The absorption theorem for affable equivalence relations

Thierry Giordano *
Department of Mathematics and Statistics
University of Ottawa
585 King Edward, Ottawa, Ontario, Canada K1N 6N5

Hiroki Matui †
Graduate School of Science
Chiba University
1-33 Yayoi-cho, Inage-ku, Chiba 263-8522, Japan

Ian F. Putnam ‡
Department of Mathematics and Statistics
University of Victoria
Victoria, B.C., Canada V8W 3P4

Christian F. Skau §
Department of Mathematical Sciences
Norwegian University of Science and Technology (NTNU)
N-7034 Trondheim, Norway

Abstract

We prove a result about extension of a minimal AF-equivalence relation R on the Cantor set X, the extension being 'small' in the sense that we modify R on a thin closed subset Y of X. We show that the resulting extended equivalence relation S is orbit equivalent to the original R, and so, in particular, S is affable. Even in the simplest case—when Y is a finite set—this result is highly non-trivial. The result itself—called the absorption theorem—is a powerful and crucial tool for the study of the orbit structure of minimal $\mathbb{Z}^n$-actions

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on the Cantor set, see Remark 4.8. The absorption theorem is a significant generalization of the main theorem proved in [GPS2]. However, we shall need a few key results from [GPS2] in order to prove the absorption theorem.

1 Introduction

We introduce some basic definitions as well as relevant notation and terminology, and we refer to [GPS2] as a general reference for background and more details. Throughout this paper we will let X, Y or Z denote compact, metrizable and zero-dimensional spaces, i.e. compact spaces which have countable bases consisting of closed-open (clopen) subsets. Equivalently, the spaces are compact, metrizable and totally disconnected spaces. In particular, if a space does not have isolated points, it is homeomorphic to the (unique) Cantor set. We will study equivalence relations, denoted by R, S, K, on these spaces that are countable, i.e. all the equivalence classes are countable (including finite).

Let $R \subset X \times X$ be a countable equivalence relation on X, and let $[x]_R$ denote the (countable) R-equivalence class, $\{y \in X \mid (x, y) \in R\}$, of $x \in X$. We say that R is *minimal*, if all the R-equivalence classes are dense in X. R has a natural *groupoid structure*. Specifically, if $(x, y), (y, z) \in R$, then the product of this *composable* pair is defined by

$$(x, y) \cdot (y, z) = (x, z).$$

The inverse of $(x, y) \in R$ is defined to be $(x, y)^{-1} = (y, x)$. Let R be given a Hausdorff, locally compact and second countable (equivalently, metrizable) topology $\mathcal{T}$, so that the product of composable pairs (with the relative topology from the product topology on $R \times R$) is continuous. Also, the inverse map is required to be a homeomorphism on R. We say that $(R, \mathcal{T})$ is a locally compact (principal) groupoid. The range map $r : R \to X$ is defined by $r((x, y)) = x$, and the source map $s : R \to X$ is defined by $s((x, y)) = y$, both maps being surjective.

Definition 1.1 (Étale equivalence relation)**.** The locally compact groupoid $(R, \mathcal{T})$ is *étale*, if $r : R \to X$ is a local homeomorphism, i.e. for every $(x, y) \in R$ there exists an open neighbourhood $U^{(x,y)} \in \mathcal{T}$ of (x, y) such that $r(U^{(x,y)})$ is open in X and $r : U^{(x,y)} \to r(U^{(x,y)})$ is a homeomorphism.

Remark 1.2. Clearly r is an open map, and one may choose $U^{(x,y)}$ to be a clopen set (and so $r(U^{(x,y)})$ is a clopen subset of X). One thus gets that $(R, \mathcal{T})$ is a locally compact, metrizable and zero-dimensional space. Also, r being a local homeomorphism implies that s is a local homeomorphism as well. Occasionally we will refer to the local homeomorphism condition as the *étale condition*, and to $U^{(x,y)}$ as an *étale neighbourhood* (around (x, y)). It is noteworthy that only rarely will the topology $\mathcal{T}$ on $R \subset X \times X$ be the relative topology $\mathcal{T}_{\text{rel}}$ from $X \times X$. In general, $\mathcal{T}$ is a finer topology than $\mathcal{T}_{\text{rel}}$. For convenience, we will sometimes write R for $(R, \mathcal{T})$ when the topology $\mathcal{T}$ is understood from the context.

It is a fact that if R is étale, then the diagonal of R, $\Delta = \Delta_X (= \{(x,x) \mid x \in X\})$, is homeomorphic to X via the map $(x,x) \mapsto x$. We will often make the identification between Δ and X. Furthermore, Δ is an open subset of R. Also, R admits an (essentially) unique left Haar system consisting of counting measures. (See [P] for this. We shall not need this last fact in this paper.)

Definition 1.3 (Isomorphism and orbit equivalence)**.** Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two étale equivalence relations on X_1 and X_2, respectively. R_1 is *isomorphic* to R_2—we will write $R_1 \cong R_2$—if there exists a homeomorphism $F : X_1 \to X_2$ satisfying the following:

(i) $(x,y) \in R_1 \Leftrightarrow (F(x), F(y)) \in R_2$.

(ii) $F \times F : R_1 \to R_2$ is a homeomorphism, where $F \times F((x,y)) = (F(x), F(y))$ for $(x,y) \in R_1$.

We say that F *implements* an isomorphism between R_1 and R_2.

We say that R_1 is *orbit equivalent* to R_2 if (i) is satisfied, and we call F an *orbit map* in this case. (The term *orbit equivalence* is motivated by the important example of étale equivalence relations coming from group actions, where equivalence classes coincide with orbits (see below).)

There is a notion of invariant probability measure associated to an étale equivalence relation $(R, \mathcal{T})$ on X. In fact, if $(x,y) \in R$, there exists a clopen neighbourhood $U^{(x,y)} \in \mathcal{T}$ of (x,y) such that both $r : U^{(x,y)} \to r(U^{(x,y)}) = A$ and $s : U^{(x,y)} \to s(U^{(x,y)}) = B$ are homeomorphism, with A a clopen neighbourhood of $x \in X$, and B a clopen neighbourhood of $y \in X$. The map $\gamma = s \circ r^{-1} : A \to B$ is a homeomorphism such that $\operatorname{graph}(\gamma) = \{(x, \gamma(x)) \mid x \in A\} \subset R$. The triple (A, γ, B) is called a (local) *graph* in R, and by obvious identifications (in fact, $U^{(x,y)} = \operatorname{graph}(\gamma)$) the family of such graphs form a basis for $(R, \mathcal{T})$. Let μ be a probability measure on X. We say that μ is R*-invariant*, if $\mu(A) = \mu(B)$ for every graph (A, γ, B) in R. If $(R_G, \mathcal{T}_G)$ is the étale equivalence relation associated with the free action of the countable group G acting as homeomorphisms on X (see Example 1.4 below), then μ is R_G-invariant if and only if μ is G*-invariant*, i.e. $\mu(A) = \mu(g(A))$ for all Borel sets $A \subset X$, and all $g \in G$. Note that if G is an amenable group there exist G-invariant, and hence R_G-invariant, probability measures. We remark that if $(R_1, \mathcal{T}_1)$ is orbit equivalent to $(R_2, \mathcal{T}_2)$ via the orbit map $F : X_1 \to X_2$, then F maps the set of R_1-invariant probability measures bijectively onto the set of R_2-invariant probability measures.

Example 1.4. Let G be a countable discrete group acting freely (i.e. $gx = x$ for some $x \in X$, $g \in G$, implies $g = e$ (the identity of the group)) as homeomorphisms on X. Let

$$R_G = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X,$$

i.e. the R_G-equivalence classes are simply the G-orbits. We topologize R_G by transferring the product topology on $X \times G$ to R_G via the map $(x, g) \mapsto (x, gx)$, which is a bijection since G acts freely. With this topology $\mathcal{T}_G$ we get that $(R_G, \mathcal{T}_G)$ is an étale equivalence relation. Observe that if G is a finite group, then R_G is compact.

2 AF and AF-able (affable) equivalence relations

Let CEER be the acronym for *compact étale equivalence relation*. We have the following general result about CEERs, cf. [GPS2, Proposition 3.2].

Proposition 2.1. *Let $(R, \mathcal{T})$ be a CEER on X, where X is a compact, metrizable and zero-dimensional space. Let $X \times X$ be given the product topology.*

(i) *$\mathcal{T}$ is the relative topology from $X \times X$.*

(ii) *R is a closed subset of $X \times X$ and the quotient topology of the quotient space X/R is Hausdorff.*

(iii) *R is uniformly finite, that is, there is a natural number N such that the number $\#([x]_R)$ of elements in $[x]_R$ is less than or equal to N for all $x \in X$.*

In [GPS2] the structure of a CEER, $(R, \mathcal{T})$ on X, is described. Figure 1 illustrates how the structure looks like: X is decomposed into a finite number of m disjoint clopen towers $T_1, T_2, \ldots, T_m$, each of these consisting of finitely many disjoint clopen sets. The equivalence classes of R are represented in Figure 1 as the family of sets consisting of points lying on the same vertical line in each tower. (In the figure we have marked the equivalence class $[x_1]_R$ of a point $x_1 \in T_1$. We also show the graph picture associated to the tower T_m of height three.)

Figure 1 also illustrates a concept that will play an important role in the sequel, namely a very special clopen partition of R, which we will refer to as a *groupoid partition*. Let A and B be two (clopen) floors in the same tower, say the tower T_1. There is a homeomorphism $\gamma : A \to B$ such that $\text{graph}(\gamma) = \{(x, \gamma(x)) \mid x \in A\} \subset R$. Let $\mathcal{O}'$ be the (finite) clopen partition of R consisting of the set of these graphs γ, and let $\mathcal{O}$ be the associated (finite) clopen partition of X (which we identify with the diagonal $\Delta = \Delta_X$). So $\mathcal{O} = \mathcal{O}' \cap \Delta$, which means that $A \in \mathcal{O}$ if $A = B$ and $\gamma : A \to B$ is the identity map. The properties of $\mathcal{O}'$ are as follows (where we define $U \cdot V$ for subsets U, V of R to be $U \cdot V = \{(x, z) \mid (x, y) \in U, (y, z) \in V \text{ for some } y \in X\}$):

(i) $\mathcal{O}'$ is a finite clopen partition of R finer than $\{\Delta, R \setminus \Delta\}$.

(ii) For all $U \in \mathcal{O}'$, the maps $r, s : U \to X$ are homeomorphisms onto their respective images, and if $U \subset R \setminus \Delta$, then $r(U) \cap s(U) = \emptyset$.

(iii) For all $U, V \in \mathcal{O}'$, we have $U \cdot V = \emptyset$ or $U \cdot V \in \mathcal{O}'$. Also, $U^{-1}(= \{(y, x) \mid (x, y) \in U\})$ is in $\mathcal{O}'$ for every U in $\mathcal{O}'$.

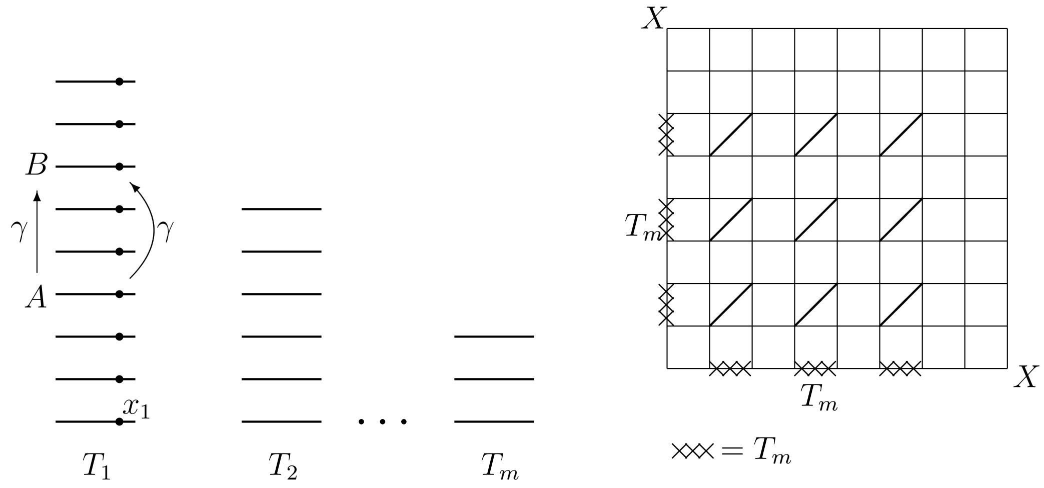

Figure 1: Illustration of the groupoid partition of a CEER; $R \subset X \times X$

(iv) With $\mathcal{O}'^{(2)} = \{(U,V) \mid U,V \in \mathcal{O}',\ U \cdot V \neq \emptyset\}$, define $(U,V) \in \mathcal{O}'^{(2)} \mapsto U \cdot V \in \mathcal{O}'$.

