices to show that $\varphi \otimes \mathrm{id} : C \otimes \mathcal{Z} \to A \otimes \mathcal{Z}$ is approximately unitarily equivalent to $\psi \otimes \mathrm{id} : C \otimes \mathcal{Z} \to A \otimes \mathcal{Z}$.

Suppose that we are given a finite subset $F \subset C \otimes \mathcal{Z}$ and $\varepsilon > 0$. We would like to show $\varphi \otimes \mathrm{id} \sim_{F,\varepsilon} \psi \otimes \mathrm{id}$. Without loss of generality, we may assume that $F$ is contained in $C \otimes \mathbb{C}$.

Let $Z, B_0, B_1, Q$ be as in Proposition 6.5. We identify $T(A \otimes Q)$, $T(A \otimes B_j)$ with $T(A)$. Put $\bar{\varphi} = \varphi \otimes \mathrm{id}$, $\bar{\psi} = \psi \otimes \mathrm{id}$. As in the proof of Proposition 6.5, we omit '$\otimes \mathbb{C}$', '$\otimes 1$' and '$\otimes \mathrm{Tr}$' to simplify notation.

By the classification theorem in [16], $C \otimes Q$ is a unital simple AT algebra with real rank zero. Thus, $C \otimes Q$ can be written as an inductive limit of $C^*$-algebras of the form $\bigoplus_{i=1}^n M_{k_i}(C(\mathbb{T}))$. By using Theorem 5.4 to $\bar{\psi} : C \otimes Q \to A \otimes Q$, $F$ and $\varepsilon/2$, we obtain a finite subset $L \subset \mathcal{K}(C \otimes Q)$, a finite subset $G_1 \subset C \otimes Q$ and $\delta_1 > 0$. By Remark 5.5, we may and do assume that $L$ is written as $L = L_0 \cup L_1$, where $L_0$ is a finite subset of $P(C \otimes Q \otimes \mathbb{K})$ and $L_1$ is a finite subset of $U_\infty(C \otimes Q)$. We may further assume that $L_0$ and $L_1$ are finite subsets of $P(C \otimes \mathbb{K})$ and $U_\infty(C)$ respectively, because Bott gives rise to a 'partial homomorphism' (see Section 2.3). Let $H_i \subset K_i(C)$ be the subgroup generated by $L_i$. Since $H_i$ is finitely generated, one can find a finite subset $G_2 \subset C \otimes Q$ and $\delta_2 > 0$ such that the following holds: For any unitary $w \in A \otimes Q$ satisfying $\|[\bar{\psi}(c), w]\| < \delta_2$ for any $c \in G_2$, there exist $\xi_i \in \operatorname{Hom}(H_i, K_{1-i}(A \otimes Q))$ such that $\xi_i([s]) = \mathrm{Bott}(\bar{\psi}, w)(s)$ for any $s \in L_i$ and $i = 0, 1$ ([19, Section 2]). We may assume that $G_2$ contains $G_1$ and that $\delta_2$ is less than $\delta_1$. As $C \otimes Q$ is generated by $C$, $B_0$ and $B_1$, one may choose finite subsets $G_3 \subset C$, $G_{3,0} \subset B_0$, $G_{3,1} \subset B_1$ and $\delta_3 > 0$ so that if a unitary $w \in A \otimes Q$ satisfies $\|[\bar{\psi}(c), w]\| < \delta_3$ for every $c \in G_3 \cup G_{3,0} \cup G_{3,1}$, then $\|[\bar{\psi}(c), w]\| < \delta_2$ holds for all $c \in G_2$. We assume that $G_3$ contains $F$ and that $\delta_3$ is less than $\varepsilon$. For each $j = 0, 1$, by the classification theorem in [16], $C \otimes B_j$ is a unital simple AH algebra with real rank zero and slow dimension growth, and so one can find a unital subalgebra $C_j \subset C \otimes B_j$ such that the following hold.

- $C_j$ is a finite direct sum of $C^*$-algebras of the form $p(C(X) \otimes M_k)p$, where $X$ is a connected finite CW complex (with dimension at most three) and $p \in C(X) \otimes M_k$ is a projection.
- There exists a finite subset $G'_{3,j} \subset C_j$ such that any elements of $G_3 \cup G_{3,j}$ are within distance $\delta_3/12$ of $G'_{3,j}$.
- There exist finite subsets $L'_{0,j} \in P(C_j \otimes \mathbb{K})$ and $L'_{1,j} \subset U_\infty(C_j)$ such that any elements of $L_i$ are within distance $1/2$ of $L'_{i,j}$ for each $i = 0, 1$. Let $L_i \ni s \mapsto s'_j \in L'_{i,j}$ be a map such that $\|s - s'_j\| < 1/2$. We further require that $L'_{1,j}$ generates $K_1(C_j)$.

