Affability of equivalence relations arising from two-dimensional substitution tilings

MATUI Hiroki

Abstract

We will show that an equivalence relation on a Cantor set arising from a two-dimensional substitution tiling by polygons is affable in the sense of Giordano, Putnam and Skau.

1 Introduction

We study an equivalence relation arising from a substitution tiling system in $\mathbb{R}^2$. The dynamics of substitution tiling systems have been studied by many authors (see [KP] and the references given there). A tiling in $\mathbb{R}^2$ gives rise to an action of $\mathbb{R}^2$ on a topological space Ω. Each element of this space can be thought of as a tiling of $\mathbb{R}^2$ and the $\mathbb{R}^2$-action is given by translation. Under suitable hypotheses, this topological space Ω becomes compact and the action of $\mathbb{R}^2$ becomes minimal. Besides, we can find a transversal to this $\mathbb{R}^2$-action and it is known to be a Cantor set. Our object of study is an equivalence relation on this Cantor set which is induced from the $\mathbb{R}^2$-action. We remark that Ω is known to be homeomorphic to the suspension of a free minimal $\mathbb{Z}^2$-action on a Cantor set X (see [SW]). This means that the equivalence relation arising from a tiling space is isomorphic to an induced equivalence relation on a certain clopen subset $U \subset X$.

The study of topological orbit equivalence between equivalence relations on a Cantor set was initiated by Giordano, Putnam and Skau in [GPS1]. They showed that equivalence relations arising from $\mathbb{Z}$-actions on Cantor sets are (topologically) orbit equivalent to AF equivalence relations. An equivalence relation is said to be AF when it can be written as an inductive limit of compact open subequivalence relations. An equivalence relation is said to be affable when it is orbit equivalent to an AF equivalence relation ([GPS2]). In a recent paper [GPS3], they also showed that equivalence relations arising from $\mathbb{Z}^2$-actions on Cantor sets are affable under a hypothesis involving the existence of sufficiently many small positive cocycles.

In the case of tilings, however, it is not known whether the equivalence relation admits sufficiently many such cocycles. It relates to a more complete understanding of the cohomology of tiling spaces. In this paper we take a more direct tack. We will prove the affability by a similar method to [M], in which any extension of a product of two Cantor minimal $\mathbb{Z}$-systems was shown to be affable. As discussed in [P], the equivalence relation arising from a substitution tiling has a canonical AF subequivalence relation. But, this AF equivalence relation is too large to apply the absorption theorem in [GPS2]. We will construct a smaller AF subequivalence relation carefully so that the absorption theorem can be applied.

2 Preliminaries

Let $\mathcal{V}$ be a finite collection of polygons in $\mathbb{R}^2$. Each element of $\mathcal{V}$ is called a prototile. Every prototile is homeomorphic to the closed unit ball and has finitely many edges and vertices. We call a translate of one of the prototile a tile. Let ω be a substitution rule and let $\lambda > 1$ be its inflation constant. Thus, for each prototile $p \in \mathcal{V}$, $\omega(p)$ is a finite collection of tiles with pairwise disjoint interiors and their union is $\lambda p = \{\lambda v : v \in p\}$. Such a pair $(\mathcal{V}, \omega)$ is called a (two-dimensional) substitution tiling system. Several examples, including the Penrose tiles, were given in [AP]. We also allow the possibility that our prototiles carry labels. Even if two tiles coincide as geometric objects, they may be distinguished. A tiling is a collection of tiles such that their union is $\mathbb{R}^2$ and their interiors are pairwise disjoint.

As in [AP], [KP] and [P], we assume that our substitution tiling system $(\mathcal{V}, \omega)$ is primitive, aperiodic and satisfies the finite pattern condition. We follow the notation of [P]. Thanks to the finite pattern condition, we can define a compact metric space Ω consisting of tilings. For each tiling $T \in \Omega$, we let $\omega(T) = \{\omega(p) : p \in T\}$. By the aperiodicity condition, this map $\omega : \Omega \to \Omega$ becomes a homeomorphism. We will also assume that our substitution system forces its border. As discussed in [KP], this loses no generality, provided we allow labelled tiles.

Without loss of generality, we may assume that every $p \in \mathcal{V}$ contains the origin in its interior. We define the puncture of any prototile $p \in \mathcal{V}$ to be the origin. For any $v \in \mathbb{R}^2$, we define the puncture of the tile $p + v$ to be v. As in [P], Ω_{punc} denotes the set of all tilings T in Ω such that the origin is a puncture of some tile $p \in T$. This set is compact, totally disconnected and has no isolated points, that is, Ω_{punc} is a Cantor set.

Define an equivalence relation R_{punc} on Ω_{punc} by

$$R_{punc} = \{(T, T + v) : T, T + v \in \Omega_{punc},\ v \in \mathbb{R}^2\}.$$

As explained in [P], this equivalence relation is given a topology so that it becomes a locally compact, Hausdorff, σ-compact, r-discrete, principal groupoid. In other words, R_{punc} is an étale equivalence relation [GPS2, Definition 2.1]. Since the action of $\mathbb{R}^2$ on Ω by translation is minimal, the equivalence relation R_{punc} becomes minimal. In [P], a subequivalence relation $R_{AF} \subset R_{punc}$ was introduced. The equivalence relation R_{AF} is a minimal AF equivalence relation with the relative topology from R_{punc}. Our main theorem asserts that R_{punc} is orbit equivalent to R_{AF}.

For each $p \in \mathcal{V}$, let $\mathrm{Punc}(p)$ denote the set of all punctures in the tiles of $\omega(p)$. Let $\mathcal{E}$ be the disjoint union of all the $\mathrm{Punc}(p)$'s for $p \in \mathcal{V}$. We define $s : \mathcal{E} \to \mathcal{V}$ and $r : \mathcal{E} \to \mathcal{V}$ as follows: When $e \in \mathcal{E}$ is contained in $\mathrm{Punc}(q)$, we put $r(e) = q$. When $p \in \mathcal{V}$ satisfies $p + e \in \omega(r(e))$, we put $s(e) = p$. We regard $(\mathcal{V}, \mathcal{E}, r, s)$ as a finite directed graph. Since the substitution is primitive, this graph is also primitive.

Put

$$X = \{(e_n)_{n=1}^{\infty} \in \mathcal{E}^{\mathbb{N}} : r(e_n) = s(e_{n+1}) \text{ for all } n \in \mathbb{N}\}.$$

It is easy to see that X is a Cantor set with the induced topology from $\mathcal{E}^{\mathbb{N}}$. Moreover, there exists a natural homeomorphism $\pi : \Omega_{punc} \to X$ defined as follows: For $T \in \Omega_{punc}$ and $n \in \mathbb{N} \cup \{0\}$, let $p_n + v_n$ be the tile in $\omega^{-n}(T)$ whose interior contains the origin, where $p_n \in \mathcal{V}$ and $v_n \in \mathbb{R}^2$. Note that v_0 equals zero because T is in Ω_{punc}. For every $n \in \mathbb{N}$, it is not hard to see that $e_n = v_{n-1} - \lambda v_n$ belongs to $\mathrm{Punc}(p_n)$. By defining $\pi(T) = (e_n)_n$, we obtain a homeomorphism $\pi : \Omega_{punc} \to X$.