Then $\mathcal{O}'$ has a principal groupoid structure with unit space equal to $\{U \in \mathcal{O}' \mid U \subset \Delta\}$ (which clearly may be identified with $\mathcal{O}$). Hence the name *groupoid partition* for $\mathcal{O}'$. (Note that if we think of U and V as maps, then $U \cdot V$ means first applying the map U and then the map V.)

Note that if we define the equivalence relation $\sim_{\mathcal{O}'}$ on $\mathcal{O}$ by $A \sim_{\mathcal{O}'} B$ if there exists $U \in \mathcal{O}'$ such that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$, then the equivalence classes $[\cdot]_{\mathcal{O}'}$ are exactly the towers in Figure 1. The heights of the various towers $T_1, T_2, \ldots, T_m$ in Figure 1 are not necessarily distinct. All the groupoid partitions of R finer than the one shown in Figure 1 are obtained by vertically dividing the various towers $T_1, T_2, \ldots, T_m$ (by clopen sets) in an obvious way.

The proof of the following proposition can be found in [GPS2, Lemma 3.4, Corollary 3.5].

Proposition 2.2. *Let $(R, \mathcal{T})$ be a CEER on X, and let $\mathcal{V}'$ and $\mathcal{V}$ be (finite) clopen partitions of R and X, respectively. There exists a groupoid partition $\mathcal{O}'$ of R which is finer than $\mathcal{V}'$, and such that $\mathcal{O} = \mathcal{O}' \cap \Delta$ is a clopen partition of X that is finer than $\mathcal{V}$.*

Definition 2.3 (AF and AF-able (affable) equivalence relations)**.** Let $\{(R_n, \mathcal{T}_n)\}_{n=0}^{\infty}$ be an ascending sequence of CEERs on X (compact, metrizable, zero-dimensional), that is, $R_n \subset R_{n+1}$ and $R_n \in \mathcal{T}_{n+1}$ (i.e. R_n is open in R_{n+1}) for $n = 0, 1, 2, \ldots$, where we set $R_0 = \Delta_X (\cong X)$, $\mathcal{T}_0$ being the topology on X. Let $(R, \mathcal{T})$ be the *inductive limit* of $\{(R_n, \mathcal{T}_n)\}$ with the inductive limit topology $\mathcal{T}$, i.e. $R = \bigcup_{n=0}^{\infty} R_n$

and $U \in \mathcal{T}$ if $U \cap R_n \in \mathcal{T}_n$ for any n. In particular, R_n is an open subset of R for all n. We say that $(R, \mathcal{T})$ is an *AF-equivalence relation* on X, and we use the notation $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$. We say that an equivalence relation S on X is *AF-able (affable)* if it can be given a topology making it an AF-equivalence relation. (Note that this is the same as to say that S is orbit equivalent to an AF-equivalence relation, cf. Definition 1.3.)

Remark 2.4. One can prove that $(R, \mathcal{T})$ is an AF-equivalence relation if and only if $(R, \mathcal{T})$ is the inductive limit of an ascending sequence $\{(R_n, \mathcal{T}_n)\}_{n=0}^{\infty}$, where all the $(R_n, \mathcal{T}_n)$ are étale and *finite* (i.e. the R_n-equivalence classes are finite) equivalence relations, not necessarily CEERs, cf. [M]. This fact highlights the analogy between AF-equivalence relations in the topological setting with the so-called *hyperfinite* equivalence relations in the Borel and measure-theoretic setting.

It can be shown that the condition that R_n is open in R_{n+1} is superfluous when R_n and R_{n+1} are CEERs (see the comment right after Definition 3.7 of [GPS2]).

We will assume some familiarity with the notion of a *Bratteli diagram* (cf. [GPS2] for details). We remind the reader of the notation we will use. Let (V, E) be a Bratteli diagram, where V is the vertex set and E is the edge set, and where V, respectively E, can be written as a countable disjoint union of finite non-empty sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \dots \quad \text{and} \quad E = E_1 \cup E_2 \cup \dots$$

with the following property: an edge e in E_n connects a vertex v in V_{n-1} to a vertex w in V_n. We write $i(e) = v$ and $t(e) = w$, where we call i the source (or initial) map and t the range (or terminal) map. So a Bratteli diagram has a natural grading, and we will say that V_n is the vertex set at level n. We require that $i^{-1}(v) \neq \emptyset$ for all $v \in V$ and $t^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. We also want our Bratteli diagram to be *standard*, i.e. $V_0 = \{v_0\}$ is a one-point set. In the sequel all our Bratteli diagrams are assumed to be standard, so we drop the term 'standard'. Let

$$X_{(V,E)} = \{(e_1, e_2, \dots) \mid e_n \in E_n,\ t(e_n) = i(e_{n+1}) \text{ for all } n \in \mathbb{N}\}$$

be the path space associated to (V, E). Equipped with the relative topology from the product space $\prod_n E_n$, $X_{(V,E)}$ is compact, metrizable and zero-dimensional. We denote the cofinality relation on (V, E) by $AF(V, E)$, that is, two paths are equivalent if they agree from some level on. We now equip $AF(V, E)$ with an AF-structure. Let $n \in \{0, 1, 2, \dots\}$. Then $AF_n(V, E)$ will denote the compact étale subequivalence relation of $AF(V, E)$ defined by the property of cofinality from level n on. That is, if $x = (e_1, e_2, \dots, e_n, e_{n+1}, \dots)$, $y = (f_1, f_2, \dots, f_n, f_{n+1}, \dots)$ is in $X_{(V,E)}$, then $(x, y) \in AF_n(V, E)$ if $e_{n+1} = f_{n+1}, e_{n+2} = f_{n+2}, \dots$, and $AF_n(V, E)$ is given the relative topology from $X_{(V,E)} \times X_{(V,E)}$, thus getting a CEER structure. Obviously $AF_n(V, E) \subset AF_{n+1}(V, E)$, and we have $AF(V, E) = \bigcup_{n=0}^{\infty} AF_n(V, E)$. We give $AF(V, E)$ the inductive limit topology, i.e. $AF(V, E) = \varinjlim AF_n(V, E)$.

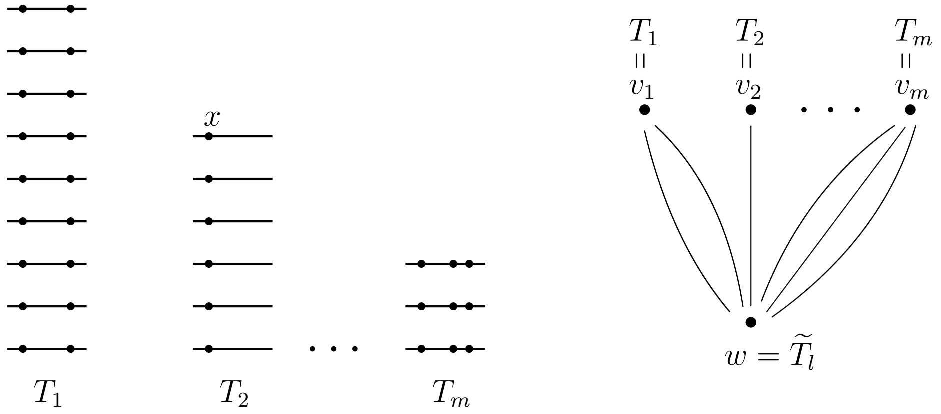

Figure 2: Illustration of the maps $i, t : E \to V$

Let $p = (e_1, e_2, \ldots, e_n)$ be a finite path from level 0 to some level n, and let $U(p)$ denote the *cylinder set* in $X_{(V,E)}$ defined by

$$U(p) = \{x = (f_1, f_2, \ldots) \in X_{(V,E)} \mid f_1 = e_1, f_2 = e_2, \ldots, f_n = e_n\}.$$

Then $U(p)$ is a clopen subset of $X_{(V,E)}$, and the collection of all cylinder sets is a clopen basis for $X_{(V,E)}$. Let $p = (e_1, e_2, \ldots, e_n)$ and $q = (e'_1, e'_2, \ldots, e'_n)$ be two finite paths from level 0 to the same level n, such that $t(e_n) = t(e'_n)$. Let $U(p, q)$ denote the intersection of $AF_n(V, E)$ with the Cartesian product $U(p) \times U(q)$. The collection of sets of the form $U(p, q)$ is a clopen basis for $AF(V, E)$. From this it follows immediately that a probability measure μ on $X_{(V,E)}$ is $AF(V, E)$-invariant (cf. Section 1) if and only if $\mu(U(p)) = \mu(U(q))$ for all such cylinder sets $U(p)$ and $U(q)$.

The following theorem is proved in [GPS2, Theorem 3.9]. However, we will explain the salient feature of the proof, as this will be important for arguments later in this paper.

Theorem 2.5. *Let $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$ be an AF-equivalence relation on X. There exists a Bratteli diagram (V, E) such that $(R, \mathcal{T})$ is isomorphic to the AF-equivalence relation $AF(V, E)$ associated to (V, E). Furthermore, (V, E) is simple if and only if $(R, \mathcal{T})$ is minimal.*

Proof sketch. For each n, choose a partition $\mathcal{P}'_n$ of R_n such that $\bigcup_{n=0}^{\infty} \mathcal{P}'_n$ generates the topology $\mathcal{T}$ of R. (We assume that R_0 equals $\Delta(= \Delta_X)$, the diagonal of X, and we will freely identify X with Δ whenever that is convenient.) Assume we have inductively obtained a groupoid partition $\mathcal{O}'_n$ of R_n which is finer than both $\mathcal{P}'_n$ and $\mathcal{O}'_{n-1} \cup \{R_n \setminus R_{n-1}\}$, cf. Proposition 2.2. Obviously $\mathcal{O}_{n+1} = \mathcal{O}'_{n+1} \cap \Delta$ is finer than

$\mathcal{O}_n = \mathcal{O}'_n \cap \Delta$, and $\bigcup_{i=0}^{\infty} \mathcal{O}_i$ generates the topology of X. Let $\mathcal{O}'_n$ be represented by m towers $T_1, T_2, \ldots, T_m$ of (not necessarily distinct) heights $h_1, h_2, \ldots, h_m$ (cf. Figure 1). We let the vertex set V_n at level n of the Bratteli diagram (V, E) be $V_n = \{v_1, v_2, \ldots, v_m\}$, where v_i corresponds to the tower T_i. So $[v_i] = [A]_{\mathcal{O}'_n}$ for some $A \in \mathcal{O}_n$, where $[\cdot]_{\mathcal{O}'_n}$ denotes the equivalence class of the relation $\sim_{\mathcal{O}'_n}$ on $\mathcal{O}'_n$, i.e. $B \sim_{\mathcal{O}'_n} C$ if there exists $V \in \mathcal{O}'_n$ such that $V^{-1} \cdot V = B$, $V \cdot V^{-1} = C$. Let $\mathcal{O}'_{n+1}$ be represented by k towers $\widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_k$ of heights $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_k$, and set $V_{n+1} = \{w_1, w_2, \ldots, w_k\}$, where w_j corresponds to the tower $\widetilde{T}_j$. Let $\mathcal{O}''_{n+1} = \{U \in \mathcal{O}'_{n+1} \mid U \subset R_n\}$. Clearly $\mathcal{O}''_{n+1} = \mathcal{O}'_{n+1} \cap R_n$. It is a simple observation that $\mathcal{O}''_{n+1}$ is a groupoid partition of R_n that is finer than $\mathcal{O}'_n$, and that $\mathcal{O}''_{n+1} \cap \Delta = \mathcal{O}_{n+1}$. The set of edges E_{n+1} between V_n and V_{n+1} is labelled by $\mathcal{O}_{n+1}$ modulo $\mathcal{O}''_{n+1}$, i.e. E_{n+1} consists of $\sim_{\mathcal{O}''_{n+1}}$ equivalence classes (denoted by $[\cdot]_{\mathcal{O}''_{n+1}}$) of $\mathcal{O}_{n+1}$. Specifically, if $A, B \in \mathcal{O}_{n+1}$, we have $A \sim_{\mathcal{O}''_{n+1}} B$ if there exists $U \in \mathcal{O}''_{n+1}$ such that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$. As we explained above, the vertex set V_n, resp. V_{n+1} (i.e. the towers $T_1, T_2, \ldots, T_m$, resp. $\widetilde{T}_1, \widetilde{T}_2, \ldots, \widetilde{T}_k$) may be identified with the $\sim_{\mathcal{O}'_n}$ (resp. $\sim_{\mathcal{O}'_{n+1}}$) equivalence classes $[\cdot]_{\mathcal{O}'_n}$ (resp. $[\cdot]_{\mathcal{O}'_{n+1}}$) of $\mathcal{O}_n$ (resp. $\mathcal{O}_{n+1}$). Let $[A]_{\mathcal{O}''_{n+1}} \in E_{n+1}$, where $A \in \mathcal{O}_{n+1}$. Then we define $t([A]_{\mathcal{O}''_{n+1}}) = [A]_{\mathcal{O}'_{n+1}}$ and $i([A]_{\mathcal{O}''_{n+1}}) = [B]_{\mathcal{O}'_n}$, where B is the unique element of $\mathcal{O}_n$ such that $A \subset B$. We give a more intuitive explanation of this by appealing to Figure 2. The R_{n+1}-equivalence class $[x]_{R_{n+1}}$ of a point x in X is marked in the tower presentation of the groupoid partition $\mathcal{O}'_n$ of R_n. Since $R_n \subset R_{n+1}$, the set $[x]_{R_{n+1}}$ breaks into a finite disjoint union of R_n-equivalence classes. Let A be the unique clopen set in $\mathcal{O}_{n+1}$ that contains x, and let $\widetilde{T}_l$ be the unique tower $w \in V_{n+1}$ corresponding to $[A]_{\mathcal{O}'_{n+1}}$. Then $t^{-1}(w)$ consists of the edges shown in Figure 2. In other words, if $[x]_{R_{n+1}}$ contains t_i distinct R_n-equivalence classes 'belonging' to the tower $v_i \in V_n$, we connect w to v_i by t_i edges. We have that $\tilde{h}_l = \sum_{i=1}^{m} t_i h_i$.