Let $\gamma_j : C_j \to C \otimes B_j$ denote the embedding map. For each $i = 0, 1$, we choose a finitely generated subgroup $H'_i \subset K_i(C)$ so that $H_i$ is contained in $H'_i$ and $\operatorname{Im} K_i(\gamma_j)$ is contained in $H'_i \otimes K_i(B_j)$ for each $j = 0, 1$. Applying Lemma 6.3 to $\bar{\varphi} \circ \gamma_j : C_j \to A \otimes B_j$, $\bar{\psi} \circ \gamma_j : C_j \to A \otimes B_j$, $G'_{3,j}$ and $\delta_3/12$, we get $\delta_{4,j} > 0$ for each $j = 0, 1$.

As in the proof of Proposition 6.5, we can find a homomorphism $\eta : H'_1 \to \mathrm{Aff}(T(A))$ such that

$$\eta(x) + \mathrm{Im}\, D_A = \Theta_{\varphi,\psi}(x) \quad \forall x \in H'_1$$

and

$$\|\tilde{\eta}_j([\gamma_j(w)])\| < \delta_{4,j} \quad \forall w \in L'_{1,j},\ j = 0, 1,$$

where $\tilde{\eta}_j \in \mathrm{Hom}(H'_1 \otimes K_0(B_j), \mathrm{Aff}(T(A)))$ denotes the homomorphism induced from $\eta$. It follows from Lemma 6.3 that there exists a unitary $u_j \in A \otimes B_j$ such that

$$\|\bar{\varphi}(c) - u_j \bar{\psi}(c) u_j^*\| < \delta_3/12 \quad \forall c \in G'_{3,j}$$

and

$$\frac{1}{2\pi\sqrt{-1}} \tau(\log(\bar{\varphi}(w)^* u_j \bar{\psi}(w) u_j^*)) = \tilde{\eta}_j([\gamma_j(w)])(\tau) \quad \forall \tau \in T(A \otimes B_j),\ w \in L'_{1,j}.$$

By choosing $G'_{3,j}$ large enough in advance, we may also assume that $\|\bar{\varphi}(w) - u_j \bar{\psi}(w) u_j^*\|$ is less than $1/2$ for every $w \in L'_{1,j}$. From the choice of $G'_{3,j}$, we obtain $\|\bar{\varphi}(c) - u_j \bar{\psi}(c) u_j^*\| < \delta_3/4$ for all $c \in G_3 \cup G_{3,j}$. If $c$ is in $G_{3,1-j}$, then $u_j \in A \otimes B_j$ commutes with $\bar{\psi}(c) \in B_{1-j}$, and so $\bar{\varphi}(c) = \bar{\psi}(c) = u_j \bar{\psi}(c) u_j^*$. Therefore

$$\|[\bar{\psi}(c), u_1^* u_0]\| < \delta_3/2 \quad \forall c \in G_3 \cup G_{3,0} \cup G_{3,1}.$$

From the choice of $G_3$, $G_{3,0}$, $G_{3,1}$ and $\delta_3$, one has

$$\|[\bar{\psi}(c), u_1^* u_0]\| < \delta_2 \quad \forall c \in G_2.$$

Then, from the choice of $G_2$ and $\delta_2$, we can find $\xi_i \in \mathrm{Hom}(H_i, K_{1-i}(A \otimes Q))$ such that $\xi_i([s]) = \mathrm{Bott}(\bar{\psi}, u_1^* u_0)(s)$ holds for any $s \in L_i$ and $i = 0, 1$. By [18, Theorem 3.6],