We denote the copies of R_{punc} and R_{AF} via the map $\pi : \Omega_{punc} \to X$ by $\mathcal{X}_\omega$ and $\mathcal{X}$, respectively. Thus,

$$\mathcal{X}_\omega = \{(x, y) \in X \times X : (\pi^{-1}(x), \pi^{-1}(y)) \in R_{punc}\}$$

and

$$\mathcal{X} = \{(x, y) \in X \times X : (\pi^{-1}(x), \pi^{-1}(y)) \in R_{AF}\}.$$

We topologize $\mathcal{X}_\omega$ and $\mathcal{X}$ by transferring the topology of R_{punc} and R_{AF} via the map $\pi \times \pi$. The aim of this paper it to show that $\mathcal{X}_\omega$ is affable. For every $n \in \mathbb{N}$, we define a subequivalence relation $\mathcal{X}_n \subset \mathcal{X}$ by

$$\mathcal{X}_n = \{((e_k)_k, (f_k)_k) \in X \times X : e_m = f_m \text{ for all } m > n\}.$$

Clearly $\mathcal{X}_n$ is a compact open subequivalence relation of $\mathcal{X}$ and $\mathcal{X} = \bigcup_{n\in\mathbb{N}} \mathcal{X}_n$.

For a prototile $p \in \mathcal{V}$ we denote the set of all edges of p by $E(p)$. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^2$ be the vertices of p, which are ordered counterclockwise. When $a_i \in E(p)$ is the edge between v_i and v_{i+1}, we define

$$\theta_p(a_i) = \frac{v_{i+1} - v_i}{|v_{i+1} - v_i|} \in \mathbb{T},$$

where the indices are understood modulo n and $\mathbb{T}$ denotes the set of unit vectors in $\mathbb{R}^2$.

We need the following hypotheses about edges of prototiles. Let $p, q \in \mathcal{V}$ be prototiles and suppose $p + v \in \omega(q)$ for $v \in \mathbb{R}^2$.

(P1) If $a \in E(p)$ is an edge between vertices u_1 and u_2, then $\lambda^{-1}a$ is contained in the boundary of $q - \lambda^{-1}v$, or $\lambda^{-1}(a \setminus \{u_1, u_2\})$ is contained in the interior of $q - \lambda^{-1}v$.

(P2) Let $a_1, a_2 \in E(p)$ with $\theta_p(a_1) = \theta_p(a_2)$. If there exist edges $b_1, b_2 \in E(q)$ such that $\lambda^{-1}(a_i + v) \subset b_i$ for $i = 1, 2$, then we have $a_1 = a_2$ and $b_1 = b_2$.

(P3) Let $a_1, a_2 \in E(p)$ with $\theta_p(a_1) \neq \theta_p(a_2)$. If there exist edges $b_1, b_2 \in E(q)$ such that $\lambda^{-1}(a_i + v) \subset b_i$ for $i = 1, 2$, then a_1 is adjacent to a_2 and b_1 is adjacent to b_2.

We further assume the following.

(P4) For any prototiles $p_1, p_2 \in \mathcal{V}$ and their edges $a_i \in E(p_i)$, the length of λa_1 is greater than the length of $2a_2$.

We remark that (P4) can be achieved by replacing ω and λ with ω^N and λ^N for sufficiently large $N \in \mathbb{N}$.

Let $e \in \mathcal{E}$. By definition, $e \in \mathrm{Punc}(r(e))$ and $s(e) + e \in \omega(r(e))$. We define

$$A(e) = \{(a, b) : a \in E(s(e)), b \in E(r(e)) \text{ and } \lambda^{-1}(a + e) \subset b\}.$$

Notice that $\theta_q(b)$ equals $\theta_p(a)$ for every $(a, b) \in A(e)$. The conditions (P2) and (P3) above can be interpreted as follows:

(P2) If $(a_i, b_i) \in A(e)$ for $i = 1, 2$ and $\theta_{s(e)}(a_1) = \theta_{s(e)}(a_2)$, then $a_1 = a_2$ and $b_1 = b_2$.

(P3) If $(a_i, b_i) \in A(e)$ for $i = 1, 2$ and $\theta_{s(e)}(a_1) \neq \theta_{s(e)}(a_2)$, then a_1 is adjacent to a_2 and b_1 is adjacent to b_2.

By (P2) and (P3), we can see that the cardinality of $A(e)$ is not greater than two. Moreover, (P3) and (P4) imply the following.

(P5) For any $q \in \mathcal{V}$ and $b \in E(q)$, there exists $e \in \mathcal{E}$ such that $r(e) = q$ and $A(e) = \{(a, b)\}$ for some $a \in E(s(e))$.

Let $x = (x_n)_n \in X$. We say that $a = (a_n)_n$ is a border of x when $(a_n, a_{n+1}) \in A(x_n)$ for every $n \in \mathbb{N}$. Note that if a is a border of x, then $\theta_{s(x_n)}(a_n) = \theta_{s(x_m)}(a_m)$ for all $n, m \in \mathbb{N}$. We write this value by $\theta(a)$.

For $t \in \mathbb{T}$, we define

$$B_t = \{x \in X : x \text{ has a border } a = (a_n)_n \text{ with } \theta(a) = t\}.$$

The following proposition will play a crucial role in Section 5.

Proposition 2.1. *For $t \in \mathbb{T}$, B_t is a closed $\mathcal{X}$-étale subset.*

Proof. Let U_m be the set of $x = (x_n)_n \in X$ for which there exist $a_i \in E(s(x_i))$ for $i = 1, 2, \dots, m$ such that $\theta_{s(x_1)}(a_1) = t$ and $(a_i, a_{i+1}) \in A(x_i)$ for every $i = 1, 2, \dots, m-1$. It can be easily seen that U_m is clopen. Since B_t is the intersection of all the U_m's, it is a closed subset.

