An introduction to topological full groups

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1 Introduction

In this section, we review several results for topological full groups of minimal $\mathbb{Z}$ actions. Main reference: [7], [20], [12].

Let $\varphi : X \to X$ be a minimal homeomorphism on a Cantor set X. We define

$$\mathsf{F}(\varphi) := \left\{ \alpha \in \mathrm{Homeo}(X) \mid \exists \text{conti. } n : X \to \mathbb{Z}, \text{ s.t. } \alpha(x) = \varphi^{n(x)}(x) \right\},$$

and call it the topological full group of (X, φ).

The following theorem was obtained as a topological analogue of results by Dye in measurable dynamics.

Theorem 1.1 ([7, Corollary 4.4]). *For $i = 1, 2$, let (X_i, φ_i) be as above. The following are equivalent.*

(1) *φ_1 is conjugate to φ_2 or φ_2^{-1}.*

(2) *$\mathsf{F}(\varphi_1)$ and $\mathsf{F}(\varphi_2)$ are isomorphic as groups.*

Later, the following properties were shown.

Theorem 1.2 ([20, 12]). (1) *The commutator subgroup of $\mathsf{F}(\varphi)$ is simple.*

(2) *The abelianization of $\mathsf{F}(\varphi)$ is $\mathbb{Z} \oplus (H_0(X, \varphi) \otimes \mathbb{Z}/2)$.*

(3) *$\mathsf{F}(\varphi)$ is finitely generated if and only if φ is expansive.*

(4) *$\mathsf{F}(\varphi)$ is amenable.*

2 Ample groupoids

Main reference: [20], [21], [22], [24].

2.1 Ample groupoids and topological full groups

A groupoid $\mathcal{G}$ is a small category in which every morphism is invertible. We identify objects with identity maps, and denote it by $\mathcal{G}^{(0)}$.

The range map and the source map $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are given by $g \mapsto gg^{-1}$ and $g \mapsto g^{-1}g$.

An element $g \in \mathcal{G}$ can be thought of as an arrow from $s(g)$ to $r(g)$.

Definition 2.1. A topological groupoid $\mathcal{G}$ is always assumed to be locally compact and Hausdorff (LCH). It is said to be étale if the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

An ample groupoid is an étale groupoid whose unit space is 0-dimensional (totally disconnected).

For $x \in \mathcal{G}^{(0)}$, the set $r(s^{-1}(x))$ is called the orbit of x. When every orbit is dense in $\mathcal{G}^{(0)}$, $\mathcal{G}$ is said to be minimal.

The isotropy bundle of $\mathcal{G}$ is $\mathrm{Iso}(\mathcal{G}) = \{g \in \mathcal{G} \mid r(g) = s(g)\}$. We say that $\mathcal{G}$ is principal if $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$. When the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$, we say that $\mathcal{G}$ is essentially principal (or effective).

A subset $U \subset \mathcal{G}$ is called a bisection if $r|U, s|U$ are injective (a “fat arrow”). Any open bisection U induces the homeomorphism $\theta_U := (r|U) \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$.

A (probability) measure μ on $\mathcal{G}^{(0)}$ is said to be $\mathcal{G}$-invariant if $\mu(r(U)) = \mu(s(U))$ holds for every compact open bisection U. The set of all $\mathcal{G}$-invariant probability measures is denoted by $M(\mathcal{G})$.

Example 2.2 (Transformation groupoids)**.** Let $\varphi : \Gamma \curvearrowright X$ be an action of a countable discrete group Γ on an LCH 0-dimensional space X. The transformation groupoid $\mathcal{G} := X \rtimes_\varphi \Gamma$ is $X \times \Gamma$ equipped with the product topology. The unit space of $\mathcal{G}$ is given by $\mathcal{G}^{(0)} = X \times \{1\}$ (where 1 is the identity of Γ), and identified with X. The groupoid operations are as follows:

$$r(x, \gamma) = (x, 1), \quad s(x, \gamma) = (\varphi_\gamma^{-1}(x), 1),$$

$$(x, \gamma) \cdot (x', \gamma') = (x, \gamma\gamma'), \quad (x, \gamma)^{-1} = (\varphi_\gamma^{-1}(x), \gamma^{-1}).$$

The groupoid $\mathcal{G}$ is principal if and only if the action φ is free, that is, φ_γ does not have any fixed points unless $\gamma = 1$. The groupoid $\mathcal{G}$ is essentially principal if and only if the action φ is topologically free, that is, $\{x \in X \mid \varphi_\gamma(x) = x\}$ has no interior points unless $\gamma = 1$. The groupoid $\mathcal{G}$ is minimal if and only if the action φ is minimal, that is, any orbit of φ is dense in X.

A measure μ on $\mathcal{G}^{(0)}$ is $\mathcal{G}$-invariant if and only if it is φ-invariant.

Hereafter, we always assume that $\mathcal{G}$ is essentially principal.

Definition 2.3 (Topological full groups)**.** Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. For a compact open bisection $U \subset \mathcal{G}$ such that $r(U) = \mathcal{G}^{(0)} = s(U)$, $\theta_U = (r|U) \circ (s|U)^{-1}$ is a homeomorphism on $\mathcal{G}^{(0)}$. We let $\mathsf{F}(\mathcal{G}) \subset \mathrm{Homeo}(\mathcal{G}^{(0)})$ be the set of those homeomorphisms, and call it the topological full group (TFG) of $\mathcal{G}$.

For $\mathcal{G} = X \rtimes_\varphi \Gamma$ with X compact,

$$\mathsf{F}(\mathcal{G}) = \left\{\alpha \in \mathrm{Homeo}(X) \mid \exists \text{conti. } n : X \to \Gamma, \text{ s.t. } \alpha(x) = \varphi_{n(x)}(x)\right\}.$$

Example 2.4 (AF groupoids)**.** cf. [28, Definition III.1.1], [8, Definition 3.7], [21, Definition 2.2] $\mathcal{K}$ is said to be elementary if $\mathcal{K}$ is principal and compact. When $\mathcal{K}$ is elementary:

- the topology on $\mathcal{K}$ agrees with the relative topology from $\mathcal{K}^{(0)} \times \mathcal{K}^{(0)}$,
- the equivalence relation $\mathcal{K}$ is uniformly finite, i.e. $\sup_x \# r^{-1}(x) < \infty$.

We say that $\mathcal{G}$ is an AF (approximately finite) groupoid if it can be written as an increasing union of open elementary subgroupoids.