$$\begin{aligned}
D_{A\otimes Q}(\xi_1([w]))(\tau) &= D_{A\otimes Q}(\mathrm{Bott}(\bar{\psi}, u_1^* u_0)(w))(\tau) \\
&= \frac{1}{2\pi\sqrt{-1}} \tau(\log(u_1^* u_0 \bar{\psi}(w) u_0^* u_1 \bar{\psi}(w)^*)) \\
&= \frac{1}{2\pi\sqrt{-1}} \big(\tau(\log(\bar{\varphi}(w)^* u_0 \bar{\psi}(w) u_0^*)) - \tau(\log(\bar{\varphi}(w)^* u_1 \bar{\psi}(w) u_1^*))\big) \\
&= \frac{1}{2\pi\sqrt{-1}} \big(\tau(\log(\bar{\varphi}(w'_0)^* u_0 \bar{\psi}(w'_0) u_0^*)) - \tau(\log(\bar{\varphi}(w'_1)^* u_1 \bar{\psi}(w'_1) u_1^*))\big) \\
&= \tilde{\eta}_0([w'_0])(\tau) - \tilde{\eta}_1([w'_1])(\tau) \\
&= \tilde{\eta}_0([w])(\tau) - \tilde{\eta}_1([w])(\tau) \\
&= \eta([w])(\tau) - \eta([w])(\tau) = 0
\end{aligned}$$

for any $w \in L_1$ and $\tau \in T(A \otimes Q)$. Thus $\mathrm{Im}\, \xi_1$ is contained in $\mathrm{Ker}\, D_{A\otimes Q}$. Since $\mathrm{Ker}\, D_{A\otimes Q} = (\mathrm{Ker}\, D_A) \otimes \mathbb{Q}$ is divisible, $\xi_1$ extends to a homomorphism $\xi_1 : H'_1 \to \mathrm{Ker}\, D_{A\otimes Q}$. Likewise $\xi_0$ extends to a homomorphism $\xi_0 : H'_0 \to K_1(A \otimes Q) = K_1(A) \otimes \mathbb{Q}$. By Lemma 6.4, we can find $\xi_{1,j} : H'_1 \to \mathrm{Ker}\, D_{A\otimes B_j}$ such that $\xi_1 = \xi_{1,1} - \xi_{1,0}$. We let $\tilde{\xi}_{1,j} : H'_1 \otimes K_0(B_j) \to \operatorname{Ker} D_{A\otimes B_j}$ denote the homomorphism induced from $\xi_{1,j}$. In the same way, one obtains $\xi_{0,j} : H'_0 \to K_1(A \otimes B_j)$ such that $\xi_0 = \xi_{0,1} - \xi_{0,0}$. We let $\tilde{\xi}_{0,j} : H'_0 \otimes K_0(B_j) \to K_1(A \otimes B_j)$ denote the homomorphism induced from $\xi_{0,j}$.

In the same way as in the proof of Proposition 6.5, for each $j = 0, 1$, we consider the following exact sequence of $C^*$-algebras:

$$0 \longrightarrow C_j \otimes C_0(\mathbb{T} \setminus \{-1\}) \xrightarrow{\iota_j} C_j \otimes C(\mathbb{T}) \xrightarrow{\pi_j} C_j \longrightarrow 0,$$

where $\pi_j$ is the evaluation at $-1 \in \mathbb{T}$. We write $S = C_0(\mathbb{T} \setminus \{-1\})$ for short. Let $\rho_j : C_j \to C_j \otimes C(\mathbb{T})$ be the homomorphism defined by $\rho_j(c) = c \otimes 1$. Then $\pi_j \circ \rho_j$ is the identity on $C_j$. This split exact sequence induces the isomorphism

$$(a, b) \mapsto KL(\rho_j)(a) + KL(\iota_j)(b)$$

from $\underline{K}(C_j) \oplus \underline{K}(C_j \otimes S)$ to $\underline{K}(C_j \otimes C(\mathbb{T}))$. Let $\omega_{i,j} : K_i(C_j \otimes S) \to K_{1-i}(C_j)$ be the canonical isomorphism for each $i, j = 0, 1$. For each $j = 0, 1$, choose $\kappa_j \in KL(C_j \otimes S, A \otimes B_j)$ so that

$$K_i(\kappa_j) = \tilde{\xi}_{1-i,j} \circ K_{1-i}(\gamma_j) \circ \omega_{i,j} \quad \forall i = 0, 1.$$