In this way, we construct the Bratteli diagram (V, E), and the map $F : X \to X_{(V,E)}$ is defined by $F(x) = (e_1, e_2, \ldots, e_{n+1}, \ldots)$, where e_{n+1} is $[A]_{\mathcal{O}''_{n+1}}$. It is now straightforward to show that F is a homeomorphism and that $F \times F : R \to AF(V, E)$ is an isomorphism. $\square$

Remark 2.6. If $n = 0$, then $\mathcal{O}'_0$ consists of one tower of height 1 ($\mathcal{O}'_0 = \{\Delta\}$, $\mathcal{O}_0 = \{X\}$). So V_0 is a one-point set, $V_0 = \{v_0\}$. We have $V_1 = \{w_1, w_2, \ldots, w_k\}$, and w_j is connected to v_0 by $\tilde{h}_j$ edges, where $\tilde{h}_j$ is the height of the tower corresponding to w_j.

3 Transverse equivalence relations

Let X be a compact, metrizable and zero-dimensional space, and let R and S be (countable) equivalence relations on X. We let $R \vee S$ denote the (countable) equivalence relation on X generated by R and S. If $S \subset R$, i.e. $R \vee S = R$, we say that S is a *subequivalence* relation of R. If S is a subequivalence relation of the étale equivalence relation $(R, \mathcal{T})$ and $S \in \mathcal{T}$ (i.e. S is open in R), then $(S, \mathcal{T}|_S)$ (i.e. S with the relative topology) is étale—a fact that is easily shown.

For equivalence relations R and S on X, we define the following subset, denoted $R \times_X S$, of the Cartesian product $R \times S$:

$$R \times_X S = \{((x,y),(y,z)) \mid (x,y) \in R, (y,z) \in S\}$$

and we define $r, s : R \times_X S \to X$ by

$$r((x,y),(y,z)) = x \quad \text{and} \quad s((x,y),(y,z)) = z.$$

Also, we define $r \times s : R \times_X S \to X \times X$ by $r \times s((x,y),(y,z)) = (x,z)$.

If R and S have topologies, we give $R \times_X S$ the relative topology from $R \times S$ (with the product topology).

Definition 3.1 (Transverse equivalence relation)**.** Let R and S be étale equivalence relations on X. We say that R and S are *transverse* to each other, denoted $R \perp S$, if the following hold:

(i) $R \cap S = \Delta_X (= \{(x,x) \mid x \in X\})$.

(ii) There is a homeomorphism $h : R \times_X S \to S \times_X R$ such that $r \circ h = r$ and $s \circ h = s$. (Note that $h^{-1} : S \times_X R \to R \times_X S$ satisfies $r \circ h^{-1} = r$ and $s \circ h^{-1} = s$.) So, for each (x,y) in R and (y,z) in S, there is a unique y' in X such that (x,y') is in S, (y',z) is in R and $h((x,y),(y,z)) = ((x,y'),(y',z))$.

An important class of examples is the following. Suppose that R is an étale equivalence relation on X, and $\alpha : G \to \mathrm{Homeo}(X)$ is an action of the countable (or finite) group G as homeomorphisms on X such that

(i) $\alpha_g \times \alpha_g(R) = R$ for all $g \in G$.

(ii) $\alpha_g \times \alpha_g : R \to R$ is a homeomorphism for all $g \in G$. ((i) and (ii) together says that α_g implements an automorphism of R for all $g \in G$.)

(iii) $(x, \alpha_g(x))$ is not in R for any x in X and $g \neq e$ in G (in particular, the action is free).

Then the equivalence relation $R_G = \{(x, \alpha_g(x)) \mid x \in X, g \in G\}$ is étale (cf. Example 1.4) and transverse to R, the map $h : R \times_X R_G \to R_G \times_X R$ being defined by

$$h((x,y),(y,\alpha_g(y))) = ((x,\alpha_g(x)),(\alpha_g(x),\alpha_g(y)))$$

for all (x,y) in R and g in G.

Lemma 3.2. *Let R and S be transverse étale equivalence relations on X. The equivalence relation on X generated by R and S, $R \vee S$, is equal to $r \times s(R \times_X S)$, respectively $r \times s(S \times_X R)$. Furthermore, the map $r \times s : R \times_X S \to R \vee S$ (respectively, $r \times s : S \times_X R \to R \vee S$) is a bijection. (We do not need the map h in Definition 3.1 (ii) to be a homeomorphism for this proof.)*

Proof. We consider the map $r \times s : R \times_X S \to R \vee S$ (it being obvious that the arguments we give apply similarly to the map $r \times s : S \times_X R \to R \vee S$, since by Definition 3.1 (ii), $r \times s(R \times_X S) = r \times s(S \times_X R)$). Clearly $r \times s(R \times_X S) \subset R \vee S$. If $(x, y) \in R$, then $((x, y), (y, y)) \in R \times_X S$, and so $(x, y) \in r \times s(R \times_X S)$. Hence $R \subset r \times s(R \times_X S)$. Likewise we show that $S \subset r \times s(R \times_X S)$. If (x, z) equals $r \times s((x, y), (y, z)) = r \times s((x, y'), (y', z))$, then $(x, y), (x, y')$ are in R, and so $(y, y') \in R$. Likewise, $(y, z), (y', z)$ are in S, and so $(y, y') \in S$. Hence $y = y'$, and so the map $r \times s$ is injective. The proof will be completed by showing that $K = r \times s(R \times_X S)$ is an equivalence relation on X.

Clearly, K is reflexive. To prove symmetry, assume $(x, z) \in K$. There exists $y \in X$ such that $(x, y) \in R$, $(y, z) \in S$, and so $((z, y), (y, x)) \in S \times_X R$. Since $r \times s(S \times_X R) = r \times s(R \times_X S)$, we get that $(z, x) \in K$. Hence K is symmetric.

To prove transitivity, assume $(x, z), (z, w) \in K$. We must show that $(x, w) \in K$. There exists $y, y' \in X$ such that $((x, y), (y, z)), ((z, y'), (y', w)) \in R \times_X S$. This implies that $((y, z), (z, y')) \in S \times_X R$. Since the map $h : R \times_X S \to S \times_X R$ is a bijection, there exists $y'' \in X$ such that $((y, y''), (y'', y')) \in R \times_X S$. We thus get that $(y', w), (y'', y') \in S$, which implies that $(y'', w) \in S$. Also, we have that $(x, y), (y, y'') \in R$, which implies that $(x, y'') \in R$. Hence $((x, y''), (y'', w)) \in R \times_X S$, which implies that $(x, w) \in r \times s(R \times_X S) = K$, which proves transitivity. $\square$

Proposition 3.3. *Let $(R, \mathcal{T})$ and $(S, \widetilde{\mathcal{T}})$ be two étale equivalence relations on X which are transverse to each other. By the bijective map (cf. Lemma 3.2) $r \times s : R \times_X S \to R \vee S$, which sends $((x, y), (y, z)) \in R \times_X S$ to (x, z), we transfer the topology on $R \times_X S (\subset R \times S)$ to $R \vee S$. With this topology, denoted $\mathcal{W}$, $R \vee S$ is an étale equivalence relation on X. In particular, if R and S are CEERs, then $(R \vee S, \mathcal{W})$ is a CEER. Furthermore, $\mathcal{W}$ is the unique étale topology on $R \vee S$, which extends $\mathcal{T}$ on R and $\widetilde{\mathcal{T}}$ on S, i.e. the relative topologies on R and S are $\mathcal{T}$ and $\widetilde{\mathcal{T}}$, respectively. Also R and S are both open subsets of $R \vee S$.*

Proof. It is easily seen that $R \times_X S$ is a closed subset of $R \times S$, and so the relative topology on $R \times_X S$, and consequently $\mathcal{W}$, is locally compact and metrizable. If R and S are CEERs, then clearly $R \times_X S$ is compact. Now R (resp. S) is the image under $r \times s$ of

$$
\begin{aligned}
&R \times_X \Delta = \{((x, y), (y, y)) \mid (x, y) \in R\} \\
&(\text{resp. } \Delta \times_X S = \{((x, x), (x, y)) \mid (x, y) \in S\}),
\end{aligned}
$$

where $\Delta = \Delta_X$ is the diagonal of $X \times X$. Now $R \times_X \Delta$ (resp. $\Delta \times_X S$) is clopen in $R \times_X S$, since Δ is clopen in R (resp. S), and so we get that R (resp. S) is clopen in $R \vee S$.

We now show the étale condition for $R \vee S$. Let $(x, z) \in R \vee S$, and let y be the unique point in X such that $((x, y), (y, z)) \in R \times_X S$. A local basis at (x, z) is the family of composition of graphs in $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ (cf. Section 1)

$$
(\widetilde{U}, \tilde{\gamma}, \widetilde{V}) \circ (U, \gamma, V) = (U \cap \gamma^{-1}(V \cap \widetilde{U}), \tilde{\gamma} \circ \gamma, \tilde{\gamma}(V \cap \widetilde{U})),
$$

where $(U,\gamma,V)\in\mathcal{T}$, $(\widetilde{U},\tilde{\gamma},\widetilde{V})\in\widetilde{\mathcal{T}}$, and $x\in U$, $y\in V\cap\widetilde{U}$, $z\in\widetilde{V}$, such that $y=\gamma(x)$, $z=\tilde{\gamma}(y)$. In fact, we may assume that $\widetilde{U}=V$, and so a local basis at (x,z) is the family of graphs

$$\left\{(U,\tilde{\gamma}\circ\gamma,V)\ \middle|\ \begin{array}{c} x\in U,\ z\in V,\ U \text{ and } V \text{ open in } X, \\ V=\tilde{\gamma}\circ\gamma(U),\ \mathrm{graph}(\gamma)\subset R,\ \mathrm{graph}(\tilde{\gamma})\subset S \end{array}\right\}.$$

Clearly each $(U,\tilde{\gamma}\circ\gamma,V)$ is an étale open neighbourhood of (x,z) in $\mathcal{W}$. We need to show that the product of composable pairs in $R\vee S$ is continuous, and also that the inverse map on $R\vee S$ is continuous. If we assume this has been established, it follows easily that any étale topology $\mathcal{E}$ on $R\vee S$ that extends $\mathcal{T}$ on R and $\widetilde{\mathcal{T}}$ on S has to be equal to $\mathcal{W}$. In fact, since $(x,y)\cdot(y,z)=(x,z)$, where $(x,y)\in R$, $(y,z)\in S$, it is easy to show that the graphs $(U,\tilde{\gamma}\circ\gamma,V)$ considered above is also a local basis for $\mathcal{E}$ at (x,z). Thus, $\mathcal{W}$ is contained in $\mathcal{E}$. The other inclusion follows easily from the étaleness of $\mathcal{E}$.