In order to show that B_t is $\mathcal{X}$-étale, take $x = (x_n)_n \in B_t$ and $y = (y_n)_n \in B_t$ with $(x, y) \in \mathcal{X}_m$. Assume that $a = (a_n)_n$ is a border of x and $b = (b_n)_n$ is a border of y such that $\theta(a) = \theta(b) = t$. Define

$$U = \{(z, w) \in \mathcal{X}_m : x_i = z_i \text{ and } y_i = w_i \text{ for all } i = 1, 2, \dots, m+1\}.$$

Then, U is a clopen neighborhood of (x, y) in $\mathcal{X}$. Suppose that $(z, w) \in U$ and $z \in B_t$. We would like to show that w also belongs to B_t. By the definition of B_t, z has a border $c = (c_n)_n$ with $\theta(c) = t$. Since $(b_{m+1}, b_{m+2}) \in A(y_{m+1})$, $(c_{m+1}, c_{m+2}) \in A(z_{m+1})$, $\theta(b) = \theta(c) = t$ and $y_{m+1} = z_{m+1}$, thanks to the condition (P2), we get $b_{m+1} = c_{m+1}$ and $b_{m+2} = c_{m+2}$. Put

$$d_n = \begin{cases} b_n & n \leq m+1 \\ c_n & n > m+1. \end{cases}$$

One can see that $d = (d_n)_n$ is a border of w with $\theta(d) = t$, which means $w \in B_t$. It follows that B_t is $\mathcal{X}$-étale. □

3 Boundaries of tilings

For a tiling $T \in \Omega$, we let $\partial(T)$ be the union of boundaries of all tiles belonging to T. We define

$$\partial_\infty(T) = \bigcap_{n \in \mathbb{N}} \lambda^n \partial(\omega^{-n}(T)).$$

It is clear that for each connected component O of $\mathbb{R}^2 \setminus \partial_\infty(T)$ the set $\{T - v \in \Omega_{punc} : v \in O\}$ is a R_{AF}-orbit. We have the following three possibilities: When $\partial_\infty(T)$ is an empty set, we say that T is of type I. In this case the R_{punc}-orbit of T is equal to the R_{AF}-orbit of it. When $\partial_\infty(T)$ is a line, we say that T is of type II. In this case the R_{punc}-orbit of T splits into two R_{AF}-orbits. In the other case, we say that T is of type III. If T is of type III, then there exists $v \in \mathbb{R}^2$ such that $\partial_\infty(T)$ equals a union of finitely many (at least two) half lines with the same end-point $v \in \mathbb{R}^2$. We call v the center of $\partial_\infty(T)$. Note that if $\partial_\infty(T)$ consists of n half lines then the R_{punc}-orbit of T splits into n R_{AF}-orbits. Evidently the type of T depends only on the R_{punc}-orbit of T. We also remark that a connected component of $\mathbb{R}^2 \setminus \partial_\infty(T)$ can be an open half-plane, even if T is of type III.

For distinct $s, t \in \mathbb{T}$, we would like to define a closed subset $B_{s,t}$ of $B_s \cap B_t$ as follows: Let $x = (x_n)_n \in B_s \cap B_t$. By definition, x has borders $a = (a_n)_n$ and $b = (b_n)_n$ with $\theta(a) = s$ and $\theta(b) = t$. Let $v_1, v_2, \dots, v_n$ be the vertices of $s(x_1)$, which are ordered counterclockwise. Assume that a_1 is the edge between v_{i-1} and v_i. From the condition (P3), b_1 is adjacent to a_1. We define $B_{s,t}$ to be the set of all $x \in B_s \cap B_t$ for which the edge b_1 agrees with the edge between v_i and v_{i+1}. Clearly $B_{s,t} \cup B_{t,s} = B_s \cap B_t$.

For $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$, we denote the real number $u_1 v_2 - u_2 v_1$ by $\det(u, v)$. We can see the following lemma by an easy geometric observation and the condition (P3).

Lemma 3.1. *Let $x = (x_n)_n, y = (y_n)_n \in X$. Let $s, t \in \mathbb{T}$ be distinct elements satisfying $\det(s, t) > 0$. Suppose that both x and y belong to $B_{s,t}$. If $r(x_n) = r(y_n)$, then we get $x_n = y_n$.*

By the lemma above, we obtain the following.

Lemma 3.2. *Let $s, t \in \mathbb{T}$ be distinct elements satisfying* $\det(s,t) > 0$.

(1) *The cardinality of $B_{s,t}$ is not greater than the cardinality of $\mathcal{V}$.*

(2) *Every $x \in B_{s,t}$ is a periodic sequence.*

Proof. (1) Let N be the cardinality of $\mathcal{V}$. Suppose that $x^{(0)}, x^{(1)}, \ldots, x^{(N)}$ belong to $B_{s,t}$. There exist distinct i and j such that $\{n \in \mathbb{N} : r(x_n^{(i)}) = r(x_n^{(j)})\}$ is infinite. It follows from the lemma above that $x^{(i)} = x^{(j)}$, which means that the cardinality of $B_{s,t}$ is not greater than N.

(2) Let $x \in B_{s,t}$. It is obvious that $x^{(m)} = (x_{n+m})_n$ also belongs to $B_{s,t}$ for every $m \in \mathbb{N}$. From (1), x must be periodic. □

Lemma 3.3. *The set $L = \{T \in \Omega_{punc} : T$ is of type III $\}$ consists of finitely many R_{punc}-orbits.*

Proof. Suppose that $T \in \Omega_{punc}$ is of type III. There exists at least one connected component O of $\mathbb{R}^2 \setminus \partial_\infty(T)$ which is congruent to a cone, that is, O is a translate of

$$\{u \in \mathbb{R}^2 : \det(s,u) > 0,\ \det(t,u) > 0\}$$

for some distinct $s, t \in \mathbb{T}$. By replacing (s,t) with (t,s) if necessary, we may assume $\det(s,t) > 0$. Choose a tile $p \in T$ whose puncture v is contained in O. Put $x = (x_n)_n = \pi(T - v)$. It is not hard to see that $(x_{n+m})_n$ belongs to $B_{s,t}$ for sufficiently large $m \in \mathbb{N}$. Since $\{t \in \mathbb{T} : B_t \neq \emptyset\}$ is a finite set, it suffices to show the following: For any distinct $s, t \in \mathbb{T}$ satisfying $\det(s,t) > 0$,

$$C = \{x = (x_n)_n \in X : \text{there exists } m \in \mathbb{N} \text{ such that } (x_{n+m})_m \in B_{s,t}\}$$

consists of finitely many $\mathcal{X}$-orbits. Assume that $x^{(0)}, x^{(1)}, \ldots, x^{(N)}$ belong to C, where N is the cardinality of $\mathcal{V}$. There exists $m \in \mathbb{N}$ such that $y^{(i)} = (x_{n+m}^{(i)})_n$ is contained in $B_{s,t}$ for all $i = 0, 1, \ldots, N$. By Lemma 3.2 (1), there must exist distinct i and j such that $y^{(i)} = y^{(j)}$, which implies that $(x^{(i)}, x^{(j)}) \in \mathcal{X}_m \subset \mathcal{X}$. Therefore C consists of at most N $\mathcal{X}$-orbits. □