Notice that the composition of $\tilde{\xi}_{1-i,j}$ and $K_{1-i}(\gamma_j)$ is well-defined, because $\operatorname{Im} K_{1-i}(\gamma_j)$ is contained in $H'_{1-i} \otimes K_0(B_j)$. Define $\tilde{\kappa}_j \in KL(C_j \otimes C(\mathbb{T}), A \otimes B_j)$ by

$$\tilde{\kappa}_j \circ KL(\rho_j) = KL(\bar{\psi} \circ \gamma_j) \quad \text{and} \quad \tilde{\kappa}_j \circ KL(\iota_j) = \kappa_j.$$

Clearly $K_0(\tilde{\kappa}_j)$ is unital. Also for any $x \in K_0(C_j \otimes C(\mathbb{T}))_+ \setminus \{0\}$, one has $K_0(\pi_j)(x) \in K_0(C_j)_+ \setminus \{0\}$, and so $\tau(K_0(\bar{\psi} \circ \gamma_j \circ \pi_j)(x)) > 0$ for every $\tau \in T(A \otimes B_j)$. Since the image of $\tilde{\xi}_{1,j}$ is contained in the kernel of $D_{A\otimes B_j}$, we obtain

$$\tau(K_0(\tilde{\kappa}_j)(x)) = \tau(K_0(\bar{\psi} \circ \gamma_j \circ \pi_j)(x)) > 0,$$

which entails $K_0(\tilde{\kappa}_j)(x) \in K_0(A \otimes B_j)_+ \setminus \{0\}$. It thus follows that $K_0(\tilde{\kappa}_j)$ is unital and strictly positive, and hence $\tilde{\kappa}_j$ is in $KL(C_j \otimes C(\mathbb{T}), A \otimes B_j)_{+,1}$. Let $\tau_0 \in T(C(\mathbb{T}))$ be the tracial state corresponding to the Haar measure on $\mathbb{T}$ and define the affine continuous map $\lambda_j : T(A \otimes B_j) \to T(C_j \otimes C(\mathbb{T}))$ by $\lambda_j(\tau) = T(\bar{\psi} \circ \gamma_j)(\tau) \otimes \tau_0$. For each minimal central projection $p \otimes 1 \in C_j \otimes C(\mathbb{T})$ and $\tau \in T(A \otimes B_j)$, it is easy to verify

$$\tau(K_0(\tilde{\kappa}_j)([p \otimes 1])) = \tau(K_0(\bar{\psi} \circ \gamma_j)([\pi_j(p \otimes 1)])) = \tau(\bar{\psi}(\gamma_j(p))) = \lambda_j(\tau)(p \otimes 1).$$

Hence the hypotheses of Corollary 2.9 are satisfied. Thanks to Corollary 2.9, there exists a unital monomorphism $\sigma_j : C_j \otimes C(\mathbb{T}) \to A \otimes B_j$ such that $KL(\sigma_j) = \tilde{\kappa}_j$ and $T(\sigma_j) = \lambda_j$. Since $KL(\sigma_j \circ \rho_j) = KL(\bar{\psi} \circ \gamma_j)$ and $T(\sigma_j \circ \rho_j) = T(\bar{\psi} \circ \gamma_j)$, Corollary 4.9 implies that $\bar{\psi} \circ \gamma_j$ and $\sigma_j \circ \rho_j$ are approximately unitarily equivalent. Hence there exists a unitary $v_j \in A \otimes B_j$ such that

$$\|[\bar{\psi}(\gamma_j(c)), v_j]\| < \delta_3/12 \quad \forall c \in G'_{3,j}$$

and

$$\begin{aligned}
\operatorname{Bott}(\bar{\psi} \circ \gamma_j, v_j)(s) &= (K_{1-i}(\sigma_j) \circ K_{1-i}(\iota_j) \circ \omega_{1-i,j}^{-1})([s]) \\
&= (K_{1-i}(\kappa_j) \circ \omega_{1-i,j}^{-1})([s]) \\
&= (\tilde{\xi}_{i,j} \circ K_i(\gamma_j) \circ \omega_{1-i,j} \circ \omega_{1-i,j}^{-1})([s]) \\
&= \tilde{\xi}_{i,j}([\gamma_j(s)])
\end{aligned}$$

for any $s \in L'_{i,j}$ and $i = 0, 1$. As before,

$$\|[\bar{\psi}(c), v_j]\| < \delta_3/4$$

holds for any $c \in G_3 \cup G_{3,0} \cup G_{3,1}$. By choosing $G'_{3,j}$ large enough and $\delta_3$ small enough in advance, we have $\mathrm{Bott}(\bar{\psi}, v_j)(s) = \mathrm{Bott}(\bar{\psi} \circ \gamma_j, v_j)(s'_j)$ for any $s \in L_i$ and $i = 0, 1$.