To prove that the map $(x,z)\mapsto(x,z)^{-1}=(z,x)$ is continuous (and hence a homeomorphism) on $R\vee S$, let $(x_n,z_n)\to(x,z)$. This means that there exist (unique) $y_n,y\in X$ such that $((x_n,y_n),(y_n,z_n))\to((x,y),(y,z))$ in $R\times_X S$. This implies that $((z_n,y_n),(y_n,x_n))\to((z,y),(y,x))$ in $S\times_X R$. Applying the map h of Definition 3.1 (ii), we conclude that there exist $y'_n,y'\in X$ such that $((z_n,y'_n),(y'_n,x_n))\to((z,y'),(y',x))$ in $R\times_X S$. This means that $(z_n,x_n)\to(z,x)$ in $R\vee S$, and we are done.

To prove that the product of composable pairs in $(R\vee S)\times(R\vee S)$ is continuous, let $((x_n,y_n),(y_n,z_n))\to((x,y),(y,z))$ in $(R\vee S)\times(R\vee S)$. We want to show that $(x_n,z_n)\to(x,z)$ in $R\vee S$. There exist (unique) $y'_n,y''_n,y',y''\in X$ such that $((x_n,y'_n),(y'_n,y_n))\to((x,y'),(y',y))$ and $((y_n,y''_n),(y''_n,z_n))\to((y,y''),(y'',z))$ in $R\times_X S$. This implies that $((y'_n,y_n),(y_n,y''_n))\to((y',y),(y,y''))$ in $S\times_X R$. Using the map h of Definition 3.1 (ii), there exist $y'''_n,y'''\in X$ such that $((y'_n,y'''_n),(y'''_n,y''_n))\to((y',y'''),(y''',y''))$ in $R\times_X S$. So we have altogether $(x_n,y'_n)\to(x,y')$, $(y'_n,y'''_n)\to(y',y''')$ in R, $(y'''_n,y''_n)\to(y''',y'')$, $(y''_n,z_n)\to(y'',z)$ in S. This implies that $(x_n,y'''_n)\to(x,y''')$ in R, and $(y'''_n,z_n)\to(y''',z)$ in S. Hence $((x_n,y'''_n),(y'''_n,z_n))\to((x,y'''),(y''',z))$ in $R\times_X S$, and so $(x_n,z_n)\to(x,z)$ in $R\vee S$. □

Henceforth, whenever $(R,\mathcal{T})$ and $(S,\widetilde{\mathcal{T}})$ are two transverse equivalence relations on X, we will give $R\vee S$ the étale topology $\mathcal{W}$ described in Proposition 3.3.

We want to prove that if R is an AF-equivalence relation on X, and S is a CEER on X such that $R\perp S$ (i.e. R and S are transverse), then $R\vee S$ is again AF. Furthermore, we will give an explicit description of the relation between the Bratteli diagram models for $R\vee S$ and R, respectively. This will be important for the proof of the absorption theorem in the next section. We shall need the following two lemmas.

Lemma 3.4. *Let $(R,\mathcal{T})$ and $(S,\widetilde{\mathcal{T}})$ be two transverse CEERs on X. Then the following hold:*

(i) $(x,y) \in R \Rightarrow \#([x]_S) = \#([y]_S)$. *(In fact, R does not have to be a CEER for (i) to hold.)*

(ii) *Let $\mathcal{O}'$ be a groupoid partition of $R \vee S$ that is finer than the (clopen) partition $\{\Delta, R \setminus \Delta, S \setminus \Delta, (R \vee S) \setminus (R \cup S)\}$. Then for any $U \in \mathcal{O}'$, there are unique elements U_R, U_S, V_R, V_S in $\mathcal{O}'$ such that $U_R, V_R \subset R$, $U_S, V_S \subset S$ and $U = U_R \cdot U_S = V_S \cdot V_R$. The partitions $\mathcal{O}'|_R (= \mathcal{O}' \cap R)$ and $\mathcal{O}'|_S (= \mathcal{O}' \cap S)$ are groupoid partitions for R and S, respectively.*

Proof. (i). Let $(x,y) \in R$ and let $[x]_S = \{x{=}x_1, x_2, \ldots, x_n\}$, $[y]_S = \{y{=}y_1, y_2, \ldots, y_m\}$, where $n, m \in \mathbb{N}$ (cf. Proposition 2.1 (iii)). Let $y_i \in [y]_S$. Then $((x,y),(y,y_i)) \in R \times_X S$. By Definition 3.1 (ii) there exists a unique $x_i \in [x]_S$ such that $h((x,y),(y,y_i)) = ((x,x_i),(x_i,y_i))$. We will prove that the map $y_i \in [y]_S \mapsto x_i \in [x]_S$ is one-to-one. In fact, assume $y_j \in [y]_S$ and that $h((x,y),(y,y_j)) = ((x,x_i),(x_i,y_j))$. Since $(x_i,y_i),(x_i,y_j) \in R$, we get that $(y_i,y_j) \in R$. Likewise $(y,y_i),(y,y_j) \in S$, and so $(y_i,y_j) \in S$. Since $R \cap S = \Delta$, we get that $y_i = y_j$, and so $j = i$. We conclude that $\#([x]_S) \geq \#([y]_S)$. Similarly, by considering the map $x_j \in [x]_S \mapsto y_j \in [y]_S$ defined by $h((y,x),(x,x_j)) = ((y,y_j),(y_j,x_j))$, we show that $\#([y]_S) \geq \#([x]_S)$. Hence we get that $\#([x]_S) = \#([y]_S)$.

(ii). Let (x,z) be any element of U. There exists a unique $y \in X$ such that $(x,y) \in R$ and $(y,z) \in S$. Let U_R, U_S be the unique elements of $\mathcal{O}'$ which contain (x,y) and (y,z), respectively. Since $(x,y) \in R$ and the partition $\mathcal{O}'$ is finer than the partition $\{R, (R \vee S) \setminus R\}$, we have $U_R \subset R$. Similarly, we have $(y,z) \in U_S \subset S$. Since $U_R \cdot U_S$ contains (x,z), it meets U and hence $U = U_R \cdot U_S$. The existence and uniqueness of V_R and V_S are shown in an analogous way.

That $\mathcal{O}'|_R$ and $\mathcal{O}'|_S$ are groupoid partition for R and S, respectively, is obvious. □

By Lemma 3.4 each tower associated to a groupoid partition $\mathcal{O}'$ of $R \vee S$ (cf. Figure 1) is decomposed into an "orthogonal" array of towers that are associated to the groupoid partitions $\mathcal{O}'|_R$ and $\mathcal{O}'|_S$ of R and S, respectively. In fact, let T be one of the "$R \vee S$-towers" associated to $\mathcal{O}'$ of height k. If $x \in T$, then $[x]_{R \vee S}$ (which has cardinality k) is a disjoint union of m R-equivalence classes, respectively n S-equivalence classes, where $k = mn$. So T is the disjoint union of m R-towers associated to the groupoid partition $\mathcal{O}'|_R$ of R, and also the disjoint union of n S-towers associated to the groupoid partition $\mathcal{O}'|_S$ of S. We have illustrated this in Figure 3, where the R-towers are drawn vertical and the S-towers are drawn horizontal. The $R \vee S$-equivalence class of x is marked, and we indicate how a given $U \in \mathcal{O}'$ associated to the tower T can be written as $U = U_R \cdot U_S = V_S \cdot V_R$, as explained in Lemma 3.4.

Lemma 3.5. *Let $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$ be an AF-equivalence relation on X, where $\Delta_X{=}\Delta{=}R_0 \subset R_1 \subset R_2 \subset \ldots$ is an ascending sequence of CEERs on X. Let $(S, \widetilde{\mathcal{T}})$ be a CEER which is transverse to R, i.e. $R \perp S$. There exists an ascending sequence*

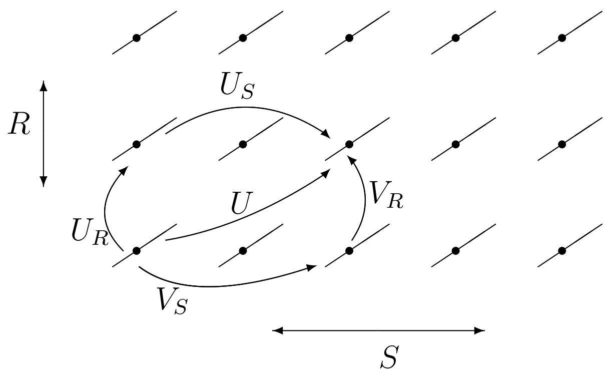

Figure 3: The decomposition of a $(R \vee S)$-tower into R-towers and S-towers

of CEERs $\{(R'_n, \mathcal{T}'_n)\}$, $\Delta{=}R'_0 \subset R'_1 \subset R'_2 \subset \dots$, *such that* $(R, \mathcal{T}) = \varinjlim(R'_n, \mathcal{T}'_n)$ *and* $R'_n \perp S$ *for all* n.

Proof. Define the following subset R'_n of R_n by

$$R'_n = \left\{ (x,y) \in R_n \;\middle|\; \begin{array}{l} \forall (y,z) \in S, \\ h((x,y),(y,z)) = ((x,y'),(y',z)) \text{ implies } (y',z) \in R_n \end{array} \right\},$$

where $h : R \times_X S \to S \times_X R$ is the map in Definition 3.1. By slight abuse of notation we may alternatively define $R'_n (\subset R_n)$ by

$$(x,y) \in R'_n \iff h(\{(x,y)\} \times_X S) \subset S \times_X R_n.$$

Clearly $R'_n \subset R'_{n+1}$ for all n. Also, we claim that $\bigcup_{n=0}^{\infty} R'_n = R$. In fact, let $(x,y) \in R$. By Proposition 2.1 (iii), the S-equivalence class of y is finite, say $[y]_S = \{y{=}y_1, y_2, y_3, \dots, y_L\}$. For each $1 \le l \le L$, there exists a unique $x_l \in X$ such that $h((x,y),(y,y_l)) = ((x,x_l),(x_l,y_l))$, where $(x_l,y_l) \in R$. We choose N sufficiently large so that $(x_l,y_l) \in R_N$ for all $1 \le l \le L$. Since $y_1 = y$, we must have $x_1 = x$ by the properties of the map h. Hence $(x,y) \in R'_N$.

We now prove that R'_n is an equivalence relation for all n. Reflexivity is obvious from the definition of R'_n, using the fact that $h((x,x),(x,z)) = ((x,z),(z,z))$. Now let $(x,y) \in R'_n (\subset R)$. By Lemma 3.4 (i), we have that $\#([x]_S) = \#([y]_S)$. By appropriate labelling, we have for any $1 \le l \le m$,

$$h((x,y),(y,y_l)) = ((x,x_l),(x_l,y_l)), \tag{$*$}$$

where $[x]_S = \{x{=}x_1, x_2, \dots, x_m\}$, $[y]_S = \{y{=}y_1, y_2, \dots, y_m\}$ and $(x_l,y_l) \in R_n$. Since $h((y,x),(x,x_l)) = ((y,y_l),(y_l,x_l))$, we conclude that $(y,x) \in R'_n$, and so R'_n is symmetric. To prove transitivity, assume $(x,y),(y,z) \in R'_n$, and let $[x]_S =$

$\{x{=}x_1, x_2, \dots, x_m\}$, $[y]_S = \{y{=}y_1, \dots, y_m\}$, $[z]_S = \{z{=}z_1, z_2, \dots, z_m\}$. The labelling is done as explained above (cf. $(*)$), so that $(x_l, y_l), (y_l, z_l) \in R_n$ for $1 \le l \le m$. Since $(x_l, z_l) \in R_n$, we get that $h((x,z),(z,z_l)) = ((x,x_l),(x_l,z_l))$ for all $1 \le l \le m$. This implies that $(x,z) \in R'_n$, thus finishing the proof that R'_n is an equivalence relation.

Note that we can use $(*)$ to deduce that the map h (or rather, its restriction), $h : R'_n \times_X S \to S \times_X R'_n$, is a bijection. In fact, let l be fixed, $1 \le l \le m$. We must show that (x_l, y_l) in $(*)$ lies in R'_n. We have that $h((y_l, x_l), (x_l, x_j)) = ((y_l, y_j), (y_j, x_j))$ for all $1 \le j \le m$. Since $(y_j, x_j) \in R_n$ for $1 \le j \le m$, we conclude that (y_l, x_l), and hence (x_l, y_l), is in R'_n.