Lemma 3.4. *Suppose that $T \in \Omega$ is of type III and the center of $\partial_\infty(T)$ is the origin. Let $p \in T$ be a tile which meets the origin and let $w \in \mathbb{R}^2$ be the puncture of p. If $\omega^{-k}(T) = \omega(T)$, then the sequence $\pi(T - w)$ has period k.*

Proof. For $n \in \mathbb{N} \cup \{0\}$, let $q_n + u_n \in \omega^{-n}(T - w)$ be the tile which contains the origin in its interior, where $q_n \in \mathcal{V}$ and $u_n \in \mathbb{R}^2$. It suffices to show that the sequence $(u_{n-1} - \lambda u_n)_n$ has period k. As p meets the origin and the origin is in $\partial_\infty(T)$, we can see that $-\lambda^{-n} w$ is in the boundary of the tile $q_n + u_n \in \omega^{-n}(T - w)$. Hence the boundary of

$$q_n + u_n + \lambda^{-n} w \in \omega^{-n}(T)$$

contains the origin. Put $p_n = q_n + u_n + \lambda^{-n} w$. By $\omega^{-k}(T) = \omega(T)$, we have $p_n \in \omega^{-n-k}(T)$, and so the interior of p_n does not intersect $\lambda^k p_n$ or p_n is contained in $\lambda^k p_n$. We obtain $p_n \subset \lambda^k p_n$, because the boundary of p_n contains the origin and p_n is a polygon. It follows that

$$\lambda^{-k}(q_n + u_n) \subset q_n + u_n + \lambda^{-n} w - \lambda^{-n-k} w.$$

From this and

$$\begin{aligned}\omega^{-n-k}(T - w) &= \omega^{-n}(T) - \lambda^{-n-k} w \\ &= \omega^{-n}(T - w) + \lambda^{-n} w - \lambda^{-n-k} w \\ &\ni q_n + u_n + \lambda^{-n} w - \lambda^{-n-k} w,\end{aligned}$$

we get $q_{n+k} = q_n$ and $u_{n+k} = u_n + \lambda^{-n}w - \lambda^{-n-k}w$. Then we have

$$\begin{aligned}
&u_{n+k-1} - \lambda u_{n+k} \\
&= u_{n-1} + \lambda^{-n+1}w - \lambda^{-n-k+1}w - \lambda(u_n + \lambda^{-n}w - \lambda^{-n-k}w) \\
&= u_{n-1} - \lambda u_n,
\end{aligned}$$

thereby completing the proof. □

Lemma 3.5. *When $T \in \Omega_{punc}$ is of type III, there exists a unique periodic sequence $x \in [\pi(T)]_{\mathcal{X}}$.*

Proof. Since the uniqueness is clear, it suffices to show the existence.

As in the proof of Lemma 3.3, we can find a connected component O of $\mathbb{R}^2 \setminus \partial_\infty(T)$ which is congruent to a cone, that is, O is a translate of $\{u \in \mathbb{R}^2 : \det(s,u) > 0,\ \det(t,u) > 0\}$ for some distinct $s, t \in \mathbb{T}$. By replacing (s,t) with (t,s) if necessary, we may assume $\det(s,t) > 0$. Choose $u \in O$ so that $S = T - u \in \Omega_{punc}$. At first we would like to show the assertion for S. Let $y = (y_n)_n = \pi(S)$. There exists $m \in \mathbb{N}$ such that $(y_{n+m})_n$ belongs to $B_{s,t}$. By virtue of Lemma 3.2 (2), we see that $(y_{n+m})_n$ is a periodic sequence. Let $k \in \mathbb{N}$ be its period. Take $l \in \mathbb{N}$ so that $kl > m$. Define $z = (z_n)_n \in X$ by $z_n = y_{n+kl}$. It is easy to see that z is a periodic sequence and $(z,y) \in \mathcal{X}$. Thus the assertion has been shown for S.

Let $v \in \mathbb{R}^2$ be the center of $\partial_\infty(T)$. The sequence $z = (z_n)_n$ has period k, and so we have

$$\{p \in T - v : p \subset (\overline{O} - v) \neq \emptyset\} = \{p \in \omega^{-k}(T - v) : p \subset (\overline{O} - v) \neq \emptyset\},$$

where $\overline{O}$ means the closure of O. Since the substitution system forces its border, we can conclude that $T - v$ equals $\omega^{-k}(T - v)$.

Let U be the connected component of $\mathbb{R}^2 \setminus \partial_\infty(T)$ which contains the origin. We notice that U corresponds to the $\mathcal{X}$-orbit of $\pi(T)$. We can find a tile $p \in T$ which meets v and whose interior is contained in U. Then we have $\pi(T - w) \in [\pi(T)]_{\mathcal{X}}$, where w is the puncture of p. The conclusion follows from the lemma above. □

By Lemma 3.3,

$$L = \{T \in \Omega_{punc} : T \text{ is of type III}\}$$

consists of finitely many R_{AF}-orbits. It follows from Lemma 3.5 that

$$M = \{x \in X : x \text{ is periodic and } \pi^{-1}(x) \text{ is of type III}\}$$

is a finite set and there exists a natural bijective correspondence between M and $\pi(L)/\mathcal{X}$. By replacing ω with ω^N for some appropriate $N \in \mathbb{N}$, we may assume that every $x \in M$ has period one. Define

$$\mathcal{C} = \{e \in \mathcal{E} : x = (e, e, \dots) \in \pi(L)\}.$$

We remark that there exists a natural bijective correspondence between $\mathcal{C}$ and $\pi(L)/\mathcal{X}$.

By the conditions (P2) and (P3), we also have the following.

Lemma 3.6. *For $e \in \mathcal{C}$, we have $A(e) = \{(a,a)\}$ or $A(e) = \{(a,a),(b,b)\}$ for some $a, b \in E(r(e))$.*

In the next section we will need the following condition.

(P6) For any $p \in \mathcal{V}$ and $a \in E(p)$, there exists $e \in \mathcal{E} \setminus \mathcal{C}$ such that $A(e) = \{(b,a)\}$ for some $b \in E(s(e))$.

Replacing ω with ω^2 if necessary, we can easily achieve this condition.

4 Construction of a subequivalence relation

In this section we would like to construct a subequivalence relation $\mathcal{X}'$ of $\mathcal{X}$.

At first we define a closed subset δ_e of $r(e) \in \mathcal{V}$ for each $e \in \mathcal{C}$ as follows. Let $v_1, v_2, \ldots, v_n \in \mathbb{R}^2$ be the vertices of $r(e)$, which are ordered counterclockwise. By definition, $\lambda^{-1}(s(e)+e)$ is a subset of $r(e)$, and $s(e)$ is equal to $r(e)$. By Lemma 3.6, we have $A(e) = \{(a,a)\}$ or $A(e) = \{(a,a),(b,b)\}$ for some $a, b \in E(r(e))$.