It is easy to see that

$$\|\bar{\varphi}(c) - u_j v_j \bar{\psi}(c) v_j^* u_j^*\| < \delta_3/4 + \delta_3/4 = \delta_3/2$$

holds for any $c \in G_3 \cup G_{3,0} \cup G_{3,1}$. In particular one has

$$\|[\bar{\psi}(c), v_1^* u_1^* u_0 v_0]\| < \delta_3 \quad \forall c \in G_3 \cup G_{3,0} \cup G_{3,1},$$

and hence

$$\|[\bar{\psi}(c), v_1^* u_1^* u_0 v_0]\| < \delta_2 < \delta_1 \quad \forall c \in G_1,$$

because $G_1$ is contained in $G_2$. Besides, when $G_3$ is sufficiently large and $\delta_3$ is sufficiently small, we get

$$\begin{aligned}\mathrm{Bott}(\bar{\psi}, v_1^* u_1^* u_0 v_0)([s]) &= \mathrm{Bott}(\bar{\psi}, v_1^*)([s]) + \mathrm{Bott}(\bar{\psi}, u_1^* u_0)([s]) + \mathrm{Bott}(\bar{\psi}, v_0)([s]) \\ &= -\tilde{\xi}_{i,1}([s'_1]) + \xi_i([s]) + \tilde{\xi}_{i,0}([s'_0]) \\ &= -\xi_{i,1}([s]) + \xi_i([s]) + \xi_{i,0}([s]) = 0\end{aligned}$$

for any $s \in L_i$ and $i = 0, 1$, where we have used [18, (e2.6)]. Therefore, by Theorem 5.4, we can find a path of unitaries $w : [0, 1] \to A \otimes Q$ such that $w(0) = v_1^* u_1^* u_0 v_0$, $w(1) = 1$ and

$$\|[\bar{\psi}(c), w(t)]\| < \varepsilon/2 \quad \forall c \in F,\ t \in [0, 1].$$

Define a unitary $U \in Z$ by $U(t) = u_1 v_1 w(t)$. It is easy to see that

$$\|\varphi(c) \otimes 1 - U(\psi(c) \otimes 1)U^*\| < \varepsilon/2 + \delta_3/2 < \varepsilon$$

holds for any $c \in F$. □

**Remark 7.2.** In the theorem above, if $A$ has real rank zero, then the image of $D_A$ is dense in $\mathrm{Aff}(T(A))$. Hence the condition $\mathrm{Im}\,\Theta_{\varphi,\psi} \subset \overline{\mathrm{Im}\,D_A}$ is trivially satisfied.

# References


[1] B. Blackadar, *K-theory for operator algebras*, Mathematical Sciences Research Institute Publications, 5. Cambridge University Press, Cambridge, 1998.

[2] B. Blackadar, A. Kumjian and M. Rørdam, *Approximately central matrix units and the structure of non-commutative tori*, K-theory 6 (1992), 267–284.

[3] L. G. Brown, R. G. Douglas and P. A. Fillmore, *Extensions of $C^*$-algebras and K-homology*, Ann. of Math. 105 (1977), 265–324.

[4] M. Dadarlat, *Approximately unitarily equivalent morphisms and inductive limit $C^*$-algebras*, $K$-Theory 9 (1995), 117–137.

[5] M. Dadarlat and T. A. Loring, *A universal multicoefficient theorem for the Kasparov groups*, Duke Math. J. 84 (1996) 355–377.

[6] G. A. Elliott, *On the classification of $C^*$-algebras of real rank zero*, J. Reine Angew. Math. 443 (1993), 179–219.

[7] G. Gong, X. Jiang and H. Su, *Obstructions to $\mathcal{Z}$-stability for unital simple $C^*$-algebras*, Canad. Math. Bull. 43 (2000) 418–426.

[8] G. Gong and H. Lin, *Classification of homomorphisms from $C(X)$ to simple $C^*$-algebras of real rank zero*, Acta Math. Sin. (Engl. Ser.) 16 (2000), 181–206.

[9] G. Gong and H. Lin, *Almost multiplicative morphisms and $K$-theory*, Internat. J. Math. 11 (2000), 983-1000.