Obviously we have $R'_n \cap S = \Delta$. So to finish the proof, it is sufficient to show that R'_n is a clopen subset of R_n. We claim that

$$R'_n \times_X S = (R_n \times_X S) \cap h^{-1}(S \times_X R_n). \qquad (**)$$

In fact, it is clear that the set on the left hand side of $(**)$ is contained in the set on the right hand side of $(**)$. Conversely, let $((x,y),(y,z)) \in (R_n \times_X S) \cap h^{-1}(S \times_X R_n)$. Then $(x,y) \in R_n$, and $h((x,y),(y,z)) = ((x,y'),(y',z))$ implies $(y',z) \in R_n$. So $(x,y) \in R'_n$, proving the other inclusion of $(**)$.

Now $R_n \times_X S$ and $S \times_X R_n$ are easily seen to be closed subsets of the Cartesian products $R_n \times S$ and $S \times R_n$, respectively (cf. Proposition 2.1 (i)). Hence they are compact and, a fortiori, closed subsets of $R \times_X S$ and $S \times_X R$, respectively. Since R_n is open in R, it follows easily that $R_n \times_X S$ and $S \times_X R_n$ are open subsets of $R \times_X S$ and $S \times_X R$, respectively. From $(**)$ we conclude that $R'_n \times_X S$ is a compact and open subset of $R \times_X S$. Now $R'_n = \pi_1(R'_n \times_X S)$, where $\pi_1 : R \times S \to R$ is the projection map. Since π_1 is a continuous and open map, we conclude that R'_n is compact and open in R, and hence in R_n. This completes the proof. $\square$

Proposition 3.6. *Let $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$ be an AF-equivalence relation on X, and let S be a CEER on X such that $R \perp S$, i.e. R and S are transverse to each other. Then $R \vee S$ is an AF-equivalence relation.*

Furthermore, there exist Bratteli diagrams (V, E) and (V', E') such that

$$R \cong AF(V, E), \quad R' = R \vee S \cong AF(V', E')$$

and so that $t : E_1 \to V_1$ is injective and $S \cong AF_1(V', E')$. Moreover, there are surjective maps (respecting gradings) $q_V : V \to V'$ and $q_E : E \to E'$ such that

(i) *$i(q_E(e)) = q_V(i(e))$, $t(q_E(e)) = q_V(t(e))$ for e in E.*

(ii) *For each v in V, $q_E : i^{-1}(\{v\}) \to i^{-1}(\{q_V(v)\})$ is a bijection.*

(iii) *For each $v \in V_n$ and $n \ge 2$, $q_E : t^{-1}(\{v\}) \to t^{-1}(\{q_V(v)\})$ is a bijection.*

The map $H : X_{(V,E)} \to X_{(V',E')}$ defined by

$$x = (e_1, e_2, e_3, \dots) \mapsto H(x) = (q_E(e_1), q_E(e_2), q_E(e_3), \dots)$$

is a homeomorphism, and H implements an embedding of $AF(V,E)$ into $AF(V',E')$ whose image is transverse to $AF_1(V',E') \cong S$. Moreover, we have

$$AF_1(V',E') \vee (H\times H)(AF(V,E)) \cong AF(V',E').$$

(Recall the notation and terminology we introduced in Section 2. In particular, $V = V_0 \cup V_1 \cup V_2 \cup \ldots$, $E = E_1 \cup E_2 \cup \ldots$, $V' = V_0' \cup V_1' \cup V_2' \cup \ldots$, $E' = E_1' \cup E_2' \cup \ldots$.)

Proof. By Lemma 3.5, we may assume that $R_n \perp S$ for all n. Now let $(\Delta_X =)\Delta = R_0 = R_1 \subset R_2 \subset R_3 \subset \ldots$ and

$$\Delta = R_0' \subset R_1' = R_1 \vee S \subset R_2' = R_2 \vee S \subset R_3' = R_3 \vee S \subset \cdots \subset R' = R \vee S = \bigcup_{n=0}^{\infty} R_n'.$$

Applying Proposition 3.3 we can conclude that $R \vee S$ is an AF-equivalence relation.

Now let $\mathcal{P}_n'$ be a groupoid partition of $R_n' = R_n \vee S$ which is finer than both $\mathcal{P}_{n-1}' \cup \{R_n' \setminus R_{n-1}'\}$ and $\{\Delta, R_n \setminus \Delta, S \setminus \Delta, R_n' \setminus (R_n \cup S)\}$, where we set $\mathcal{P}_0' = \{\Delta\}$. We require that $\bigcup_{n=0}^{\infty} \mathcal{P}_n'$ generates the topology of $R \vee S$. (This can be achieved by successive applications of Proposition 2.2.) By Lemma 3.4 we get that $\mathcal{O}_n' = \mathcal{P}_n'|_{R_n} (= \mathcal{P}_n' \cap R_n)$ is a groupoid partition of R_n for $n \geq 0$, such that $\mathcal{O}_n'$ is finer than both $\mathcal{O}_{n-1}' \cup \{R_n \setminus R_{n-1}\}$ and $\{\Delta, R_n \setminus \Delta\}$. Furthermore, $\bigcup_{n=0}^{\infty} \mathcal{O}_n'$ generates the topology of R. Similarly, $\mathcal{Q}_n' = \mathcal{P}_n'|_S (= \mathcal{P}_n' \cap S)$ is a groupoid partition of S for $n \geq 1$, such that $\mathcal{Q}_n'$ is finer than both $\mathcal{Q}_{n-1}'$ and $\{\Delta, S \setminus \Delta\}$, and $\bigcup_{n=0}^{\infty} \mathcal{Q}_n'$ generates the topology of S. (We set $\mathcal{Q}_0' = \{\Delta\}$.) By Lemma 3.4, every $U \in \mathcal{P}_n'$ can be uniquely written as $U = U_R \cdot U_S$, where $U_R \in \mathcal{O}_n'$ and $U_S \in \mathcal{Q}_n'$, and we may suggestively write $\mathcal{P}_n' = \mathcal{O}_n' \cdot \mathcal{Q}_n'$. For each $n \geq 0$ we have that

$$(\mathcal{P}_n' \cap \Delta =)\mathcal{P}_n'|_\Delta = \mathcal{O}_n'|_\Delta = \mathcal{Q}_n'|_\Delta = \mathcal{P}_n$$

is a clopen partition of X, with

$$\Delta = \mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \ldots$$

and $\bigcup_{n=0}^{\infty} \mathcal{P}_n$ being a basis for X.

Combining all this—following the description given in the proof sketch of Theorem 2.5— we construct the Bratteli diagrams (V',E') and (V,E), so that $R' = R \vee S \cong AF(V',E')$ and $R \cong AF(V,E)$, respectively, and such that the conditions stated in the proposition are satisfied. For brevity we will omit some of the details, which are routine verifications, and focus on the main ingredients of the proof. (We will use the same notation that we used in the proof sketch of Theorem 2.5.)

First we observe that $S \cong AF_1(V',E')$. In fact, since $\mathcal{P}_0' = \mathcal{Q}_0' = \{\Delta\}$, and $\mathcal{P}_1' = \mathcal{Q}_1'$ is a groupoid partition of $\mathcal{R}_1' = \Delta \vee S = S$, the edge set E_1' is related in an obvious way to the towers $\{T_1, T_2, \ldots, T_m\}$ associated to $\mathcal{Q}_1'$ (cf. Figures 1 & 2 and Remark 2.6). The towers associated to $\mathcal{Q}_n'$, $n > 1$, are obtained by subdividing (vertically) the towers $\{T_1, T_2, \ldots, T_m\}$, and from this it is easily seen that S is isomorphic to $AF_1(V', E')$. Since $R_0 = R_1 = \{\Delta\}$, and $\mathcal{O}'_0 = \{\Delta\}$, $\mathcal{O}'_1 = \mathcal{O}'_1|_\Delta = \mathcal{P}_1$, we deduce that $t : E_1 \to V_1$ is injective and that the obviously defined maps, $q_V : V_0 \to V'_0$, $q_V : V_1 \to V'_1$, $q_E : E_1 \to E'_1$, satisfy condition (i) for $e \in E_1$, and (ii) for $v \in V_0$. (For instance, if $e \in E_1$ corresponds to $A \in \mathcal{P}_1$, then it is mapped to $e' \in E'_1$, which corresponds to the "floor" A, that lies in one of the towers $\{T_1, T_2, \dots, T_m\}$.)

Let $n \geq 1$. Assume that we have defined $q_V : V_i \to V'_i$, $i = 0, 1, \dots, n$, and $q_E : E_j \to E'_j$, $j = 1, 2, \dots, n$, such that (i) is true for $e \in E_j$, $j = 1, 2, \dots, n$, and (ii) is true for $v \in V_i$, $i = 0, 1, \dots, n-1$. Assume also that $H((e_1, e_2, \dots, e_n)) = (q_E(e_1), q_E(e_2), \dots, q_E(e_n))$ is a bijection between finite paths of length n from the top vertices of (V, E) and (V', E'), respectively.

We now define $q_E : E_{n+1} \to E'_{n+1}$. Let $e = [A]_{\mathcal{O}''_{n+1}} \in E_{n+1}$ for some $A \in \mathcal{P}_{n+1} (= \mathcal{P}'_{n+1}|_\Delta = \mathcal{O}'_{n+1}|_\Delta)$, where $\mathcal{O}''_{n+1} = \{U \in \mathcal{O}'_{n+1} \mid U \subset R_n\}$. We define $q_E(e) = [A]_{\mathcal{P}''_{n+1}} \in E'_{n+1}$, where $\mathcal{P}''_{n+1} = \{W \in \mathcal{P}'_{n+1} \mid W \subset R'_n\}$. Since $\mathcal{O}''_{n+1} \subset \mathcal{P}''_{n+1}$, we get that $q_E : E_{n+1} \to E'_{n+1}$ is well-defined and surjective. We define the map $q_V : V_{n+1} \to V'_{n+1}$ by $q_V(v) = [B]_{\mathcal{P}'_{n+1}} \in V'_{n+1}$, where $v = [B]_{\mathcal{O}'_{n+1}} \in V_{n+1}$ for some $B \in \mathcal{P}_{n+1}$. Since $\mathcal{O}'_{n+1} \subset \mathcal{P}'_{n+1}$, we get that the map q_V is well-defined and surjective. It is easy to see that (i) is satisfied for all $e \in E_{n+1}$.

We show that the maps defined in (ii) and (iii) are surjective. Let $v = [B]_{\mathcal{O}'_n} \in V_n$, where $B \in \mathcal{P}_n$ (resp. $w = [\widetilde{B}]_{\mathcal{O}'_{n+1}} \in V_{n+1}$, where $\widetilde{B} \in \mathcal{P}_{n+1}$). So $q_V(v) = [B]_{\mathcal{P}'_n} \in V'_n$ (resp. $q_V(w) = [\widetilde{B}]_{\mathcal{P}'_{n+1}} \in V'_{n+1}$). Let $e' = [A]_{\mathcal{P}''_{n+1}} \in i^{-1}(\{q_V(v)\}) \subset E'_{n+1}$, where $A \in \mathcal{P}_{n+1}$ (resp. $f' = [\widetilde{A}]_{\mathcal{P}''_{n+1}} \in t^{-1}(\{q_V(w)\}) \subset E'_{n+1}$, where $\widetilde{A} \in \mathcal{P}_{n+1}$). This means that $A \sim_{\mathcal{P}''_n} A' \subset B$, for some $A' \in \mathcal{P}_{n+1}$ (resp. $\widetilde{A} \sim_{\mathcal{P}'_{n+1}} \widetilde{B}$, or, equivalently, $[\widetilde{A}]_{\mathcal{P}'_{n+1}} = [\widetilde{B}]_{\mathcal{P}'_{n+1}}$). Let $e = [A']_{\mathcal{O}''_{n+1}} \in E_{n+1}$ (resp. $f = [\widetilde{A}]_{\mathcal{O}''_{n+1}} \in E_{n+1}$). Then clearly $i(e) = v$ and $q_E(e) = e'$, proving that $q_E : i^{-1}(\{v\}) \to i^{-1}(\{q_V(v)\})$ is surjective. Clearly $q_E(f) = f'$. Also, since $n + 1 \geq 2$, we have that $\mathcal{P}''_{n+1}|_{R'_1} \succ \mathcal{Q}'_1 (= \mathcal{P}'_1 = \mathcal{P}''_1)$, and so we may choose $\widetilde{A}$ such that $\widetilde{A}$ and $\widetilde{B}$ are contained in the same set $C \in \mathcal{P}_1$. This fact, together with $\widetilde{A} \sim_{\mathcal{P}'_{n+1}} \widetilde{B}$, implies that $\widetilde{A} \sim_{\mathcal{O}'_{n+1}} \widetilde{B}$. In fact, if $U = U_R \cdot U_S \in \mathcal{P}'_{n+1}$ such that $U^{-1} \cdot U = \widetilde{A}$, $U \cdot U^{-1} = \widetilde{B}$, with $U_R \in \mathcal{O}'_{n+1}$, $U_S \in \mathcal{Q}'_{n+1}$, then $U = U_R$, since U_S must be the identity map on $\widetilde{B}$. Hence we get that $[\widetilde{A}]_{\mathcal{O}'_{n+1}} = [\widetilde{B}]_{\mathcal{O}'_{n+1}} = w$, and so we have proved that the map $q_E : t^{-1}(\{w\}) \to t^{-1}(\{q_V(w)\})$ is surjective.