Let us consider the case of $A(e) = \{(a,a)\}$. Suppose that a is the edge between v_i and v_{i+1}, where the indices are understood modulo n. Since $\lambda^{-1}(a+e) \subset a$, the two points $\lambda^{-1}(v_i+e)$ and $\lambda^{-1}(v_{i+1}+e)$ lie on the edge a. In this case we let δ_e be the closed interval between v_i and $\lambda^{-1}(v_{i+1}+e)$. Note that δ_e is a subset of a.

Next, let us consider the case of $A(e) = \{(a,a),(b,b)\}$. We may assume that a is the edge between v_{i-1} and v_i, and that b is the edge between v_i and v_{i+1}. In this case we let δ_e be the edge a.

By using these δ_e's, we would like to define continuous functions $\mu_n^e : X \to \{0,1\}$ for $e \in \mathcal{C}$ and $n \in \mathbb{N}$. Let $x = (x_n)_n \in X$. At first we define μ_1^e by

$$\mu_1^e(x) = \begin{cases} 1 & r(x_1) = r(e) \text{ and } \lambda^{-1}(a + x_1) \subset \delta_e \text{ for some } a \in E(s(x_1)) \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 2, 3, \ldots$, we define μ_n^e by

$$\mu_n^e(x) = \begin{cases} \mu_{n-1}^e(x) & x_n = e \\ 1 & x_n \neq e, r(x_n) = r(e) \text{ and } \lambda^{-1}(a + x_n) \subset \delta_e \text{ for some } a \in E(s(x_n)) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that μ_n^e is well-defined and continuous.

The following is an immediate consequence of the definition of μ_n^e.

Lemma 4.1. *For $e \in \mathcal{C}$ and $(x,y) \in \mathcal{X}_n$, if*

$$\mu_n^e(x) = \mu_n^e(y)$$

then we have

$$\mu_m^e(x) = \mu_m^e(y)$$

for all $m > n$.

For every $n \in \mathbb{N}$, we define a subset $\mathcal{X}'_n$ of $\mathcal{X}_n$ by

$$\mathcal{X}'_n = \{(x,y) \in \mathcal{X}_n : \mu_n^e(x) = \mu_n^e(y) \text{ for all } e \in \mathcal{C}\}.$$

Lemma 4.2. *For every $n \in \mathbb{N}$, $\mathcal{X}'_n$ is a compact open subequivalence relation of $\mathcal{X}_n$, and $\mathcal{X}'_n$ is contained in $\mathcal{X}'_{n+1}$.*

Proof. It is obvious that $\mathcal{X}'_n$ is a subequivalence relation of $\mathcal{X}_n$. Since μ_n^e is continuous, $\mathcal{X}'_n$ is compact and open. From the lemma above we can see $\mathcal{X}'_n \subset \mathcal{X}'_{n+1}$. □

Define

$$\mathcal{X}' = \bigcup_{n \in \mathbb{N}} \mathcal{X}'_n.$$

By the lemma above, $\mathcal{X}'$ is an AF equivalence relation on X.

Lemma 4.3. *Suppose that $T \in \Omega_{punc}$ is not of type III. Then we have $[\pi(T)]_{\mathcal{X}} = [\pi(T)]_{\mathcal{X}'}$.*

Proof. Let $x = (x_n)_n = \pi(T)$. Take $y = (y_n)_n \in [x]_{\mathcal{X}}$ arbitrarily. We would like to show that y belongs to $[x]_{\mathcal{X}'}$. There exists $m \in \mathbb{N}$ such that $(x, y) \in \mathcal{X}_m$. Let $e \in \mathcal{C}$. If $x_l = e$ for every $l > m$, then x is equivalent to $(e, e, \dots)$ in $\mathcal{X}$. It follows that $\pi^{-1}(x) = T$ is of type III. Hence we can find $l > m$ such that $x_l \neq e$. Combined with $x_l = y_l$, this implies $\mu_l^e(x) = \mu_l^e(y)$. By Lemma 4.1 the proof is completed. □

Lemma 4.4. *The equivalence relation $\mathcal{X}'$ is minimal.*

Proof. Let $x = (x_n)_n \in X$. It suffices to show that $[x]_{\mathcal{X}'}$ is dense in X. If $\pi^{-1}(T)$ is not of type III, by the lemma above, $[x]_{\mathcal{X}'} = [x]_{\mathcal{X}}$. Since $\mathcal{X}$ is minimal, we can deduce that $[x]_{\mathcal{X}'} = [x]_{\mathcal{X}}$ is dense in X.

Let us assume that $T = \pi^{-1}(x)$ is of type III. There exist $e \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $x_n = e$ for all $n > m$. Take a finite path $(y_1, y_2, \dots, y_k)$ in $(\mathcal{V}, \mathcal{E})$ arbitrarily. It suffices to find $x' \in [x]_{\mathcal{X}'}$ whose initial segments agree with $(y_1, y_2, \dots, y_k)$. Since the directed graph $(\mathcal{V}, \mathcal{E})$ is primitive, we can find $N \in \mathbb{N}$ such that for any $p \in \mathcal{V}$ there exists a path of length N which starts from $r(y_k)$ and ends at p. Let l be a natural number greater than m and $k + N$.

We would like to consider the case of $\mu_l^e(x) = 1$. We can find $f \in \mathcal{E}$ such that $f \neq e$, $r(f) = s(e)$ and $\lambda^{-1}(a + f) \subset \delta_e$ for some $a \in E(s(f))$. Take a finite path $(y_{k+1}, y_{k+2}, \dots, y_{l-1})$ in $(\mathcal{V}, \mathcal{E})$ so that $r(y_k) = s(y_{k+1})$ and $r(y_{l-1}) = s(f)$. Define

$$x' = (y_1, y_2, \dots, y_{l-1}, f, e, e, \dots) \in X.$$

By the choice of f we have $\mu_l^e(x') = 1$. Therefore we can conclude that $(x, x') \in \mathcal{X}'_{l+1}$.

In the case of $\mu_l^e(x) = 0$, we choose $f \in \mathcal{E}$ so that $r(f) = s(e)$ and $\lambda^{-1}(a + f) \subset \delta_e$ does not hold for any $a \in E(s(f))$. Then the proof goes in a similar fashion to the preceding paragraph. □

It is well-known that both R_{punc} and R_{AF} are uniquely ergodic (see [P] for example). It follows that both $\mathcal{X}_\omega$ and $\mathcal{X}$ are also uniquely ergodic. The next lemma claims that $\mathcal{X}'$ is uniquely ergodic, too.