It is known that any AF groupoids are represented by Bratteli diagrams (see [8, Theorem 3.9]). We provide a brief explanation of it. A directed graph $B = (V, E)$ is called a Bratteli diagram when $V = \bigsqcup_{n=0}^\infty V_n$ and $E = \bigsqcup_{n=1}^\infty E_n$ are disjoint unions of finite sets of vertices and edges with maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$ both of which are surjective. Let

$$X_B := \left\{ e = (e_n)_n \in \prod_n E_n \mid e_n \in E_n,\ t(e_n) = i(e_{n+1}) \quad \forall n \in \mathbb{N} \right\}.$$

The set X_B endowed with the relative topology is called the infinite path space of B. Define an equivalence relation (i.e. principal groupoid) $\mathcal{K}_m$ by

$$\mathcal{K}_m = \{(e, f) \in X_B \times X_B \mid e_n = f_n \quad \forall n \geq m\}.$$

Then, $\mathcal{K}_m$ equipped with the relative topology from $X_B \times X_B$ is a compact principal ample groupoid. Clearly one has $\mathcal{K}_m \subset \mathcal{K}_{m+1}$. Set $\mathcal{G} := \bigcup_m \mathcal{K}_m$. Endowed with the inductive limit topology, $\mathcal{G}$ becomes an AF groupoid. Conversely, Theorem 3.9 of [8] states that any AF groupoid arises in such a way.

The AF groupoid $\mathcal{G}$ is minimal if and only if for any $n \in \mathbb{N}$ there exists $m > n$ such that for any $v \in V_n$ and $w \in V_m$ there exists a path from v to w.

2.2 TFG of AF groupoids

Let $B = (V, E)$ be such as Example 2.4. For paths p, q from V_0 to V_m with $t(p) = t(q)$, we can define $\tau_{p,q} \in \mathrm{Homeo}(X_B)$ as follows: for $(e_n)_n \in X_B$, if its initial segment is either p or q, then exchange it; otherwise, do nothing. Then, $\tau_{p,q}$ is in $\mathsf{F}(\mathcal{G})$. For a fixed $m \in \mathbb{N}$, we let G_m be the subgroup generated by these $\tau_{p,q}$'s. Clearly,

$$G_m \cong \bigoplus_{v \in V_m} \mathfrak{S}_{h(v)},$$

where $h(v)$ denotes the number of paths from V_0 to v, and

$$G_m \subset G_{m+1}, \quad \mathsf{F}(\mathcal{G}) = \bigcup_m G_m.$$

Theorem 2.5 ([20, Proposition 3.2])**.** *Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. The following are equivalent.*

(1) *$\mathcal{G}$ is an AF groupoid.*

(2) *$\mathsf{F}(\mathcal{G})$ is locally finite.*

Example 2.6. Let $\varphi : \mathbb{Z} \curvearrowright X$ be minimal and let $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$. Pick $y \in X$. Define

$$\mathcal{H} := \mathcal{G} \setminus \{(\varphi^m(y), n) \in \mathcal{G} \mid m \leq 0 < m-n \text{ or } m-n \leq 0 < m\}.$$

Then $\mathcal{H}$ is known to be a minimal AF groupoid.

2.3 Homology groups and index map

For an LCH 0-dimensional space X, we let $\mathbb{Z}[X]$ denote the compactly supported $\mathbb{Z}$-valued continuous functions.

When $\pi : X \to Y$ is a local homeomorphism, we can define a homomorphism $\pi_* : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ by

$$\pi_*(f)(y) := \sum_{\pi(x)=y} f(x).$$

Let $\mathcal{G}$ be an ample groupoid. For $n \in \mathbb{N}$, we write $\mathcal{G}^{(n)}$ for the space of composable strings of n elements in $\mathcal{G}$, that is,

$$\mathcal{G}^{(n)} := \{(g_1, g_2, \ldots, g_n) \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for all } i = 1, 2, \ldots, n-1\}.$$

For $n \geq 2$ and $i = 0, 1, \ldots, n$, we let $d_i^{(n)} : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ be a map defined by

$$d_i^{(n)}(g_1, g_2, \ldots, g_n) := \begin{cases} (g_2, g_3, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, g_2, \ldots, g_{n-1}) & i = n. \end{cases}$$

When $n = 1$, we let $d_0^{(1)}, d_1^{(1)} : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)}$ be the source map and the range map, respectively. Clearly the maps $d_i^{(n)}$ are local homeomorphisms. Define the homomorphisms $\delta_n : \mathbb{Z}[\mathcal{G}^{(n)}] \to \mathbb{Z}[\mathcal{G}^{(n-1)}]$ by

$$\delta_n := \sum_{i=0}^{n} (-1)^i d_{i*}^{(n)}.$$

The abelian groups $\mathbb{Z}[\mathcal{G}^{(n)}]$ together with the boundary operators δ_n form a chain complex.

Definition 2.7 ([5, Section 3.1], [21, Definition 3.1])**.** For $n \geq 0$, we let $H_n(\mathcal{G})$ be the homology groups of the chain complex above, i.e. $H_n(\mathcal{G}) := \operatorname{Ker} \delta_n / \operatorname{Im} \delta_{n+1}$, and call them the homology groups of $\mathcal{G}$.

Remark 2.8 ([1, 18])**.** (1) $\mathcal{G}^{(n)}$ and $d_i^{(n)}$ form a semi-simplicial space, which gives the classifying space $B\mathcal{G}$. We have $H_*(\mathcal{G}) \cong H_*(B\mathcal{G})$.

(2) $\mathbb{Z}[\mathcal{G}]$ is a ring with the convolution product

$$(f_1 * f_2)(g) := \sum_{hk=g} f_1(h) f_2(k),$$

and $\mathbb{Z}[\mathcal{G}^{(0)}]$ is a left and right $\mathbb{Z}[\mathcal{G}]$-module via

$$(fm)(x) := \sum_{s(g)=x} f(g^{-1}) m(r(g)), \ (mf)(x) := \sum_{r(g)=x} m(s(g)) f(g^{-1}).$$

It is known that $H_*(\mathcal{G})$ is isomorphic to $\operatorname{Tor}_*^{\mathbb{Z}[\mathcal{G}]}(\mathbb{Z}[\mathcal{G}^{(0)}], \mathbb{Z}[\mathcal{G}^{(0)}])$.

The map $\delta_1 : \mathbb{Z}[\mathcal{G}] \to \mathbb{Z}[\mathcal{G}^{(0)}]$ is

$$\delta_1(f)(x) = \sum_{s(g)=x} f(g) - \sum_{r(g)=x} f(g),$$

and

$$H_0(\mathcal{G}) = \mathbb{Z}[\mathcal{G}^{(0)}] / \operatorname{Im} \delta_1.$$

The map $\delta_2 : \mathbb{Z}[\mathcal{G}^{(2)}] \to \mathbb{Z}[\mathcal{G}]$ is

$$\delta_2(f)(g) = \sum_{k=g} f(h,k) - \sum_{hk=g} f(h,k) + \sum_{h=g} f(h,k),$$

and

$$H_1(\mathcal{G}) = \operatorname{Ker} \delta_1 / \operatorname{Im} \delta_2.$$

For later use, we observe the following.

Lemma 2.9. (1) *For compact open bisections $U, V \subset \mathcal{G}$ such that $s(U) = r(V)$, let $W := (U \times V) \cap \mathcal{G}^{(2)}$. Then $\delta_2(1_W) = 1_U - 1_{UV} + 1_V$.*

(2) *For any compact open $A \subset \mathcal{G}^{(0)}$, $[1_A] = 0$ in $H_1(\mathcal{G})$.*

(3) *For any compact open bisection $U \subset \mathcal{G}^{(0)}$, $[1_U + 1_{U^{-1}}] = 0$ in $H_1(\mathcal{G})$.*

Example 2.10. When $\mathcal{G}$ is the transformation groupoid of $\varphi : \Gamma \curvearrowright X$, $H_n(\mathcal{G})$ is canonically isomorphic to $H_n(\Gamma, \mathbb{Z}[X])$.