We prove that (ii) holds for $v \in V_n$. Since $q_E : i^{-1}(\{v\}) \to i^{-1}(\{q_V(v)\})$ is surjective, we need to prove injectivity. So let $e_1, e_2 \in i^{-1}(\{v\})$, and assume $q_E(e_1) = q_E(e_2)$. We must show that $e_1 = e_2$. Now $e_1 = [A]_{\mathcal{O}''_{n+1}}$, $e_2 = [B]_{\mathcal{O}''_{n+1}}$ for some $A, B \in \mathcal{P}_{n+1}$. Since $q_E(e_1) = q_E(e_2)$, we have that $[A]_{\mathcal{P}''_{n+1}} = [B]_{\mathcal{P}''_{n+1}}$. Hence there exists $U \in \mathcal{P}'_{n+1}$ such that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$, and $U \subset R_n \vee S = R'_n$. Since $e_1, e_2 \in i^{-1}(\{v\})$, we have that $A \subset A_1 \in \mathcal{P}_n$, $B \subset B_1 \in \mathcal{P}_n$, such that $[A_1]_{\mathcal{O}'_n} = [B_1]_{\mathcal{O}'_n}$; that is, there exists $U_1 \in \mathcal{O}'_n \subset \mathcal{P}'_n$, such that $U_1^{-1} \cdot U_1 = A_1$, $U_1 \cdot U_1^{-1} = B_1$. Since $\mathcal{P}'_{n+1}|_{R'_n}$ is finer than $\mathcal{P}'_n$, we must have $U \subset U_1 (\subset R_n)$. This means that $U \in \mathcal{O}''_{n+1}$, and so $e_1 = [A]_{\mathcal{O}''_{n+1}} = [B]_{\mathcal{O}''_{n+1}} = e_2$.

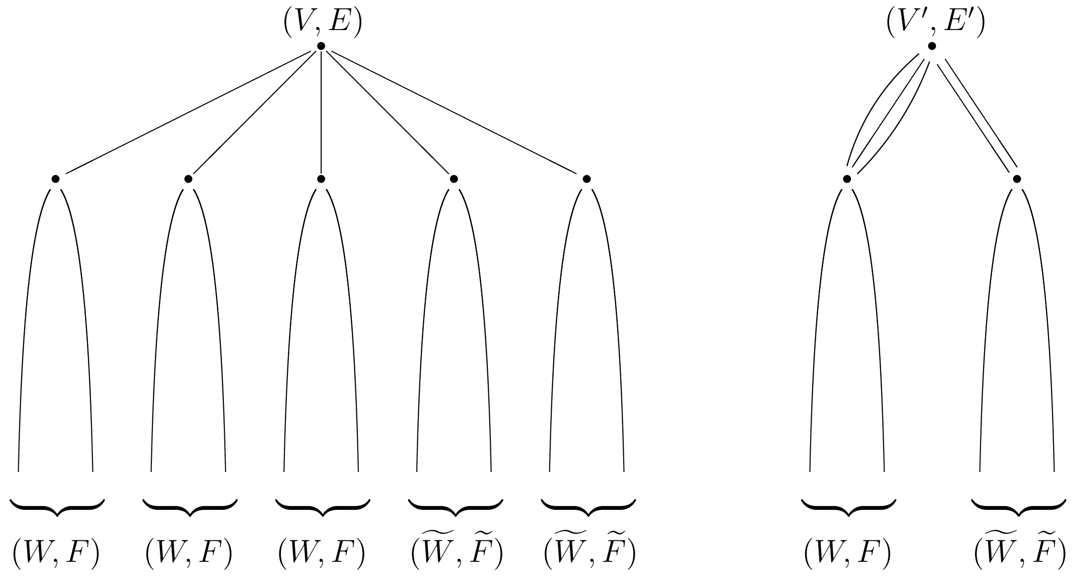

Figure 4: Illustrating the content of Proposition 3.6

In a similar way we prove that (iii) holds. In fact, let $e_1, e_2 \in t^{-1}(\{v\})$, $v \in V_{n+1}$, such that $q_E(e_1) = q_E(e_2)$. We must show that $e_1 = e_2$. Again we write $e_1 = [A]_{\mathcal{O}''_{n+1}}$, $e_2 = [B]_{\mathcal{O}''_{n+1}}$ for some $A, B \in \mathcal{P}_{n+1}$. Since $e_1, e_2 \in t^{-1}(\{v\})$, we have $[A]_{\mathcal{O}'_{n+1}} = [B]_{\mathcal{O}'_{n+1}}$, that is, there exists $U_1 \in \mathcal{O}'_{n+1} \subset \mathcal{P}'_{n+1}$ such that $U_1^{-1} \cdot U_1 = A$, $U_1 \cdot U_1^{-1} = B$. Since $q_E(e_1) = q_E(e_2)$, we have $[A]_{\mathcal{P}''_{n+1}} = [B]_{\mathcal{P}''_{n+1}}$, that is, there exists $U \in \mathcal{P}'_{n+1}$ such that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$, and $U \subset R_n \vee S = R'_n$. This implies that $U = U_1$. Since $U_1 \subset R_{n+1}$, $U \subset R_n \vee S$ and $S \cap R_{n+1} = \Delta$, we must have $U_1 \subset R_n$. Hence $U_1 \in \mathcal{O}''_{n+1}$, and so $e_1 = e_2$.

It is now straightforward to verify that $H : X_{(V,E)} \to X_{(V',E')}$ is a homeomorphism that implements an isomorphism between $AF(V,E)$ and its image $H \times H(AF(V,E))$ in $AF(V',E')$. The last assertion of the proposition is now routinely verified. $\square$

Remark 3.7. It is helpful to illustrate by a figure what Proposition 3.6 says, and which at the same time gives the heuristics of the proof. Furthermore, the illustration will be useful for easier comprehending the proof of the main theorem in the next section. In Figure 4 we have drawn the diagrams of (V,E) and (V',E'). The replicate diagrams (W,F) (resp. $(\widetilde{W},\widetilde{F})$) are also drawn, and S, the CEER transverse to $R = AF(V,E)$, as well as the maps q_V and q_E, should be obvious from the figure. One sees that what is going on is a "glueing" process in the sense that distinct R-equivalence classes are "glued" together by S to form R'-equivalence classes.

4 The absorption theorem

Definition 4.1 (R-étale and R-thin sets)**.** Let $(R, \mathcal{T})$ be an étale equivalence relation on the compact, metrizable and zero-dimensional space X. Let Y be a closed subset of X. We say that Y is *R-étale* if the *restriction* $R \cap (Y \times Y)$ of R to Y, denoted by $R|_Y$, is an étale equivalence relation in the relative topology.

We say that Y is R-thin if $\mu(Y) = 0$ for all R-invariant probability measures μ (cf. Section 1).

Remark 4.2. It can be proved (cf. Theorem 3.11 of [GPS2]) that if $(R, \mathcal{T})$ is an AF-equivalence relation and Y is R-étale, then $R|_Y$ is an AF-equivalence relation on Y. Furthermore, there exist a Bratteli diagram (V, E) and a Bratteli subdiagram (W, F) such that $R \cong AF(V, E)$, $R|_Y \cong AF(W, F)$. (By a Bratteli subdiagram (W, F) of (V, E) we mean that F is a subset of E such that $i(F) = \{v_0\} \cup t(F)$, that is, (W, F) is a Bratteli diagram, where $W = i(F)$ and $W_0 = V_0$. We say that F induces an (edge) subdiagram of (V, E). Observe that, in general, if (W, F) is a subdiagram of (V, E), then $R|_Y \cong AF(W, F)$, where $R = AF(V, E)$ and $Y = X_{(W,F)}$.)

We state a result from [GPS2] that will be crucial in proving the absorption theorem.

Theorem 4.3 (Lemma 4.15 of [GPS2])**.** *Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two minimal AF-equivalence relations on the Cantor sets X_1 and X_2, respectively. Let Y_i be a closed R_i-étale and R_i-thin subset of X_i, $i = 1, 2$. Assume*

(i) *$R_1 \cong R_2$.*

(ii) *There exists a homeomorphism $\alpha : Y_1 \to Y_2$ which implements an isomorphism between $R_1|_{Y_1}$ and $R_2|_{Y_2}$.*

Then there exists a homeomorphism $\tilde{\alpha} : X_1 \to X_2$ which implements an isomorphism between R_1 and R_2, such that $\tilde{\alpha}|_{Y_1} = \alpha$, i.e. $\tilde{\alpha}$ is an extension of α.

The following lemma is a technical result—easily proved—that we shall need for the proof of the absorption theorem. In the sequel we will use the term "microscoping" (of a Bratteli diagram) in the restricted sense called "symbol splitting" in [GPS1, Section 3]. (Microscoping is a converse operation to that of "telescoping".)

Lemma 4.4. *Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set X. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of natural numbers. Then there exists a (simple) Bratteli diagram (V, E) such that $R \cong AF(V, E)$, and for each $n \geq 1$*

(i) *$\#(V_n) \geq a_n$.*

(ii) *For all $v \in V_{n-1}$ and all $w \in V_n$, $\#(\{e \in E_n | i(e){=}v, t(e){=}w\}) \geq b_n$.*

Proof. Let $R \cong AF(W,F)$ for some (simple) Bratteli diagram (W,F). By a finite number of telescopings and microscopings of the diagram (W,F), we get (V,E) with the desired properties (cf. [GPS1, Section 3]). □

Remark 4.5. We make the general remark that telescoping or microscoping a Bratteli diagram do not alter any of the essential properties attached to the diagram, like the associated path space and the AF-equivalence relation. In fact, there is a natural map between the path spaces of the original and the new Bratteli diagrams which implements an isomorphism between the AF-equivalence relations associated to the two diagrams. However, the versatility that these operations (i.e. telescopings and microscopings) give us in changing a given Bratteli diagram into one which is more suitable for our purpose—like having enough "room" to admit appropriate subdiagrams—is very helpful. This will be utilized extensively in the proof of the absorption theorem. Note that a subdiagram (W,F) of a Bratteli diagram (V,E) is being telescoped or microscoped (in an obvious way) simultaneously as these operations are applied to (V,E). For this reason we will sometimes, when it is convenient, retain the old notation for the new diagrams, and this should not cause any confusion.

We can now state and prove the main result of this paper.

Theorem 4.6 (The absorption theorem)**.** *Let $R = (R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set X, and let Y be a closed R-étale and R-thin subset of X. Let $K = (K, \mathcal{S})$ be a compact étale equivalence relation on Y. Assume $K \perp R|_Y$, i.e. K is transverse to $R|_Y$.*

Then there is a homeomorphism $h : X \to X$ such that

(i) *$h \times h(R \vee K) = R$, where $R \vee K$ is the equivalence relation on X generated by R and K. In other words, $R \vee K$ is orbit equivalent to R, and, in particular, $R \vee K$ is affable.*

(ii) *$h(Y)$ is R-étale and R-thin.*

(iii) *$h|_Y \times h|_Y : (R|_Y) \vee K \to R|_{h(Y)}$ is a homeomorphism.*

Proof. Roughly speaking, the idea of the proof is to define an (open) AF-subequivalence relation $\overline{R}$ of R, thereby setting the stage for applying Theorem 4.3 (with $R_1 = R_2 = \overline{R}$), and in the process "absorbing" Y (and thereby K) so that $R \vee K$ becomes R. To define $\overline{R}$ we will manipulate Bratteli diagrams, applying Lemma 4.4 together with Proposition 3.6. The proof is rather technical, and to facilitate the understanding and get the main idea of the proof it will be helpful to have a very special, but telling, example in mind. We refer to Remark 4.7 for details on this.