Lemma 4.5. *If ν is a $\mathcal{X}'$-invariant probability measure on X, then ν equals the unique $\mathcal{X}$-invariant probability measure.*

Proof. It follows from Lemma 4.4 that ν is nonatomic. By Lemma 3.3 and Lemma 4.3,

$$\{x \in X : [x]_{\mathcal{X}} \neq [x]_{\mathcal{X}'}\}$$

consists of finitely many $\mathcal{X}'$-orbits. Hence ν is $\mathcal{X}$-invariant. □

Lemma 4.6. *Let $t_1, t_2 \in \mathbb{T}$ be distinct elements. For*

$$U = \{(x_n)_n \in X : \textit{the cardinality of } A(x_1) \textit{ is one}\},$$

we have $\mathcal{X}' \cap ((U \cap B_{t_1}) \times (U \cap B_{t_2})) = \emptyset$.

Proof. For a proof by contradiction, assume that $\mathcal{X}' \cap ((U \cap B_{t_1}) \times (U \cap B_{t_2}))$ contains (x, y). Let $x = (x_n)_n$ and $y = (y_n)_n$. Since t_1 is distinct from t_2, $\pi^{-1}(x)$ is of type III. By Lemma 3.5, there exist $e \in \mathcal{C}$ and $m \in \mathbb{N}$ such that $x_n = y_n = e$ for all $n > m$. Besides, $A(e)$ should be $\{(a, a), (b, b)\}$ for some $a, b \in E(r(e))$. We may assume that $\delta_e = a$, $\theta_{r(e)}(a) = t_1$ and $\theta_{r(e)}(b) = t_2$. Then, for any $n > m$, one can see that $\mu_n^e(x) = 1$ and $\mu_n^e(y) = 0$ by an inductive calculation. But this contradicts $(x, y) \in \mathcal{X}'$. □

5 Affability

Let $p \in \mathcal{V}$ be a prototile and let $a \in E(p)$ be an edge with $\theta_p(a) = t$. Thanks to the conditions (P5) and (P6), we can choose $e_i \in \mathcal{E}$ and $a_i \in E(s(e_i))$ for $i = 1, 2, 3$ so that the following are satisfied.

- $r(e_1) = s(e_2)$, $r(e_2) = s(e_3)$ and $r(e_3) = p$.
- $A(e_1) = \{(a_1, a_2)\}$, $A(e_2) = \{(a_2, a_3)\}$ and $A(e_3) = \{(a_3, a)\}$.
- e_3 does not belong to $\mathcal{C}$.

Define a clopen subset U_a of X by

$$U_a = \{(x_n)_n \in X : x_1 = e_1, x_2 = e_2 \text{ and } x_3 = e_3\}.$$

Let us consider the prototile $s(e_3)$ and its edge a_3. Since the substitution system forces its border and satisfies the condition (P4), there exist $v \in \mathbb{R}^2$, $q \in \mathcal{V}$ and $b \in E(q)$ such that the following hold.

- If $S \in \Omega$ contains $s(e_3)$, then $\omega(S)$ contains $q + v$.
- $\theta_q(b) = -t$ and $\lambda^{-1}(b + v) \subset a_3$.

From the condition (P5), we can find $f \in \mathcal{E}$ such that $r(f) = q$ and $A(f) = \{(b', b)\}$ for some $b' \in s(f)$.

Put $u_a = f + \lambda v - \lambda e_2 - e_1 \in \mathbb{R}^2$. For $T \in \pi^{-1}(U_a)$ we would like to consider $T - u_a$. By the definition of U_a, the tiling $\omega^{-2}(T)$ contains the tile $s(e_3) - \lambda^{-1}e_2 - \lambda^{-2}e_1$, which implies

$$q + v - e_2 - \lambda^{-1}e_1 \in \omega^{-1}(T).$$

From $s(f) + f \in \omega(q)$ we get

$$s(f) + f + \lambda v - \lambda e_2 - e_1 = s(f) + u_a \in T.$$

Hence the map sending $\pi(T) \in U_a$ to $\pi(T - u_a) \in X$ is well-defined. We denote this map by $\beta_a : U_a \to X$.

Let $T' = T - u_a$. The tile $s(f) \in T'$ contains the origin in its interior, The first coordinate of $\pi(T')$ is f and $A(f) = \{(b', b)\}$. We define $w = -\lambda^{-1}e_3 - \lambda^{-3}f - \lambda^{-2}v \in \mathbb{R}^2$. Then we have

$$\begin{aligned}
\omega^{-3}(T') &= \omega^{-3}(T) - \lambda^{-3}u_a \\
&\ni r(e_3) - \lambda^{-1}e_3 - \lambda^{-2}e_2 - \lambda^{-3}e_1 - \lambda^{-3}u_a \\
&= p - \lambda^{-1}e_3 - \lambda^{-3}f - \lambda^{-2}v \\
&= p + w.
\end{aligned}$$

Moreover we can see

$$\begin{aligned}
a + w &\supset \lambda^{-1}(a_3 + e_3) + w \\
&= \lambda^{-1}(a_3 - \lambda^{-2}f - \lambda^{-1}v) \\
&\supset \lambda^{-1}(\lambda^{-1}(b + v) - \lambda^{-2}f - \lambda^{-1}v) \\
&= \lambda^{-2}(b - \lambda^{-1}f) \\
&\supset \lambda^{-2}(\lambda^{-1}(b' + f) - \lambda^{-1}f) \\
&= \lambda^{-3}b'.
\end{aligned}$$

We can summarize these arguments as follows.

Lemma 5.1. *Let $p \in \mathcal{V}$ and $a \in E(p)$. Put $t = \theta_p(a)$. Define a clopen subset $U_a \subset X$ and a map $\beta_a : U_a \to X$ as above.*

(1) *β_a is a homeomorphism from U_a to $\beta_a(U_a)$ and $\{(x, \beta_a(x)) \in X \times X : x \in U_a\}$ is a clopen subset of $\mathcal{X}_\omega$.*

(2) *If $x = (x_n)_n \in U_a$, then $x_3 \notin \mathcal{C}$.*

(3) *If $x = (x_n)_n \in U_a$, then $A(x_1) = \{(a_1, a_2)\}$, $A(x_2) = \{(a_2, a_3)\}$ and $A(x_3) = \{(a_3, a)\}$ for some a_1, a_2, a_3 with $\theta_{s(x_1)}(a_1) = t$.*

(4) *If $y = (y_n)_n \in \beta_a(U_a)$, then $A(y_1) = \{(b', b)\}$ for some b', b with $\theta_{s(y_1)}(b') = -t$. Moreover, there exists $w \in \mathbb{R}^2$ such that $p + w \in \omega^{-3}(\pi^{-1}(y))$ and $\lambda^{-3}b' \subset a + w$.*