[10] U. Haagerup, *Every quasitrace on an exact $C^*$-algebra is a trace*, preprint, 1991.

[11] P. de la Harpe and G. Skandalis, *Déterminant associé à une trace sur une algébre de Banach*, Ann. Inst. Fourier (Grenoble) 34 (1984), 241–260.

[12] X. Jiang and H. Su, *On a simple unital projectionless $C^*$-algebra*, Amer. J. Math. 121 (1999), 359–413.

[13] A. Kishimoto and A. Kumjian, *The Ext class of an approximately inner automorphism, II*, J. Operator Theory, 46 (2001), 99-122.

[14] H. Lin, *An introduction to the classification of amenable $C^*$-algebras*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[15] H. Lin, *Embedding an AH-algebra into a simple $C^*$-algebra with prescribed $KK$-data*, $K$-Theory 24 (2001), 135–156.

[16] H. Lin, *Classification of simple $C^*$-algebras with tracial topological rank zero*, Duke Math. J. 125 (2004), 91–119.

[17] H. Lin, *Classification of homomorphisms and dynamical systems*, Trans. Amer. Math. Soc. 359 (2007), 859–895. arXiv:math/0404018

[18] H. Lin, *Asymptotically unitary equivalence and asymptotically inner automorphisms*, Amer. J. Math. 131 (2009), 1589-1677. arXiv:math/0703610

[19] H. Lin, *Approximate homotopy of homomorphisms from $C(X)$ into a simple $C^*$-algebra*, Mem. Amer. Math. Soc. 205 (2010), no. 963. arXiv:math/0612125

[20] H. Lin, *Approximate unitary equivalence in simple $C^*$-algebras of tracial rank one*, preprint. arXiv:0801.2929

[21] H. Lin and Z. Niu, *Lifting $KK$-elements, asymptotical unitary equivalence and classification of simple $C^*$-algebras*, Adv. Math. 219 (2008), 1729–1769. arXiv:0802.1484

[22] T. A. Loring, *$K$-theory and asymptotically commuting matrices*, Canad. J. Math. 40 (1988), 197–216.

[23] S. Mardešić, *On covering dimension and inverse limits of compact spaces*, Illinois J. Math. 4 (1960), 278–291.

[24] H. Matui and Y. Sato, *$\mathcal{Z}$-stability of crossed products by strongly outer actions*, preprint. arXiv:0912.4804

[25] K. E. Nielsen and K. Thomsen, *Limits of circle algebras*, Exposition. Math. 14 (1996), 17–56.

[26] P. W. Ng and W. Winter, *Commutative $C^*$-subalgebras of simple stably finite $C^*$-algebras with real rank zero*, Indiana Univ. Math. J. 57 (2008), 3209–3239. arXiv:math/0701035

[27] N. C. Phillips, *A classification theorem for nuclear purely infinite simple $C^*$-algebras*, Doc. Math. 5 (2000), 49–114.

[28] M. Rørdam, *On the structure of simple $C^*$-algebras tensored with a UHF-algebra*, J. Funct. Anal. 100 (1991), 1–17.

[29] M. Rørdam, *On the structure of simple $C^*$-algebras tensored with a UHF-algebra, II*, J. Funct. Anal. 107 (1992), 255–269.

[30] M. Rørdam, *Classification of nuclear, simple $C^*$-algebras*, Classification of nuclear $C^*$-algebras. Entropy in operator algebras, 1–145. Encyclopaedia Math. Sci. 126, Springer, Berlin, 2002.

[31] M. Rørdam, *The stable and the real rank of $\mathcal{Z}$-absorbing $C^*$-algebras*, Internat. J. Math. 15 (2004), 1065–1084. arXiv:math/0408020

[32] M. Rørdam and W. Winter, *The Jiang-Su algebra revisited*, J. Reine. Angew. Math. 642 (2010), 129–155. arXiv:0801.2259

[33] C. Schochet, *Topological methods for $C^*$-algebras, IV, Mod $p$ homology*, Pacific J. Math. 114 (1984) 447–468.

[34] A. S. Toms and W. Winter, *Strongly self-absorbing $C^*$-algebras*, Trans. Amer. Math. Soc. 359 (2007), 3999–4029. arXiv:math/0502211

[35] W. Winter, *Localizing the Elliott conjecture at strongly self-absorbing $C^*$-algebras*, preprint. arXiv:0708.0283