Let (W,F) and (W',F') be two (fixed) Bratteli diagrams such that

$$R|_Y \cong AF(W,F), \quad R|_Y \vee K \cong AF(W',F')$$

and let

$$q_W : W \to W', \quad q_F : F \to F', \quad H : X_{(W,F)} \to X_{(W',F')}$$

be maps satisfying the conditions of Proposition 3.6. By Theorem 3.11 of [GPS2] there exist a (simple) Bratteli diagram (V, E) and a subdiagram $(\widetilde{W}, \widetilde{F})$ such that we may assume from the start that $X = X_{(V,E)}$, $Y = X_{(\widetilde{W},\widetilde{F})}$, $R = AF(V, E)$, $R|_Y = AF(\widetilde{W}, \widetilde{F})$. By a finite number of telescopings and microscopings applied to (V, E), and hence to $(\widetilde{W}, \widetilde{F})$, we may assume that for all $n \geq 1$, $\#(\widetilde{W}_n) \leq \frac{1}{2}\#(V_n)$, and for $v_0 \in V_0 = \widetilde{W}_0$, $w \in \widetilde{W}_n$,

$$\#(\{\text{paths in } X_{(\widetilde{W},\widetilde{F})} \text{ from } v_0 \text{ to } w\}) \leq \frac{1}{2}\#(\{\text{paths in } X_{(V,E)} \text{ from } v_0 \text{ to } w\}),$$

(cf. Lemma 4.12 of [GPS2]). Furthermore, by applying Lemma 4.4 we may assume the following holds for all $n \geq 1$:

(1) $\#(V_n) \geq \#(\widetilde{W}_n) + 1 + \sum_{k=1}^{n-1} \#(W_k)$

(2) For $v \in V_{n-1}$, $w \in V_n$, we have

$$\#(\{e \in E_n \mid i(e) = v,\ t(e) = w\}) \geq 2\sum_{k=1}^{n-1} \#(F_k) \ \left(\geq 2\sum_{k=1}^{n-1} \#(W_k)\right).$$

(Note that the inequalities in (1) and (2) hold, if we substitute W'_k for W_k and F'_k for F_k, cf. Proposition 3.6.) Using this we are going to construct a subdiagram of (V, E) which will consist of countable replicas of (W', F'). This subdiagram will be instrumental in defining the AF-subequivalence relation $\overline{R}$ of R that we alluded to above. We first choose $x_\infty = (e_1, e_2, \dots) \in X_{(V,E)}$ such that $t(e_n) \notin \widetilde{W}_n$ for all n. At level n we choose a replica of (W', F') "emanating" from the vertex $t(e_n)$. More precisely, we let $(W', F')_n$ denote the subdiagram of (V, E) consisting of the edges $e_1, e_2, \dots, e_n$ and then (a replica of) (W', F') starting at the vertex $t(e_n)$. This can be done by (1) and (2). Also, by (1) we have enough "room" so that we may choose the various $(W', F')_n$'s such that at each level $n \geq 1$, the vertex sets belonging to $(\widetilde{W}, \widetilde{F}), (W', F')_1, (W', F')_2, \dots, (W', F')_n$ are pairwise disjoint. In Figure 5, we have illustrated this.

By (2) it is easily seen that the subdiagram (L, G) of (V, E) whose edge set consists of the edge set $\widetilde{F}$ and the union of the edge sets belonging to $(W', F')_n$, $n \geq 1$, is a *thin subdiagram* of (V, E), i.e. $\mu(X_{(L,G)}) = 0$ for all R-invariant probability measures μ (cf. Remark 4.5). Likewise, the subdiagram (L', G') whose edge set consists of the union of the edge sets belonging to $(W', F')_n$, $n \geq 1$, is a thin subdiagram.

We now construct a new Bratteli diagram $(\overline{V}, \overline{E})$ from (V, E) such that there exists a homeomorphism $\overline{H} : X_{(\overline{V},\overline{E})} \to X_{(V,E)}$ implementing an isomorphism between

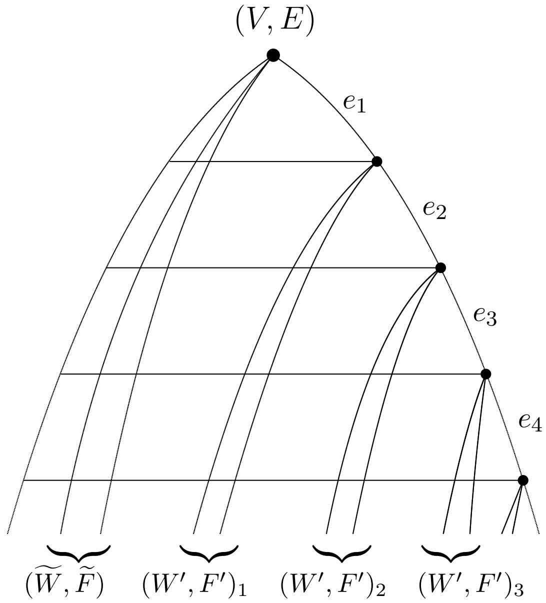

Figure 5: Constructing subdiagrams of (V, E)

$AF(\overline{V}, \overline{E})$ and the AF-subequivalence relation $\overline{R}$ of R that we want. The Bratteli diagram $(\overline{V}, \overline{E})$ will lend itself to apply Theorem 4.3 by, loosely speaking, transforming the subdiagrams $(W', F')_n$ in (V, E) to replicas of (W, F), thus making it possible to "absorb" the compact étale equivalence relation K. To construct $(\overline{V}, \overline{E})$ from (V, E) we first replace each vertex v in V_n (for $n \geq 2$) which belong to the union of the vertex sets of $(W', F')_1, (W', F')_2, \ldots, (W', F')_{n-1}$, by the vertices in $q_W^{-1}(\{v\})$. We retain the other vertices in V_n, and thus we get $\overline{V}_n$. We set $\overline{V}_0 = V_0$ and $\overline{V}_1 = V_1$. There is an obvious map $q_{\overline{V}} : \overline{V} \to V$, respecting gradings, which is surjective, and which can be considered to be an extension of the map $q_W : W \to W'$. Now we transfer in an obvious sense the subdiagram $(\widetilde{W}, \widetilde{F})$ of (V, E), again denoting it by $(\widetilde{W}, \widetilde{F})$, and so $\overline{E}$ will contain the edge set $\widetilde{F}$. Similarly, in an obvious sense, we replace $(W', F')_n$ by $(W, F)_n$, $n \geq 1$, where $(W, F)_n$ is defined in a similar way as we defined $(W', F')_n$. The new edge set $\overline{E}$ contains the collection of edges in $(W, F)_n$, $n \geq 1$, and so, in particular, the edges $e_1, e_2, \ldots$ lie in $\overline{E}$. (We will denote the path $(e_1, e_2, \ldots)$ again by x_∞.) Furthermore, if $e \in E$ is such that the vertices $i(e)$ and $t(e)$ do not lie in L', then we retain e, and so $e \in \overline{E}$. Let $e \in E \setminus G'$, and let $v = i(e)$ and $w = t(e)$. If $v \in L'$, we replace e by $\#(q_{\overline{V}}^{-1}(\{v\}))$ edges sourcing at each of the vertices $q_{\overline{V}}^{-1}(\{v\})$ and ranging at the same vertex, where this vertex can be chosen to be any vertex in $q_{\overline{V}}^{-1}(\{w\})$. If $v \notin L'$ and $w \in L'$, then we replace e by an edge sourcing at v and ranging at an arbitrary vertex in $q_{\overline{V}}^{-1}(\{w\})$. However, we require that the collection of these new edges will range at every vertex in $q_{\overline{V}}^{-1}(\{w\})$, as we consider all edges $e \in E \setminus G'$ such that $i(e) = v$ and $t(e) = w$, $v \in V_{n-1}$ and

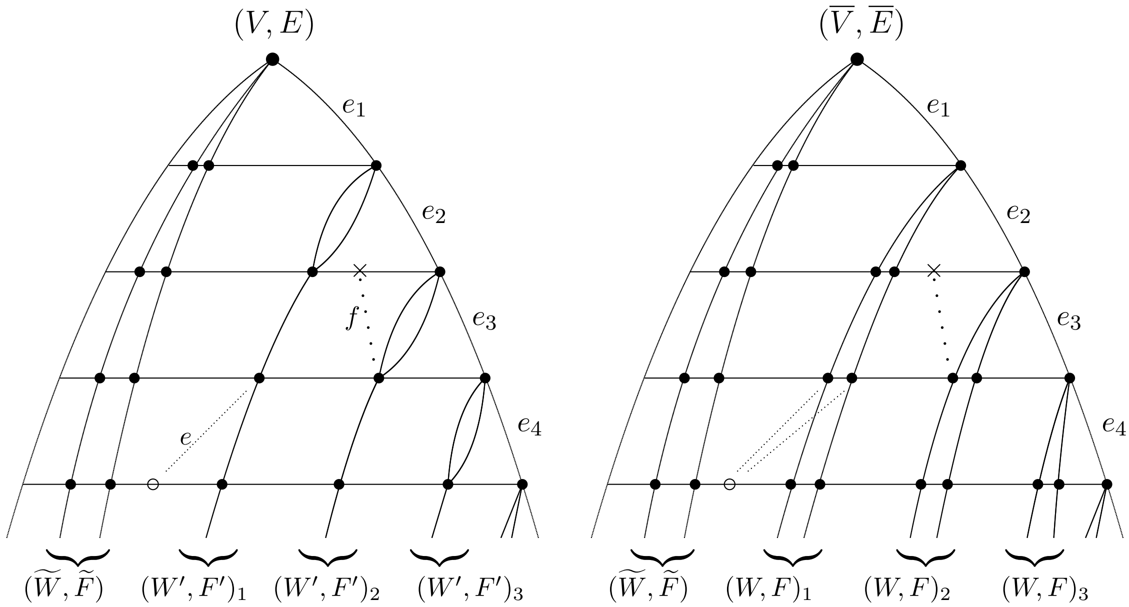

Figure 6: Construction of the AF-subequivalence relation $\overline{R} \cong AF(\overline{V},\overline{E})$ of $R = AF(V,E)$, cf. Remark 4.7

$w \in V_n$ being fixed ($n \geq 1$). By (2) there are sufficiently many edges e so that this can be achieved. So we have defined the edge set $\overline{E}$, and we have done this in such a way that $(\overline{V},\overline{E})$ is a simple Bratteli diagram by ensuring that for all $v \in \overline{V}_{n-1}$, $w \in \overline{V}_n$, there exists $e \in \overline{E}$ such that $i(e) = v$, $t(e) = w$. There is an obvious map $q_{\overline{E}} : \overline{E} \to E$, respecting gradings, which is surjective, and which can be considered to be an extension of the map $q_F : F \to F'$. By the definition of $q_{\overline{E}}$ and $q_{\overline{V}}$ we have that $i(q_{\overline{E}}(e)) = q_{\overline{V}}(i(e))$, $t(q_{\overline{E}}(e)) = q_{\overline{V}}(t(e))$ for $e \in \overline{E}$. Also, using Proposition 3.6 (ii), it is easy to show that $q_{\overline{E}} : i^{-1}(\{v\}) \to i^{-1}(\{q_{\overline{V}}(v)\})$ is a bijection for each $v \in \overline{V}$. We deduce that the map $(f_1, f_2, \ldots, f_n) \mapsto (q_{\overline{E}}(f_1), q_{\overline{E}}(f_2), \ldots, q_{\overline{E}}(f_n))$ establishes a bijection between the finite paths of length n from the top vertices of $(\overline{V},\overline{E})$ and (V,E), respectively. We define the map $\overline{H} : X_{(\overline{V},\overline{E})} \to X_{(V,E)}$ by

$$x = (f_1, f_2, f_3, \ldots) \mapsto \overline{H}(x) = (q_{\overline{E}}(f_1), q_{\overline{E}}(f_2), q_{\overline{E}}(f_3), \ldots).$$

It is now routinely checked that $\overline{H}$ is a homeomorphism, and that $\overline{H} \times \overline{H}$ maps $AF(\overline{V},\overline{E})$ isomorphically onto an AF-subequivalence relation $\overline{R}$ on $X_{(V,E)}$. Observe that $\overline{H}$ maps $X_{(W,F)_n}$ onto $X_{(W',F')_n}$ for all $n \geq 1$. (In fact, by obvious identifications, this map is the same as $H : X_{(W,F)} \to X_{(W',F')}$ introduced earlier.) Also, $\overline{H}$ maps $X_{(\widetilde{W},\widetilde{F})} (\subset X_{(\overline{V},\overline{E})})$ onto $X_{(\widetilde{W},\widetilde{F})} (\subset X_{(V,E)})$, and $\overline{H}(x_\infty) = x_\infty$.