(5) *For $x \in U_a$, $x \in B_t$ if and only if $\beta_a(x) \in B_{-t}$.*

Proof. Only (5) needs a proof. We use the notation used in the argument above. Take $x \in U_a$ and put $T = \pi^{-1}(x)$. We notice that the tiling $\omega^{-3}(T)$ contains the tile $p - \lambda^{-1}e_3 - \lambda^{-2}e_2 - \lambda^{-3}e_1$ and a is an edge of p. Hence it is not hard to see that $x \in B_t$ if and only if

$$a - \lambda^{-1}e_3 - \lambda^{-2}e_2 - \lambda^{-3}e_1 \subset \lambda^n \partial(\omega^{-3-n}(T))$$

for all $n \in \mathbb{N}$. We also notice that the tiling $T - u_a$ contains the tile $s(f)$ and b' is an edge of $s(f)$. It follows that $\beta_a(x) \in B_{-t}$ if and only if

$$b' \subset \lambda^n \partial(\omega^{-n}(T - u_a))$$

for all $n \in \mathbb{N}$. Therefore we have $\beta_a(x) \in B_{-t}$ if and only if

$$\lambda^{-3}(b' - u_a) \subset \lambda^n \partial(\omega^{-3-n}(T))$$

for all $n \in \mathbb{N}$. From

$$\lambda^{-3}(b' - u_a) \subset a - \lambda^{-1}e_3 - \lambda^{-2}e_2 - \lambda^{-3}e_1$$

and the condition (P1), we get $x \in B_t$ if and only if $\beta_a(x) \in B_{-t}$. □

Lemma 5.2. *Let $p_1, p_2 \in \mathcal{V}$ and let $a_i \in E(p_i)$ for $i = 1, 2$.*

(1) *If a_1 is not equal to a_2, then $U_{a_1} \cap U_{a_2} = \emptyset$.*

(2) *If a_1 is not equal to a_2, then $\beta_{a_1}(U_{a_1}) \cap \beta_{a_2}(U_{a_2}) = \emptyset$.*

Proof. (1) is evident from Lemma 5.1 (3). Let us prove (2). Suppose that $y = (y_n)_n \in X$ belongs to both $\beta_{a_1}(U_{a_1})$ and $\beta_{a_2}(U_{a_2})$. By Lemma 5.1 (4), we have $A(y_1) = \{(c, d)\}$ for some c and d, and $-\theta_{s(y_1)}(c)$ must be equal to both $\theta_{p_1}(a_1)$ and $\theta_{p_2}(a_2)$. Thus, we can put $t = -\theta_{s(y_1)}(c) = \theta_{p_1}(a_1) = \theta_{p_2}(a_2)$. Besides, there exists $w_i \in \mathbb{R}^2$ such that $p_i + w_i \in \omega^{-3}(\pi^{-1}(y))$ and $\lambda^{-3}c \subset a_i + w_i$ for each $i = 1, 2$. Since $\omega^{-3}(\pi^{-1}(y))$ is a tiling, we can conclude $p_1 = p_2$ and $a_1 = a_2$. □

Let U_ω be the union of all U_a's, that is,

$$U_\omega = \bigcup_{a \in E(p), p \in \mathcal{V}} U_a.$$

For $x \in U_a \subset U_\omega$ we define $\bar{\beta}(x) = \beta_a(x)$. By the lemma above $\bar{\beta}$ is a well-defined homeomorphism from U_ω to $\bar{\beta}(U_\omega)$. Moreover $\{(x, \bar{\beta}(x)) : x \in U_\omega\}$ is a clopen subset of $\mathcal{X}_\omega$.

Choose a finite subset F of $\mathbb{T}$ so that the following are satisfied:

- If B_t is not empty, then either of t or $-t$ belongs to F.
- If $t \in F$, then $-t \notin F$.

We define closed subsets B and B^* of X by

$$B = U_\omega \cap \bigcup_{t \in F} B_t$$

and

$$B^* = \bar{\beta}(U_\omega) \cap \bigcup_{t \in F} B_{-t}.$$

(3) and (4) of Lemma 5.1 tells us that if $x \in U_a$, $a \in E(p)$ and $\theta_p(a) = t$, then $U_a \cap B_s = \emptyset$ and $\beta_a(U_a) \cap B_{-s} = \emptyset$ for all $s \neq t$. Moreover, from Lemma 5.1 (5), one can see $\bar{\beta}(B) = B^*$. We denote the restriction of $\bar{\beta}$ to B by β.

Lemma 5.3. *Both B and B^* are closed $\mathcal{X}'$-étale thin subsets.*

Proof. At first let us prove that B and B^* are thin for $\mathcal{X}'$. Let ν be the unique $\mathcal{X}$-invariant probability measure on X. We remark that ν is also the unique $\mathcal{X}_\omega$-invariant probability measure. By Lemma 4.5 it suffices to show $\nu(B \cup B^*) = 0$. But this follows immediately from [P, Theorem 2.1].

We next show that B is étale for $\mathcal{X}'$. Note that B is equal to a disjoint union $\bigcup_{t \in F} U_\omega \cap B_t$. Suppose that (x, y) belongs to $\mathcal{X}' \cap (B \times B)$. Let $x \in U_\omega \cap B_{t_1}$ and $y \in U_\omega \cap B_{t_2}$. By Lemma 4.6 and Lemma 5.1 (3), t_1 must equal t_2. Thus, x and y lie in some $U_\omega \cap B_t$. From Proposition 2.1, $U_\omega \cap B_t$ is $\mathcal{X}$-étale. Since $\mathcal{X}'$ is an open subequivalence relation of $\mathcal{X}$, we can deduce that B is $\mathcal{X}'$-étale.

Similarly we can prove that B^* is also $\mathcal{X}'$-étale. The proof is completed. □

Lemma 5.4. *We have $\mathcal{X}' \cap (B \times B^*) = \emptyset$.*

Proof. For a proof by contradiction, suppose that $\mathcal{X}' \cap (B \times B^*)$ contains (x, y). Let $x \in U_\omega \cap B_{t_1}$ and $y \in \beta(U_\omega) \cap B_{-t_2}$, where t_1 and t_2 belong to F. By Lemma 5.1 (3), (4) and Lemma 4.6, t_1 must equal $-t_2$. But this contradicts the choice of F. □

Lemma 5.5. *The homeomorphism $\beta : B \to B^*$ induces an isomorphism between $\mathcal{X}' \cap (B \cap B)$ and $\mathcal{X}' \cap (B^* \times B^*)$.*

Proof. Since $\{(x, \bar{\beta}(x)) : x \in U_\omega\}$ is a clopen subset of $\mathcal{X}_\omega$,

$$\bar{\beta} \times \bar{\beta} : \mathcal{X}_\omega \cap (U_\omega \cap U_\omega) \to \mathcal{X}_\omega \cap (\bar{\beta}(U_\omega) \times \bar{\beta}(U_\omega))$$

is a well-defined isomorphism. Because $\mathcal{X}'$ is an open subequivalence relation of $\mathcal{X}_\omega$ and β is a restriction of $\bar{\beta} : U_\omega \to \bar{\beta}(U_\omega)$ to B, it suffices to show that β induces a bijective correspondence between $\mathcal{X}' \cap (B \times B)$ and $\mathcal{X}' \cap (B^* \times B^*)$.