In particular, when $\Gamma = \mathbb{Z}$,

$$H_n(\mathcal{G}) = \begin{cases} \mathbb{Z}[X]/\{f - f \circ \varphi \mid f \in \mathbb{Z}[X]\} & n = 0 \\ \{f \in \mathbb{Z}[X] \mid f = f \circ \varphi\} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

If φ is minimal, then $H_1(\mathcal{G}) = \mathbb{Z}$.

Example 2.11. When $\mathcal{G}$ is an AF groupoid, $H_0(\mathcal{G})$ is the dimension group of the Bratteli diagram:

$$\lim_{m \to \infty} \left(\mathbb{Z}^{V_m} \to \mathbb{Z}^{V_{m+1}} \right),$$

and $H_n(\mathcal{G}) = 0$ for $n \geq 1$.

Definition 2.12 (Index map,[21, Definition 7.1])**.** For $\alpha \in \mathsf{F}(\mathcal{G})$, a compact open bisection $U \subset \mathcal{G}$ satisfying $\alpha = \theta_U$ uniquely exists. It is easy to see that 1_U is in $\operatorname{Ker} \delta_1$. We define a map $I : \mathsf{F}(\mathcal{G}) \to H_1(\mathcal{G})$ by $I(\alpha) := [1_U]$ and call it the index map.

By Lemma 2.9 (1), I is a homomorphism. We put $\mathsf{K}(\mathcal{G}) := \operatorname{Ker} I$. Also, we denote by $\mathsf{D}(\mathcal{G})$ the commutator subgroup of $\mathsf{F}(\mathcal{G})$. Thus, we have

$$\mathsf{D}(\mathcal{G}) \triangleleft \mathsf{K}(\mathcal{G}) \triangleleft \mathsf{F}(\mathcal{G}).$$

Example 2.13. When $\mathcal{G}$ arises from a minimal homeomorphism on a Cantor set X,

$$\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = \mathbb{Z}, \quad \mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) = H_0(\mathcal{G}) \otimes \mathbb{Z}/2.$$

For $\alpha(x) = \varphi^{n(x)}(x)$ in $\mathsf{F}(\mathcal{G})$, it is known that $I(\alpha) \in H_1(\mathcal{G}) \cong \mathbb{Z}$ is computed as

$$I(\alpha) = \int n(x) \, d\mu(x),$$

where μ is in $M(\mathcal{G})$.

Example 2.14. Suppose that $\mathcal{G}$ is an AF groupoid. One has $\mathsf{K}(\mathcal{G}) = \mathsf{F}(\mathcal{G})$ because $H_1(\mathcal{G}) = 0$. Recall

$$\mathsf{F}(\mathcal{G}) = \bigcup_m G_m, \quad G_m \cong \bigoplus_{v \in V_m} \mathfrak{S}_{h(v)}$$

(see Section 2.2). It follows that

$$\mathsf{D}(\mathcal{G}) \cong \bigcup_m \bigoplus_{v \in V_m} \mathfrak{A}_{h(v)}$$

and

$$\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong \lim_m \left((\mathbb{Z}/2)^{V_m} \to (\mathbb{Z}/2)^{V_{m+1}} \right) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2.$$

Moreover, if $\mathcal{G}$ is minimal, then $\mathsf{D}(\mathcal{G})$ is simple ($\mathfrak{A}_k$ is simple when $k \geq 5$).

Remark 2.15 (comparison maps)**.** Let $\mathcal{G}$ be an ample groupoid.

(1) It is easy to see that there exists a homomorphism $\mu_0 : H_0(\mathcal{G}) \to K_0(C_r^*(\mathcal{G}))$ such that $\mu_0([1_A]) = [1_A]$ for every compact open set $A \subset \mathcal{G}^{(0)}$.

(2) It is known that there exists a homomorphism $\mu_1 : H_1(\mathcal{G}) \to K_1(C_r^*(\mathcal{G}))$ such that $\mu_1([1_U]) = [1_U]$ for every compact open bisection $U \subset \mathcal{G}$ satisfying $r(U) = \mathcal{G}^{(0)} = s(U)$. See [1].

Problem 2.16. Does there exist $\mu_* : H_*(\mathcal{G}) \to K_*(C_r^*(\mathcal{G}))$ for $* \geq 2$?

2.4 Reconstruction

Theorem 2.17 ([30],[22, Theorem 3.10])**.** *For $i = 1, 2$, let $\mathcal{G}_i$ be a minimal ample groupoid with compact unit space. The following are equivalent.*

(1) $\mathcal{G}_1 \cong \mathcal{G}_2$.

(2) $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$.

(3) $\mathsf{K}(\mathcal{G}_1) \cong \mathsf{K}(\mathcal{G}_2)$.

(4) $\mathsf{D}(\mathcal{G}_1) \cong \mathsf{D}(\mathcal{G}_2)$.

Remark 2.18. The assumption of minimality can be relaxed (see [30]).

The theorem above ensures TFG's are a rich source of interesting infinite groups.

3 Simplicity

Main reference: [21], [22], [24].

3.1 Two classes

Definition 3.1 ([21, Definition 6.2]). Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. We say that $\mathcal{G}$ is almost finite (abbreviated as a.f.) if for any compact subset $C \subset \mathcal{G}$ and $\varepsilon > 0$ there exists an elementary subgroupoid $\mathcal{K} \subset \mathcal{G}$ such that

$$\frac{\#(C\mathcal{K}x \setminus \mathcal{K}x)}{\#(\mathcal{K}x)} < \varepsilon$$

for all $x \in \mathcal{G}^{(0)}$.

This definition says that any compact subset C is almost 'covered' by an elementary subgroupoid $\mathcal{K}$.

This also reminds us of the notion of tracially AF C^*-algebras.

By definition, AF groupoids are almost finite.

Theorem 3.2 ([15, 26]). *When $\Gamma \curvearrowright X$ is a free action of an elementary amenable group on a Cantor set, its transformation groupoid $\mathcal{G} = X \rtimes \Gamma$ is almost finite.*

We remark that for the $\mathcal{G}$ above, $C_r^*(\mathcal{G})$ is $\mathcal{Z}$-stable ([14, Theorem 12.4]).

Definition 3.3 ([22, Definition 4.9]). Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. We say that $\mathcal{G}$ is purely infinite (p.i.) if for every clopen set $A \subset \mathcal{G}^{(0)}$, there exist compact open bisections $U, V \subset \mathcal{G}$ such that $s(U) = s(V) = A$, $r(U) \sqcup r(V) \subset A$.

When $\mathcal{G}$ is purely infinite, it is easy to see that $\mathsf{F}(\mathcal{G})$ contains the free group $\mathbb{Z} * \mathbb{Z}$ ([22, Proposition 4.10]). Also, $C_r^*(\mathcal{G})$ is purely infinite ([29, Theorem 4.1]).