We claim that

$$R = \overline{R} \vee K_1 \vee K_2 \vee \ldots, \tag{$*$}$$

where K_n, $n \geq 1$, is the CEER on $X_{(W',F')_n}$ that corresponds to K via the obvious map between $X_{(W',F')}$ and $X_{(W',F')_n}$. (We note for later use that, similarly, we have an obvious map between $X_{(W,F)}$ and $X_{(W,F)_n}$.) Clearly the right hand side of $(*)$ is contained in the left hand side. Now $\overline{H}|_{(W,F)_n}$ implements an embedding of $AF((W,F)_n)(\cong AF(W,F) \cong R|_Y)$ into $AF((W',F')_n)(\cong AF(W',F') \cong (R|_Y) \vee K)$ which is transverse to K_n, and such that $AF((W',F')_n) \cong K_n \vee (\overline{H} \times \overline{H})(AF((W,F)_n))$ (cf. Proposition 3.6). To prove that $(*)$ holds, let $(x,y) \in R$, and let $\overline{x}, \overline{y} \in X_{(\overline{V},\overline{E})}$ such that $\overline{H}(\overline{x}) = x$, $\overline{H}(\overline{y}) = y$. If the paths x and y agree from level n on in $X_{(V,E)}$, then by the definition of $\overline{H}$ we must have that $q_{\overline{E}}(f_m) = q_{\overline{E}}(f'_m)$ and $q_{\overline{V}}(i(f_m)) = q_{\overline{V}}(i(f'_m))$ for all $m > n$, where $\overline{x} = (f_1, f_2, \dots)$, $\overline{y} = (f'_1, f'_2, \dots)$. We first observe that $\overline{x}$ and $\overline{y}$ are cofinal paths in $X_{(\overline{V},\overline{E})}$, that is, $(\overline{x}, \overline{y}) \in AF(\overline{V}, \overline{E})$, if and only if $i(f_m) = i(f'_m)$ for some $m > n$. Now it follows directly from the way we constructed the Bratteli diagram $(\overline{V}, \overline{E})$ from (V,E) that if $q_{\overline{E}}(f_m)(= q_{\overline{E}}(f'_m))$ does not belong to the edge set of a fixed $(W',F')_N$ for all $m > n$, then this situation will occur, and so $(\overline{H}(\overline{x}), \overline{H}(\overline{y})) = (x,y) \in \overline{R}$. If on the other hand $q_{\overline{E}}(f_m)$ belongs to the edge set of a fixed $(W',F')_N$ for all $m > n$, then both f_m and f'_m belong to the edge set of a fixed $(W,F)_N$ for all $m > n$. We can then find paths $\tilde{x}$ and $\tilde{y}$ in $X_{(W,F)_N}(\subset X_{(\overline{V},\overline{E})})$ which are cofinal with $\overline{x}$ and $\overline{y}$, respectively, from level n on. Hence we will have that $(\overline{H}(\tilde{x}), \overline{H}(\tilde{y})) \in AF((W',F')_N)$. Now $(\overline{x}, \tilde{x}), (\overline{y}, \tilde{y}) \in AF(\overline{V}, \overline{E})$, and so $(\overline{H}(\overline{x}), \overline{H}(\tilde{x})), (\overline{H}(\overline{y}), \overline{H}(\tilde{y})) \in \overline{R}$, i.e. $(x, \overline{H}(\tilde{x})), (y, \overline{H}(\tilde{y})) \in \overline{R}$. Combining all this we get that $(x,y) \in \overline{R} \vee K_N \subset \overline{R} \vee K_1 \vee K_2 \vee \dots$, and so we have proved that $(*)$ holds. (We remark that $\overline{R}$ is open in R, since it is a general fact that if $\overline{S} \subset S$ are étale equivalence relations on X, $\overline{S}$ having the relative topology from S, then $\overline{S}$ is open in S.)

Let $(\overline{L}, \overline{G})$ be the subdiagram of $(\overline{V}, \overline{E})$ whose edge set consists of $\widetilde{F}$ and the union of the edge sets belonging to $(W,F)_1, (W,F)_2, \dots$, and let $(\overline{L'}, \overline{G'})$ be the subdiagram of $(\overline{V}, \overline{E})$ whose edge set consists of the union of the edge sets belonging to $(W,F)_1, (W,F)_2, \dots$. (Note that $(\overline{L}, \overline{G})$ and $(\overline{L'}, \overline{G'})$ are analogous to the subdiagrams (L,G) and (L',G'), respectively, of (V,E) that we defined above.) By (2) it follows that $(\overline{L'}, \overline{G'})$ is a thin subdiagram. Also, it is easily seen that $(\widetilde{W}, \widetilde{F})$ is thin in $(\overline{V}, \overline{E})$, and hence $(\overline{L}, \overline{G})$ is thin in $(\overline{V}, \overline{E})$. (Observe that both $X_{(\overline{L},\overline{G})}$ and $X_{(\overline{L'},\overline{G'})}$ are homeomorphic to $(Y \times \mathbb{N}) \cup \{x_\infty\}$, where $\mathbb{N} = \{1,2,3,\dots\}$ (with discrete topology), and x_∞ is the point at infinity of the one-point compactification of $Y \times \mathbb{N}$.) Clearly $Z = X_{(\overline{L},\overline{G})}$ and $Z' = X_{(\overline{L'},\overline{G'})}$ are $\widetilde{R}$-étale closed subsets of $X_{(\overline{V},\overline{E})}$ (where we for convenience write $\widetilde{R}$ for $AF(\overline{V}, \overline{E})$). We will define a homeomorphism $\alpha : Z \to Z'$ which implements an isomorphism between $\widetilde{R}|_Z(\cong AF(\overline{L}, \overline{G}))$ and $\widetilde{R}|_{Z'}(\cong AF(\overline{L'}, \overline{G'}))$. At the same time we want $\alpha \times \alpha$ to map K (on $Y = X_{(\widetilde{W},\widetilde{F})} \subset X_{(\overline{V},\overline{E})}$) isomorphically to K_1 (on $X_{(W,F)_1}$), and K_n (on $X_{(W,F)_n}$) isomorphically to K_{n+1} (on $X_{(W,F)_{n+1}}$) for all $n \geq 1$. (Retaining the previous notation, we mean by K_n on $X_{(W,F)_n}$ the CEER corresponding to K_n on $X_{(W',F')_n}$—$\overline{H}^{-1} \times \overline{H}^{-1}$ transferring K_n on $X_{(W',F')_n}$ to K_n on $X_{(W,F)_n}$.) There is an obvious way to define $\alpha : X_{(W,F)_n} \to X_{(W,F)_{n+1}}$ satisfying our requirements, using the fact that $(W,F)_n$ and $(W,F)_{n+1}$ are essentially replicas of (W,F). Now consider

$Y = X_{(\widetilde{W},\widetilde{F})}$, where $X_{(\widetilde{W},\widetilde{F})} \subset X_{(\overline{V},\overline{E})}$. We have $AF(\widetilde{W},\widetilde{F}) \cong \widetilde{R}|_Y$ (which clearly may be identified with $R|_Y$), and $\widetilde{R}|_Y \cong \widetilde{R}|_{X_{(W,F)_1}} (\cong AF(W,F))$. Furthermore, $(R|_Y) \vee K \cong (\widetilde{R}|_{X_{(W,F)_1}}) \vee K_1$. The various isomorphism maps are naturally related, and it follows that there exists $\alpha : X_{(\widetilde{W},\widetilde{F})} \to X_{(W,F)_1}$ satisfying our requirements. (We omit the easily checked details.) Furthermore, note that α implements an isomorphism between $(\widetilde{R}|_Y) \vee K$ and $(\widetilde{R}|_{X_{(W,F)_1}}) \vee K_1$. Also, $\overline{H} \times \overline{H}$ implements an isomorphism between the latter and $R|_{X_{(W',F')_1}}$. We define eventually $\alpha : Z \to Z'$ by patching together the various α's above, letting $\alpha(x_\infty) = x_\infty$, and it is straightforward to verify that this α satisfies all our requirements. By Theorem 4.3 there exists an extension $\overline{\alpha} : X_{(\overline{V},\overline{E})} \to X_{(\overline{V},\overline{E})}$ of α which implements an automorphism of $\widetilde{R} = AF(\overline{V},\overline{E})$. Let $h = \overline{H} \circ \overline{\alpha} \circ \overline{H}^{-1}$. Then h is a homeomorphism on $X = X_{(V,E)}$. Recall that $\overline{H} \times \overline{H}(\widetilde{R}) = \overline{R}$. By $(*)$ we have that

$$R \vee K = \overline{R} \vee K \vee K_1 \vee K_2 \vee \ldots . \tag{$**$}$$

We note that

$$\begin{aligned} h \times h(\overline{R}) &= (\overline{H} \times \overline{H}) \circ (\overline{\alpha} \times \overline{\alpha}) \circ (\overline{H}^{-1} \times \overline{H}^{-1})(\overline{R}) \\ &= (\overline{H} \times \overline{H}) \circ (\overline{\alpha} \times \overline{\alpha})(\widetilde{R}) \\ &= (\overline{H} \times \overline{H})(\widetilde{R}) \\ &= \overline{R}. \end{aligned}$$

Also,

$$\begin{aligned} h \times h(K) &= (\overline{H} \times \overline{H}) \circ (\overline{\alpha} \times \overline{\alpha}) \circ (\overline{H}^{-1} \times \overline{H}^{-1})(K) \\ &= (\overline{H} \times \overline{H}) \circ (\overline{\alpha} \times \overline{\alpha})(K) \\ &= (\overline{H} \times \overline{H})(K_1) \\ &= K_1, \end{aligned}$$

where we make the obvious identifications, referred to above, with K (respectively K_1) as CEERs on $X_{(\widetilde{W},\widetilde{F})} (\subset X_{(V,E)})$ and $X_{(\widetilde{W},\widetilde{F})} (\subset X_{(\overline{V},\overline{E})})$ (respectively, $X_{(W',F')_1} (\subset X_{(V,E)})$ and $X_{(W,F)_1} (\subset X_{(\overline{V},\overline{E})})$). Similarly we show that $h \times h(K_n) = K_{n+1}$ for $n \geq 1$. By $(**)$ we get

$$\begin{aligned} h \times h(R \vee K) &= (h \times h)(\overline{R}) \vee (h \times h)(K) \vee (h \times h)(K_1) \vee \ldots \\ &= \overline{R} \vee K_1 \vee K_2 \vee \ldots \\ &= R. \end{aligned}$$

This completes the proof of the main assertion, (i), of the theorem.

Assertions (ii) and (iii) are now immediately clear since $h(Y) = X_{(W',F')_1}$ and $h|_Y$ (by definition of $\overline{\alpha}|_Y$) implements an isomorphism between $(R|_Y) \vee K$ and $R|_{h(Y)} (\cong AF((W',F')_1))$. This finishes the proof of the theorem. □

Remark 4.7. In order to understand the idea behind the proof of the absorption theorem better, it is instructive to look at the simplest (non-trivial) case, namely when $Y = \{y_1, y_2\}$ consists of two points y_1 and y_2, such that $(y_1, y_2) \notin R$. (Even in this simple case, the conclusion one can draw from the absorption theorem is highly non-trivial.) The compact étale equivalence relation K on Y that is transverse to $R|_Y (= \Delta_Y)$ is the following: $K = \Delta_Y \cup \{(y_1, y_2), (y_2, y_1)\}$. The Bratteli diagrams $(\widetilde{W}, \widetilde{F})$ and (W, F) for $R|_Y$ are trees consisting of two paths with no vertices in common, except the top one. The Bratteli diagram (W', F') for $(R|_Y) \vee K$ starts with two edges forming a loop, and then a single path. In Figure 6 the scenario in this case is illustrated. (We have indicated how one constructs the new Bratteli diagram $(\overline{V}, \overline{E})$ from (V, E) (where $R = AF(V, E)$) by exhibiting two specific edges e and f in E and how they give rise to new edges $q_{\overline{E}}^{-1}(\{e\})$ and $q_{\overline{E}}^{-1}(\{f\})$, respectively, in $\overline{E}$.) The example considered here corresponds to a very special case of transversality arising from an action of the (finite) group $G = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, where $\alpha_1 : Y \to Y$ sends y_1 to y_2 and y_2 to y_1 (cf. the comments after Definition 3.1). In the paper [GPS2] the general $\mathbb{Z}/2\mathbb{Z}$-action case is considered (even though it is formulated slightly different there).

Remark 4.8. Theorem 4.6 is the key technical ingredient in the proof of the following result:

Theorem ([GMPS]). *Every minimal action of $\mathbb{Z}^2$ on a Cantor set is orbit equivalent to an AF-equivalence relation, and consequently also orbit equivalent to a minimal $\mathbb{Z}$-action.*

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