Take $x, y \in B$. It suffices to show that $(x, y) \in \mathcal{X}'$ if and only if $(\beta(x), \beta(y)) \in \mathcal{X}'$. When $\pi^{-1}(x)$ is of type I, we have nothing to do because of $[x]_{\mathcal{X}_\omega} = [x]_{\mathcal{X}'}$.

Assume that $T = \pi^{-1}(x)$ is of type II. In this case $\partial_\infty(T)$ is a line and it divides the plane into two open half-planes. By Lemma 4.3 we have $[x]_{\mathcal{X}} = [x]_{\mathcal{X}'}$, and so it suffices to show $(x, y) \in \mathcal{X}$ if and only if $(\beta(x), \beta(y)) \in \mathcal{X}$. If $(x, y) \in \mathcal{X}$, then there exists $t \in F$ such that $x, y \in B_t$. It follows that $\beta(x), \beta(y) \in B_{-t}$, which means $(\beta(x), \beta(y)) \in \mathcal{X}$. In the same way we can show that $(\beta(x), \beta(y)) \in \mathcal{X}$ implies $(x, y) \in \mathcal{X}$.

Suppose that $T = \pi^{-1}(x)$ is of type III. Thus $\partial_\infty(T)$ consists of finitely many half lines. Let $v \in \mathbb{R}^2$ be the center of $\partial_\infty(T)$. Note that if $p \in T$ does not meet v, then p meets at most one half line in $\partial_\infty(T)$. We denote the canonical bijective correspondence from $[x]_{\mathcal{X}_\omega}$ to T by ρ, that is, if the puncture of $p \in T$ is $u \in \mathbb{R}^2$ then $\rho(\pi(T-u)) = p$. Suppose that (x, y) belongs to $\mathcal{X}'$. Let $p = \rho(x)$ and $q = \rho(y)$. By Lemma 5.1 (2), we notice that neither p nor q meets v. From $(x, y) \in \mathcal{X}'$ we can deduce that there exists a half line ℓ in $\partial_\infty(T)$ such that both p and q meet ℓ from the same side of ℓ. It follows that both $\rho(\beta(x))$ and $\rho(\beta(y))$ meet ℓ but they lie on the opposite side from p and q across the half line ℓ. Hence we can see that $(\beta(x), \beta(y)) \in \mathcal{X}'$. Next, assume that (x, y) belongs to $\mathcal{X}_\omega$ but does not belong to $\mathcal{X}'$. There exist half lines ℓ_1 and ℓ_2 in $\partial_\infty(T)$ such that p meets ℓ_1 and q meets ℓ_2. If ℓ_1=ℓ_2, then p and q never lie on the same side of $\ell_1 = \ell_2$. But, if p is located on the opposite side from q across the half line $\ell_1 = \ell_2$, then $F \subset \mathbb{T}$ must contain both t and $-t$ for some $t \in \mathbb{T}$, which is a contradiction. Therefore ℓ_1 and ℓ_2 should be different. Since $\rho(\beta(x))$ meets ℓ_1 and $\rho(\beta(y))$ meets ℓ_2, we can conclude that $(\beta(x), \beta(y))$ never belongs to $\mathcal{X}'$. Consequently we have $(x, y) \in \mathcal{X}'$ if and only if $(\beta(x), \beta(y)) \in \mathcal{X}'$, thereby completing the proof. □

Let $\widetilde{\mathcal{X}}$ be the equivalence relation generated by $\mathcal{X}'$ and $\{(x, \beta(x)) \in X \times X : x \in B\}$. It is obvious that $\widetilde{\mathcal{X}}$ is a subequivalence relation of $\mathcal{X}_\omega$.

Lemma 5.6. *The equivalence relation $\widetilde{\mathcal{X}}$ is orbit equivalent to $\mathcal{X}$.*

Proof. From Lemma 4.4, 5.3, 5.4, 5.5, we can apply the absorption theorem [GPS2, Theorem 4.18] to $\mathcal{X}'$ and $\beta : B \to B^*$. It follows that $\widetilde{\mathcal{X}}$ is orbit equivalent to $\mathcal{X}'$. Both $\mathcal{X}'$ and $\mathcal{X}$ are minimal AF equivalence relations. Moreover they are uniquely ergodic with the same invariant probability measure ν by Lemma 4.5. By [GPS1, Theorem 2.3], we can conclude that $\widetilde{\mathcal{X}}$ is orbit equivalent to $\mathcal{X}$. □

Now we are ready to prove the main theorem.

Theorem 5.7. *Let $(\mathcal{V}, \omega)$ be a substitution tiling system in $\mathbb{R}^2$ which is primitive, aperiodic and satisfies the finite pattern condition. Suppose that each prototile is a polygon and the conditions (P1), (P2) and (P3) are satisfied. Then the equivalence relation R_{punc} on Ω_{punc} is orbit equivalent to R_{AF}. In particular R_{punc} is affable.*

Proof. Let $x = \pi(T) \in X$. When T is of type I, the $\mathcal{X}_\omega$-equivalence class $[x]_{\mathcal{X}_\omega}$ is equal to the $\mathcal{X}'$-equivalence class $[x]_{\mathcal{X}'}$. Hence $[x]_{\mathcal{X}_\omega} = [x]_{\widetilde{\mathcal{X}}}$. Let us assume that T is of type II. The $\mathcal{X}_\omega$-orbit $[x]_{\mathcal{X}_\omega}$ splits into two $\mathcal{X}'$-orbits. As described in the proof of Lemma 5.5, these two orbits are glued by $\beta : B \to B^*$. It follows that $[x]_{\mathcal{X}_\omega}$ agrees with $[x]_{\widetilde{\mathcal{X}}}$. If T is of type III, then $[x]_{\mathcal{X}_\omega}$ may not agree with $[x]_{\widetilde{\mathcal{X}}}$. By Lemma 3.3, however, there are only finitely many R_{punc}-orbits of type III. In addition, every $\mathcal{X}_\omega$-orbit splits into finitely many $\mathcal{X}'$-orbits. Therefore, we can find $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in $X \times X$ such that $\mathcal{X}_\omega$ is generated by $\widetilde{\mathcal{X}}$ and $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$. It follows from Lemma 5.6 and [GPS2, Corollary 4.17] that $\mathcal{X}_\omega$ is orbit equivalent to $\mathcal{X}$. Consequently the equivalence relation R_{punc} on Ω_{punc} is orbit equivalent to R_{AF}. □

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e-mail: matui@math.s.chiba-u.ac.jp
Graduate School of Science and Technology,
Chiba University,
1-33 Yayoi-cho, Inage-ku,
Chiba 263-8522,
Japan.