Example 3.4 (SFT groupoids). Let (V, E) be a finite directed graph and let A be its adjacency matrix of (V, E). We assume that A is irreducible and is not a permutation matrix. Define

$$X := \{(x_k)_{k\in\mathbb{N}} \in E^{\mathbb{N}} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\}.$$

With the product topology, X is a Cantor set. The shift σ on X is called the one-sided irreducible shift of finite type (SFT) associated with the graph (V, E) (or the matrix A).

The SFT groupoid $\mathcal{G}_{(V,E)}$ (or $\mathcal{G}_A$) is the graph groupoid:

$$\mathcal{G}_{(V,E)} := \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ n = k - l,\ \sigma^k(x) = \sigma^l(y)\}$$

with the topology generated by the sets

$$\{(x, k-l, y) \in \mathcal{G}_{(V,E)} \mid x \in P,\ y \in Q,\ \sigma^k(x) = \sigma^l(y)\},$$

where $P, Q \subset X$ are open and $k, l \in \mathbb{N}$. The groupoid structure is given by

$$(x, n, y) \cdot (y, n', y') = (x, n+n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

We identify X with the unit space $\mathcal{G}^{(0)}_{(V,E)}$ via $x \mapsto (x, 0, x)$.

It is easy to see that $\mathcal{G}_{(V,E)}$ is minimal and purely infinite.

When V is a singleton, σ is the full shift. The TFG of $\mathcal{G}_{(V,E)}$ is isomorphic to the Higman-Thompson group.

The homology groups of the SFT groupoid $\mathcal{G}_{(V,E)}$ was computed in [21].

Theorem 3.5 ([21, Theorem 4.14])**.** *One has*

$$H_n(\mathcal{G}_{(V,E)}) \cong \begin{cases} \mathrm{Coker}(I - A^t) & n = 0 \\ \mathrm{Ker}(I - A^t) & n = 1 \\ 0 & n \geq 2, \end{cases}$$

where the matrix A acts on the abelian group $\mathbb{Z}^V$ by multiplication.

The classification of SFT groupoids was given in [19].

Theorem 3.6 ([19, Theorem 3.6])**.** *Two SFT groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are isomorphic if and only if there exists an isomorphism $\Phi : H_0(\mathcal{G}_A) \to H_0(\mathcal{G}_B)$ such that $\Phi([1_{X_A}]) = [1_{X_B}]$ and $\det(I - A) = \det(I - B)$.*

Example 3.7. Let $\Gamma := \langle a, b \rangle$ be the free group and consider the boundary action $\Gamma \curvearrowright \partial\Gamma$. We may identify $\partial\Gamma$ with

$$\{(x_k)_k \in E^{\mathbb{N}} \mid aa^{-1},\ a^{-1}a,\ bb^{-1},\ b^{-1}b \text{ do not apper}\},$$

where $E := \{a, a^{-1}, b, b^{-1}\}$. Then the transformation groupoid $\mathcal{G}$ of $\Gamma \curvearrowright \partial\Gamma$ is canonicaly isomorphic to the SFT groupoid $\mathcal{G}_A$ of the martix

$$A := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Hence we have

$$H_n(\mathcal{G}) \cong \begin{cases} \mathbb{Z}^2 & n = 0, 1 \\ 0 & n \geq 2. \end{cases}$$

Problem 3.8. Does there exist a minimal ample amenable groupoid which is neither almost finite nor purely infinite?

3.2 Commutator subgroups

Theorem 3.9 ([22, Theorem 4.7, Theorem 4.16])**.** *Let $\mathcal{G}$ be a minimal ample groupoid. If $\mathcal{G}$ is either almost finite or purely infinite, then $\mathsf{D}(\mathcal{G})$ is simple.*

We give a sketchy proof for the purely infinite case. Assume that $\mathcal{G}$ is minimal and purely infinite. Let $N \lhd \mathsf{D}(\mathcal{G})$ be a non-trivial normal subgroup.

Lemma 3.10. *For any $\tau \in N$ and $\alpha \in \mathsf{F}(\mathcal{G})$, we have $\alpha\tau\alpha^{-1} \in N$.*

Proof of a part of Theorem 3.9. Let $\tau \in N \setminus \{1\}$. There exists a non-empty clopen set $A \subset \mathcal{G}^{(0)}$ such that $A \cap \tau(A) = \emptyset$.

It suffices to show $[\alpha, \beta] \in N$ for any $\alpha, \beta \in \mathsf{F}(\mathcal{G})$. We can find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathsf{F}(G)$ such that

$$\alpha = \alpha_1\alpha_2, \quad \beta = \beta_1\beta_2$$
$$\mathrm{supp}(\alpha_i) \neq \mathcal{G}^{(0)}, \quad \mathrm{supp}(\beta_i) \neq \mathcal{G}^{(0)}.$$

Thanks to the lemma above, we may assume $\mathrm{supp}(\alpha) \neq \mathcal{G}^{(0)}$ and $\mathrm{supp}(\beta) \neq \mathcal{G}^{(0)}$ from the start.

Since $\mathcal{G}$ is purely infinite and minimal, there exists $\gamma \in \mathsf{F}(\mathcal{G})$ such that

$$\gamma(\mathrm{supp}(\alpha)) \cap \mathrm{supp}(\beta) = \emptyset \quad \text{and} \quad \mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma) \neq \mathcal{G}^{(0)}.$$

Also, there exists $\sigma \in \mathsf{F}(\mathcal{G})$ such that

$$\sigma(\mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma)) \subset A.$$

By the lemma above, $\tilde{\tau} := \sigma^{-1}\tau\sigma$ is in N. It follows from $A \cap \tau(A) = \emptyset$ that

$$\mathrm{supp}(\alpha) \cap \tilde{\tau}(\mathrm{supp}(\gamma)) = \emptyset.$$

Hence $\tilde{\gamma} := [\gamma, \tilde{\tau}]$ satisfies

$$\tilde{\gamma}(\mathrm{supp}(\alpha)) \cap \mathrm{supp}(\beta) = \emptyset,$$

that is, $\tilde{\gamma}\alpha\tilde{\gamma}^{-1}$ commutes with β. Again, by the lemma above, $\tilde{\gamma}$ is in N. Therefore

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} = \alpha(\tilde{\gamma}\alpha^{-1}\tilde{\gamma}^{-1})\beta(\tilde{\gamma}\alpha\tilde{\gamma}^{-1})\alpha^{-1}\beta^{-1} = [[\alpha, \tilde{\gamma}], \beta]$$

is in N. □

3.3 Abelianization

Theorem 3.11 ([21, Theorem 7.5], [22, Theorem 5.2])**.** *If $\mathcal{G}$ is either almost finite or purely infinite, then the index map $I : \mathsf{F}(\mathcal{G}) \to H_1(\mathcal{G})$ is surjective.*

Proof. We give a sketchy proof for the purely infinite case. Suppose that $\mathcal{G}$ is purely infinite.

Let $f \in \mathrm{Ker}\,\delta_1$. By Lemma 2.9 (3), we may assume that there exist compact open bisections $C_1, C_2, \ldots, C_n$ such that

$$f = 1_{C_1} + 1_{C_2} + \cdots + 1_{C_n}.$$

By $\delta_1(f) = 0$, we have

$$\sum_{i=1}^{n} 1_{r(C_i)} = \sum_{j=1}^{n} 1_{s(C_j)}.$$

Hence we can find clopen subsets $A_{ij} \subset \mathcal{G}^{(0)}$ for $i, j = 1, 2, \ldots, n$ satisfying

$$\bigsqcup_{i=1}^{n} A_{ij} = r(C_j) \quad \text{and} \quad \bigsqcup_{j=1}^{n} A_{ij} = s(C_i).$$

(How to find A_{ij}: Take a clopen partition $(D_l)_l$ generated by $r(C_i)$'s and $s(C_j)$'s. For each l,

$$\#\{i \mid D_l \subset r(C_i)\} = \#\{j \mid D_l \subset s(C_j)\},$$

and so there exists a bijective correspondence between them. Let us denote it by $i \sim_l j$. Set

$$A_{ij} := \bigsqcup_{i \sim_l j} D_l,$$

which is a desired one.)

Since $\mathcal{G}$ is purely infinite, there exist compact open bisections $U_1, U_2, \ldots, U_n$ such that $s(U_i) = r(C_i)$ and the sets $r(U_i)$ are mutually disjoint. Put

$$V_{ij} := U_i C_i A_{ij} U_j^{-1}.$$

We can check that V_{ij} are compact open bisections such that

$$r(V_{ij}) = r(U_i C_i A_{ij}) \quad \text{and} \quad s(V_{ij}) = s(A_{ij} U_j^{-1}).$$

Therefore, $V := \bigcup_{i,j} V_{ij}$ is also a compact open bisection and

$$r(V) = \bigsqcup_{i} r(U_i C_i) = \bigsqcup_{i} r(U_i) = \bigsqcup_{j} s(U_j^{-1}) = s(V).$$

On one hand, by Lemma 2.9 (1)(3),

$$\begin{aligned}
[1_V] &= \left[\sum_{i,j} 1_{V_{ij}}\right] = \left[\sum_{i,j} 1_{U_i C_i A_{ij} U_j^{-1}}\right] = \left[\sum_{i,j} \left(1_{U_i C_i A_{ij}} + 1_{A_{ij} U_j^{-1}}\right)\right] \\
&= \left[\sum_{i} 1_{U_i C_i} + \sum_{j} 1_{U_j^{-1}}\right] = \left[\sum_{i} (1_{U_i} + 1_{C_i}) + \sum_{j} 1_{U_j^{-1}}\right] = [f].
\end{aligned}$$

Set $W := V \sqcup (\mathcal{G}^{(0)} \setminus s(V))$. Then one has $I(\theta_W) = [f]$ as desired. □

Let us recall

$$\mathsf{D}(\mathcal{G}) \triangleleft \mathsf{K}(\mathcal{G}) \triangleleft \mathsf{F}(\mathcal{G}).$$

Example 3.12. (1) When $\mathcal{G}$ is an AF groupoid, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = 0$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ (see Example 2.14).

(2) When $\mathcal{G}$ arises from a minimal action $\mathbb{Z} \curvearrowright X$, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = \mathbb{Z}$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ ([20, Theorem 4.8]).

(3) When $\mathcal{G}$ is an SFT groupoid, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = H_1(\mathcal{G})$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ ([22, Corollary 6.24]).

When $\mathcal{G}$ is minimal, a homomorphism

$$\zeta : H_0(\mathcal{G}) \otimes \mathbb{Z}/2 \to \mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G})$$

was constructed by V. Nekrashevych [27, Section 7]. The map ζ can be described as follows. When $U \subset \mathcal{G}$ is a compact open bisection such that $r(U) \cap s(U) = \emptyset$,

$$\tau(x) := \begin{cases} \theta_U(x) & x \in s(U) \\ \theta_U^{-1}(x) & x \in r(U) \\ x & \text{else} \end{cases}$$

is a transposition. Then $\zeta([1_{s(U)}])$ equals the equivalence class of τ.

Theorem 3.13 ([18, Corollary E])**.** *Let $\mathcal{G}$ be a minimal ample groupoid. If $\mathcal{G}$ is either almost finite or purely infinite, then there exists an exact sequence:*

$$H_2(\mathsf{D}(\mathcal{G})) \longrightarrow H_2(\mathcal{G}) \longrightarrow H_0(\mathcal{G}) \otimes \mathbb{Z}/2 \xrightarrow{\zeta} H_1(\mathsf{F}(\mathcal{G})) \xrightarrow{I} H_1(\mathcal{G}) \longrightarrow 0$$

Notice that $H_1(\mathsf{F}(\mathcal{G}))$ is isomorphic to the abelianization $\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$.

Li [18] discovered a close connection between homology of TFG and groupoid homology. The theorem above is one consequence from his deep analysis.

Example 3.14 ([23])**.** Fix $m \in \mathbb{N}$. Let $\mathcal{G}$ be the SFT groupoid of the full shift over $m+1$ symbols. By Theorem 3.5,

$$H_k(\mathcal{G}) = \begin{cases} \mathbb{Z}/m & k = 0 \\ 0 & k \geq 1. \end{cases}$$

Hence, the Künneth theorem implies

$$H_k(\mathcal{G} \times \mathcal{G}) = \begin{cases} \mathbb{Z}/m & k = 0, 1 \\ 0 & k \geq 2 \end{cases}$$

and

$$H_k(\mathcal{G} \times \mathcal{G} \times \mathcal{G}) = \begin{cases} \mathbb{Z}/m & k = 0 \\ (\mathbb{Z}/m)^2 & k = 1 \\ \mathbb{Z}/m & k = 2 \\ 0 & k \geq 3. \end{cases}$$

Let us describe the generator of $H_1(\mathcal{G} \times \mathcal{G}) \cong \mathbb{Z}/m$. Define $\beta \in \mathrm{Homeo}(\mathcal{G}^{(0)} \times \mathcal{G}^{(0)})$ (the baker's map) as follows:

$$\beta\left((x_n)_n, (y_n)_n\right) := \left((x_2 x_3 \dots), (x_1 y_1 y_2 \dots)\right).$$

Then β is in $\mathsf{F}(\mathcal{G}\times\mathcal{G})$ and $I(\beta)$ generates $H_1(\mathcal{G}\times\mathcal{G})\cong\mathbb{Z}/m$.

Now we assume m is even, and let $t\in H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}))$ be the image of the generator of $H_0\otimes\mathbb{Z}/2\cong\mathbb{Z}/2$ by the map ζ.

Define $\tau\in\mathsf{F}(\mathcal{G}\times\mathcal{G})$ by

$$\tau\left((x_n)_n,(y_n)_n\right):=\left((y_1x_2x_3\dots),(x_1y_2y_3\dots)\right).$$

Thus, τ is a transposition whose support is $\{(x,y)\mid x_1\neq y_1\}$. It follows that the equivalence class $[\tau]$ of τ in $H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}))$ equals

$$\frac{m(m+1)}{2}t=\begin{cases}0 & m\in 4\mathbb{Z}\\ t & m\in 2\mathbb{Z}\setminus 4\mathbb{Z}.\end{cases}$$

Let us consider $\tau\beta$. This sends (x,y) to

$$\left((x_1x_3x_4\dots),(x_2y_1y_2\dots)\right).$$

Therefore $\tau\beta$ is a product of $m+1$ elements with mutually disjoint support and each of them is conjugate to β. Hence $[\tau\beta]=[\beta^{m+1}]$ (i.e. $\tau\beta^{-m}$ belongs to $\mathsf{D}(\mathcal{G}\times\mathcal{G})$), and so we have

$$m[\beta]=\frac{m(m+1)}{2}t\in H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G})).$$

Now we consider $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$. We let

$$\beta_{12}:=\beta\times\mathrm{id}\in\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G})$$

and

$$\beta_{23}:=\mathrm{id}\times\beta\in\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}).$$

Similarly, β_{13} is defined. Thinking of t as an element in $H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}))$, we have

$$m[\beta_{12}]=m[\beta_{23}]=m[\beta_{31}]=\frac{m(m+1)}{2}t.$$

When $m\in 2\mathbb{Z}\setminus 4\mathbb{Z}$, this is equal to t. On the other hand, one has $\beta_{23}\beta_{12}=\beta_{13}$. Consequently we obtain $t=0$. Thus, the map

$$\zeta: H_0(\mathcal{G}\times\mathcal{G}\times\mathcal{G})\otimes\mathbb{Z}/2\to H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}))$$

is zero.

4 Amenability

Main reference: [12], [11].

K. Juschenko and N. Monod obtained the following remarkable result.

Theorem 4.1 ([12, Theorem A]). *When $\varphi : \mathbb{Z} \curvearrowright X$ is minimal, the TFG of $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$ is amenable.*

In the proof of this theorem, the notion of extensive amenability plays the central role. This property was first introduced (without a name) in [12], and studied further in [13, 11].

We recall the definition of extensive amenability from [11, Definition 1.1]. Let $\alpha : G \curvearrowright Z$ be an action of a discrete group G on a set Z. Set

$$P(Z) := \bigoplus_Z \mathbb{Z}/2 = \{f : Z \to \mathbb{Z}_2 \mid \mathrm{supp}(f) \text{ is finite}\}.$$

The action α naturally extends to $\alpha : G \curvearrowright P(Z)$. We say that $\alpha : G \curvearrowright Z$ is extensively amenable if there exists a G-invariant mean (i.e. finitely additive probability measure) m on $P(Z)$ such that $m(\{1_F \in P(Z) \mid E \subset F\}) = 1$ for any finite subset $E \subset Z$. In [12, Lemma 3.1], it was shown that $\alpha : G \curvearrowright Z$ is extensively amenable if and only if the action of $P(Z) \rtimes G$ on $P(Z)$ admits an invariant mean.

We denote by $W(\mathbb{Z})$ the group of all permutations g of $\mathbb{Z}$ for which the quantity $\sup\{|g(j) - j| \mid j \in \mathbb{Z}\}$ is finite. In [12, Theorem C], it was shown that the natural action $W(\mathbb{Z}) \curvearrowright \mathbb{Z}$ is extensively amenable. (This part is technically quite hard.) It follows that the action of $P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $P(\mathbb{Z})$ admits an invariant mean.

Let $\varphi : \mathbb{Z} \curvearrowright X$ be minimal and let $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$. We would like to show that $\mathsf{F}(\mathcal{G})$ is amenable. Fix a point $y \in X$. For $\alpha \in \mathsf{F}(\mathcal{G})$, we can define $\tilde{\alpha} \in W(\mathbb{Z})$ so that $\alpha(\varphi^j(y)) = \varphi^{\tilde{\alpha}(j)}(y)$. Define a map $\pi : \mathsf{F}(\mathcal{G}) \to P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ by $\pi(\alpha) = (1_\mathbb{N} + 1_{\tilde{\alpha}(\mathbb{N})}, \tilde{\alpha})$ for $\alpha \in \mathsf{F}(\mathcal{G})$. It is routine to check that π is an injective homomorphism. Since the action of $P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $P(\mathbb{Z})$ admits an invariant mean, in order to show the amenability of $\mathsf{F}(\mathcal{G})$, it suffices to prove that the stabiliser in $\pi(\mathsf{F}(\mathcal{G}))$ of 1_E is amenable for any finite subset $E \subset \mathbb{Z}$.

Lemma 4.2 ([12, Lemma 4.1]). *In the setting above, for any finite subset $E \subset \mathbb{Z}$, the stabiliser*

$$S := \{\alpha \in \mathsf{F}(\mathcal{G}) \mid \pi(\alpha)(1_E) = 1_E\}$$

is locally finite, and hence amenable.

Proof. By definition, $\pi(\alpha)(1_E) = 1_\mathbb{N} + 1_{\tilde{\alpha}(\mathbb{N})} + 1_{\tilde{\alpha}(E)}$, which implies

$$\alpha \in S \iff \tilde{\alpha}(\mathbb{N}\Delta E) = \mathbb{N}\Delta E,$$

where Δ means the symmetric difference. Put

$$k := \#(E \cap \mathbb{N}) - \#(E \setminus \mathbb{N}) \in \mathbb{Z}.$$

We can find a transposition $\tau \in \mathsf{F}(\mathcal{G})$ satisfying

$$\{\tau(\varphi^j(y)) \mid j > k\} = \{\varphi^j(y) \mid j \in \mathbb{N}\Delta E\}.$$

Then, one has

$$(\tau\varphi^k)(\mathrm{Orb}^+_\varphi(y)) = \{\varphi^j(y) \mid j \in \mathbb{N}\Delta E\}.$$

Hence

$$\alpha \in S \iff ((\tau\varphi^k)^{-1}\alpha(\tau\varphi^k))(\mathrm{Orb}^+_\varphi(y)),$$

which means that S is conjugate to the TFG of $\mathcal{H}$ mentioned in Example 2.6. Therefore, S is locally finite, and hence amenable. □

In such a way, Theorem 4.1 is proved.

In [13, 11], the notion of extensive amenability is used to prove amenability of various kinds of groups. Among others, it was shown that all subgroups of the group of interval exchange transformations that have angular components of rational rank ≤ 2 are amenable ([11, Theorem 5.1]). In particular, when $\varphi : \mathbb{Z}^2 \curvearrowright X$ is a free minimal action arising from two irrational rotations on the circle (see [9, Example 30]), the TFG of $X \rtimes_\varphi \mathbb{Z}^2$ is amenable. On the other hand, it is known that there exists a free minimal action $\varphi : \mathbb{Z}^2 \curvearrowright X$ on a Cantor set such that its TFG contains the non-abelian free group ([6]). It may be a rather complicated problem to determine when the TFG is amenable for $\varphi : \mathbb{Z}^2 \curvearrowright X$.

As a generalization of Theorem 4.1, Szőke obtained the following result.

Theorem 4.3 ([32])**.** *Let Γ be a finitely generated group and let X be the Cantor set.*

(1) *If Γ is virtually cyclic, then for any minimal action $\varphi : \Gamma \curvearrowright X$, its TFG is amenable.*

(2) *If Γ is not virtually cyclic, then there exists a free minimal action $\varphi : \Gamma \curvearrowright X$ whose TFG contains the free group.*

Problem 4.4. When is $\mathsf{F}(\mathcal{G})$ amenable?

As mentioned before, when $\mathcal{G}$ is purely infinite, $\mathsf{F}(\mathcal{G})$ contains $\mathbb{Z} * \mathbb{Z}$, and hence is not amenable. But, it is meaningful to question weak versions of amenability.

Theorem 4.5 ([10],[22, Theorem 6.7])**.** *When $\mathcal{G}_A$ is an SFT groupoid, $\mathsf{F}(\mathcal{G}_A)$ has the Haagerup property.*

5 Finiteness

Main reference: [27], [22], [16].

Theorem 5.1 ([27])**.** *Suppose that $\mathcal{G}$ is minimal and either almost finite or purely infinite. If $\mathcal{G}$ is expansive, then $\mathsf{D}(\mathcal{G})$ is finitely generated.*

When $\mathcal{G}$ arises from $\varphi : \Gamma \curvearrowright X$, $\mathcal{G}$ is expansive if and only if φ is expansive.

Example 5.2 ([20, 22]). (1) When $\varphi : \mathbb{Z} \curvearrowright X$ is minimal and expansive, its $\mathsf{D}(\mathcal{G})$ is simple, amenable and finitely generated.

(2) AF groupoids never be expansive.

(3) An SFT groupoid is expansive, and so its $\mathsf{D}(\mathcal{G})$ (and also $\mathsf{F}(\mathcal{G})$) is finitely generated. Moreover, $\mathsf{F}(\mathcal{G})$ is of type F_∞.

X. Li [16] proved that $\mathsf{F}(\mathcal{G})$ is of type F_∞ (in particular, finitely presented) when $\mathcal{G}$ is a product of SFT groupoids.

6 Stein groupoids

Main reference: [25].

Definition 6.1 (Stein's groups, [31]). Let $\Lambda \subset (0, \infty)$ be a countable multiplicative subgroup and let $\Gamma \subset \mathbb{R}$ be a countable $\mathbb{Z}\Lambda$-module, which is dense in $\mathbb{R}$. Let $\ell \in \Gamma \cap (0, \infty)$. Stein's group $V(\Gamma, \Lambda, \ell)$ is the group consisting of piecewise linear bijections f of $[0, \ell)$ satisfying the following:

- f is right continuous,
- f has finitely many discontinuous or nondifferential points, all in Γ,
- f has slopes only in Λ.

Set $V_\lambda := V(\langle \lambda \rangle, \mathbb{Z}[\lambda, \lambda^{-1}], 1)$.

Example 6.2. The following are known to be of type F_∞.

- For $n \in \mathbb{N} \setminus \{1\}$, V_n ([2]).
- For $n_1, \ldots, n_k \in \mathbb{N} \setminus \{1\}$, $V(\langle n_1, \ldots, n_k \rangle, \mathbb{Z}[1/(n_1 \ldots n_k)], \ell)$ ([31]).
- For $\lambda = \sqrt{2} + 1, (\sqrt{5} + 1)/2$, V_λ ([3, 4]).

Theorem 6.3 ([25, Theorem 5.10]). *For $i = 1, 2$, suppose that $\Lambda_i \subset (0, \infty)$ is finitely generated and* $\operatorname{rank} \Gamma_i \geq 2$. *The following conditions are equivalent.*

(1) *$V(\Gamma_1, \Lambda_1, \ell_1)$ is isomorphic to $V(\Gamma_2, \Lambda_2, \ell_2)$ as discrete groups.*

(2) *$\Lambda_1 = \Lambda_2$ and there exists $s > 0$ such that $\Gamma_1 = s\Gamma_2$ and $\ell_1 - s\ell_2$ is zero in $H_0(\Lambda_1, \Gamma_1)$.*

Notice that (2) $\Longrightarrow$ (1) is obvious.

Stein's groups are realized as TFG of ample groupoids (this was first observed by O. Tanner [33]).

Consider

$$\mathbb{R}_\Gamma := (\mathbb{R} \setminus \Gamma) \sqcup \{t_+, t_- \mid t \in \Gamma\}$$

endowed with the natural total order ($t_- < t_+$ for all $t \in \Gamma$). Then, $\mathbb{R}_\Gamma$ with the order topology is a totally disconnected LCH space. For any $t, s \in \Gamma$ with $t < s$, the interval

$$(t_-, s_+) = [t_+, s_-]$$

is compact and open in $\mathbb{R}_\Gamma$, and these intervals form a basis for the topology on $\mathbb{R}_\Gamma$.

The group $\Gamma \rtimes \Lambda$ naturally acts on $\mathbb{R}_\Gamma$. We let

$$\mathcal{S}(\Gamma, \Lambda) := \mathbb{R}_\Gamma \rtimes (\Gamma \rtimes \Lambda)$$

and call it the Stein groupoid.

It is easy to see that Stein's group $V(\Gamma, \Lambda, \ell)$ is isomorphic to the TFG of the ample groupoid $\mathcal{S}(\Gamma, \Lambda)|[0_+, \ell_-]$. Here, $\mathcal{G}|Y := r^{-1}(Y) \cap s^{-1}(Y)$ is the reduction of $\mathcal{G}$ to $Y \subset \mathcal{G}^{(0)}$. Thanks to the reconstruction theorem (Theorem 2.17), the proof of Theorem 6.3 is reduced to the classification of $\mathcal{S}(\Gamma, \Lambda)$ or its reduction.

Example 6.4 ([17])**.** For $\lambda > 1$, let $\mathcal{S}_\lambda := \mathcal{S}(\langle \lambda \rangle, \mathbb{Z}[\lambda, \lambda^{-1}])$. Li [17] computed the homology of $\mathcal{S}_\lambda$ for many values of λ.

(1) For $\lambda = n \in \mathbb{N}$,

$$H_*(\mathcal{S}_n) = \begin{cases} \mathbb{Z}/(n-1) & * = 0 \\ 0 & * \geq 1. \end{cases}$$

(2) For $\lambda = \sqrt{2} + 1$,

$$H_*(\mathcal{S}_\lambda) = \begin{cases} \mathbb{Z}/2 & * = 0, 1 \\ 0 & * \geq 2. \end{cases}$$

(3) For $\lambda = (\sqrt{5} + 1)/2$,

$$H_*(\mathcal{S}_\lambda) = \begin{cases} \mathbb{Z}/2 & * = 1 \\ 0 & * = 0, \geq 2. \end{cases}$$

(4) For $\lambda = \sqrt{n}$ ($n \neq$ square),

$$H_*(\mathcal{S}_n) = \begin{cases} \mathbb{Z}/(n-1) & * = 0 \\ \mathbb{Z}/(n+1) & * = 1 \\ 0 & * \geq 2. \end{cases}$$

(5) For any transcendental λ,

$$H_*(\mathcal{S}_\lambda) = \bigoplus_{i=0}^{\infty} \mathbb{Z} \quad \forall *\, .$$

How do we recover the data Γ and Λ from $\mathcal{S} = \mathcal{S}(\Gamma, \Lambda)$?
There exists a canonical homomorphism:

$$\mathcal{S} = \mathbb{R}_\Gamma \rtimes \Gamma \rtimes \Lambda \longrightarrow \Gamma \rtimes \Lambda \longrightarrow \Lambda.$$

Lemma 6.5. (1) *For any $x \in \mathbb{R}_\Gamma$, $\mathcal{S}_x \to \Lambda$ is injective, where $\mathcal{S}_x := r^{-1}(x) \cap s^{-1}(x)$.*

(2) *For any $t \in \Gamma$, $\mathcal{S}_{t_\pm} \to \Lambda$ is an isomorphism.*

Proof. (1) If (x, t, λ) is in the kernel, then $\lambda = 1$. If $(x, t, 1)$ is in $\mathcal{S}_x$, then $t = 0$.
(2) For any $\lambda \in \Lambda$, $(t_\pm, (1 - \lambda)t, \lambda)$ is in $\mathcal{S}_{t_\pm}$. □

Thus, $\mathcal{S}$ remembers (at least) the isomorphism class of Λ.

Definition 6.6. Let $G \curvearrowright X$ be an action of a group G on a totally disconnected LCH space X. We say that $X \rtimes G$ is H^1-rigid if $X \rtimes G \to G$ induces $H^1(G) \cong H^1(X \rtimes G)$.

From now on, we assume $\Lambda \cong \mathbb{Z}^N$ and $\operatorname{rank} \Gamma \geq 2$.

Proposition 6.7. *$\mathcal{S} = \mathbb{R}_\Gamma \rtimes (\Gamma \rtimes \Lambda)$ is H^1-rigid.*

Proof. First, one can prove that $\mathcal{H} := \mathbb{R}_\Gamma \rtimes \Gamma$ is H^1-rigid using $\operatorname{rank} \Gamma \geq 2$.

Any $\lambda \in \Lambda$ induces an automorphism of $\mathcal{H}$. The cohomology long exact sequence implies:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H^1(\Gamma \rtimes_\lambda \mathbb{Z}) & \longrightarrow & H^1(\Gamma) & \xrightarrow{\mathrm{id} - H^1(\lambda)} & H^1(\Gamma) \\
 & & \| & & \downarrow{\scriptstyle H^1(\pi')} & & \downarrow{\scriptstyle H^1(\pi)}\,\cong & & \downarrow{\scriptstyle H^1(\pi)}\,\cong \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H^1(\mathcal{H} \rtimes_\lambda \mathbb{Z}) & \longrightarrow & H^1(\mathcal{H}) & \xrightarrow{\mathrm{id} - H^1(\lambda)} & H^1(\mathcal{H}).
\end{array}$$

It follows that $H^1(\pi')$ is also an isomorphism. As $\Lambda \cong \mathbb{Z}^N$, we can repeat this argument, and conclude that $\mathcal{S} = \mathcal{H} \rtimes \Lambda$ is H^1-rigid. □

To proceed further, we need the concept of the ratio set. Let $\mathcal{G}$ be an ample groupoid and let μ be a measure on $\mathcal{G}^{(0)}$. For $g \in \mathcal{G}$ and $\lambda \in (0, \infty)$, we write

$$g_* \,\mathrm{d}\mu = \lambda \,\mathrm{d}\mu$$

if there exists a compact open bisection $U \subset \mathcal{G}$ such that $g \in U$ and $\mu(r(V)) = \lambda\mu(s(V))$ holds for all clopen subset $V \subset U$ (i.e. $\theta_{U*}\mu = \lambda\mu$). We set

$$R(\mathcal{G}, \mu) := \{\lambda \in (0, \infty) \mid g_* \,\mathrm{d}\mu = \lambda \,\mathrm{d}\mu \quad \text{for some } g \in \mathcal{G}\}.$$

and call it the ratio set for $(\mathcal{G}, \mu)$.

The following proposition says that we can recover Λ from $\mathcal{S}$.

Proposition 6.8. *Suppose that a homomorphism $\xi : \mathcal{S} \to \mathbb{Z}^N$ satisfies the following.*

(1) $\operatorname{Ker}\xi$ *admits a unique invariant measure up to scalar multiplication.*

(2) *The essential range of ξ is $\mathbb{Z}^N$.*

Then, letting ν be a $\operatorname{Ker}\xi$-invariant measure, one has $R(\mathcal{S}, \nu) = \Lambda$.

Proof. By the H^1-rigidity, there exists $\zeta \in \operatorname{Hom}(\Gamma \rtimes \Lambda, \mathbb{Z}^N)$ such that ξ is cohomologous to $\zeta \circ \pi$, where $\pi : \mathcal{S} \to \Gamma \rtimes \Lambda$. Thus, we can find $f \in C(\mathbb{R}_\Gamma, \mathbb{Z}^N)$ satisfying

$$\xi = \zeta \circ \pi + (f \circ r - f \circ s).$$

By (2), ζ is surjective. By (1), ζ factors through Λ and $\zeta|\Lambda$ is injective. Let $\omega : \mathbb{Z}^N \to \Lambda$ be the inverse of $\zeta|\Lambda$. Let μ be an $\mathcal{H}$-invariant measure on $\mathbb{R}_\Gamma$. Then,

$$d\nu(x) := \omega(f(x))^{-1} \, d\mu(x)$$

gives a $\operatorname{Ker}\xi$-invariant measure. Now, it is easy to see $R(\mathcal{S}, \nu) = \Lambda$. □

Theorem 6.9 ([25, Theorem 5.8])**.** *For $i = 1, 2$, let $\mathcal{S}_i := \mathcal{S}(\Gamma_i, \Lambda_i)$, where Λ_i is finitely generated and* $\operatorname{rank}\Gamma_i \geq 2$. *The following conditions are equivalent.*

(1) $\mathcal{S}_1 \cong \mathcal{S}_2$.

(2) $\Lambda_1 = \Lambda_2$ *and there exists $s > 0$ such that $\Gamma_1 = s\Gamma_2$.*

Proof. (2) $\Longrightarrow$ (1) is obvious.

To show the converse, let $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ be an isomorphism. We may assume $\Lambda_1 \cong \Lambda_2 \cong \mathbb{Z}^N$. Choose an isomorphism $\Lambda_1 \cong \mathbb{Z}^N$ and let ξ be

$$\mathcal{S}_1 \longrightarrow \Gamma_1 \rtimes \Lambda_1 \longrightarrow \Lambda_1 \longrightarrow \mathbb{Z}^N.$$

We can apply the proposition above to $\xi \circ \Phi^{-1}$ and obtain $\Lambda_1 = \Lambda_2$. With some extra effort, one can also get $\Gamma_1 = s\Gamma_2$. □

Notice that the condition (2) in Theorem 6.9 implies the conjugacy of $(\Gamma_i \rtimes \Lambda_i) \curvearrowright \mathbb{R}_{\Gamma_i}$.

Now Theorem 6.3 follows from Theorem 6.9.

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