An introduction to topological full groups

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1 Introduction

In this section, we review several results for topological full groups of minimal $\mathbb{Z}$ actions. Main reference: [3].

Topological full group Let $\varphi : X \to X$ be a minimal homeomorphism on a Cantor set X. We define

$$\mathsf{F}(\varphi) := \left\{\alpha \in \mathrm{Homeo}(X) \mid \exists \text{conti. } n : X \to \mathbb{Z}, \text{ s.t. } \alpha(x) = \varphi^{n(x)}(x)\right\},$$

and call it the topological full group of (X, φ). This is indeed a group: for $\alpha(x) = \varphi^{n(x)}(x)$ and $\beta(x) = \varphi^{m(x)}(x)$ in $\mathsf{F}(\varphi)$, one has

$$(\alpha \circ \beta)(x) = \varphi^{n(\beta(x))}(\beta(x)) = \varphi^{n(\beta(x))}(\varphi^{m(x)}(x))$$

and $n \circ \beta + m$ is continuous.

The following theorem was obtained as a topological analogue of results by Dye in measurable dynamics.

Theorem 1.1 ([3, Corollary 4.4]). *For $i = 1, 2$, let (X_i, φ_i) be as above. The following are equivalent.*

(1) *φ_1 is conjugate to φ_2 or φ_2^{-1}.*

(2) *$\mathsf{F}(\varphi_1)$ and $\mathsf{F}(\varphi_2)$ are isomorphic as groups.*

(3) *There exists an isomorphism $\pi : C(X_1) \rtimes_{\varphi_1} \mathbb{Z} \to C(X_2) \rtimes_{\varphi_2} \mathbb{Z}$ such that $\pi(C(X_1)) = C(X_2)$.*

AF system Pick $y \in X$. We denote the forward orbit and backward orbit of y by

$$\mathrm{Orb}^+_\varphi(y) := \{\varphi^k(y) \mid k > 0\},$$

$$\mathrm{Orb}^-_\varphi(y) := \{\varphi^k(y) \mid k \leq 0\}.$$

Let $\mathsf{F}(\varphi)_y$ be the subgroup consisting of $\alpha \in \mathsf{F}(\varphi)$ such that $\alpha(\mathrm{Orb}^+_\varphi(y)) = \mathrm{Orb}^+_\varphi(y)$. The subgroup $\mathsf{F}(\varphi_y)$ corresponds to the orbit breaking subalgebra $C^*(C(X), uC_0(X \setminus \{y\}))$, where u is the implementing unitary.

It is known that $\mathsf{F}(\varphi)_y$ is an increasing union of finite direct sums of symmetric groups. In particular, $\mathsf{F}(\varphi)_y$ is locally finite.

Theorem 1.2 ([3, Corollary 4.11])**.** *For $i = 1, 2$, let (X_i, φ_i) and $y_i \in X_i$ be as above. The following are equivalent.*

(1) *φ_1 and φ_2 are strongly orbit equivalent.*

(2) *$\mathsf{F}(\varphi_1)_{y_1}$ and $\mathsf{F}(\varphi_2)_{y_2}$ are isomorphic as groups.*

(3) *The C^*-algebras $C(X_1) \rtimes_{\varphi_1} \mathbb{Z}$ and $C(X_2) \rtimes_{\varphi_2} \mathbb{Z}$ are isomorphic.*

For $\alpha \in \mathsf{F}(\varphi)$, we define

$$K := \mathrm{Orb}^-_\varphi(y) \cap \alpha^{-1}(\mathrm{Orb}^+_\varphi(y)), \quad L := \mathrm{Orb}^+_\varphi(y) \cap \alpha^{-1}(\mathrm{Orb}^-_\varphi(y))$$

$$I(\alpha) := \#K - \#L \in \mathbb{Z}.$$

Clearly $I(\alpha) = 0$ for all $\alpha \in \mathsf{F}(\varphi)_y$ and $I(\varphi^k) = k$ for $k \in \mathbb{Z}$. Also, one can verify that I is a homomorphism.

Lemma 1.3. *If $I(\alpha) = 0$, then there exists a transposition $\tau \in \mathsf{F}(\varphi)$ such that $\alpha\tau \in \mathsf{F}(\varphi)_y$.*

Proof. There exists a bijection $\pi : K \to L$. We can construct τ so that:

- $\tau(z) = \pi(z)$ for all $z \in K$,
- $\tau(z) = \pi^{-1}(z)$ for all $z \in L$,
- τ preserves $\mathrm{Orb}^-_\varphi(y) \setminus K$ and $\mathrm{Orb}^+_\varphi(y) \setminus L$.

Then $\alpha\tau$ is in $\mathsf{F}(\varphi)_y$. □

Index map Let μ be a φ-invariant probability measure on X. We define a map $I_\mu : \mathsf{F}(\varphi) \to \mathbb{R}$ as follows. For $\alpha \in \mathsf{F}(\varphi)$ such that $\alpha(x) = \varphi^{n(x)}(x)$, we let

$$I_\mu(\alpha) := \int n \; d\mu \in \mathbb{R}.$$

When $\beta(x) = \varphi^{m(x)}(x)$,

$$I_\mu(\alpha \circ \beta) = \int (n \circ \beta + m) \; d\mu = \int n \; d\mu + \int m \; d\mu = I(\alpha) + I(\beta),$$

and so I_μ is a homomorphism.

Proposition 1.4. *We have $I = I_\mu$.*

Proof. By Lemma 1.3,

$$\operatorname{Ker} I = \{\text{products of finite order elements}\} \subset \operatorname{Ker} I_\mu,$$

and $I(\varphi) = 1 = I_\mu(\varphi)$. Therefore $I = I_\mu$. □

We call $I : \mathsf{F}(\varphi) \to \mathbb{Z}$ the index map. This $\mathbb{Z}$ comes from $\mathbb{Z} \cong K_1(C(X) \rtimes_\varphi \mathbb{Z})$.

Remark 1.5. We can construct a representation $\rho : C(X) \rtimes_\varphi \mathbb{Z} \to B(\ell^2(\mathbb{Z}))$ by

$$(\rho(f)\xi)(n) := f(\varphi^{-n}(y))\xi(n)$$

$$(\rho(u)\xi)(n) := \xi(n+1).$$

The projection p onto the subspace $\{\xi \mid \xi(n) = 0 \quad \forall n \leq 0\}$ commutes with the image of ρ up to compact operators.

For $\alpha(x) = \varphi^{n(x)}(x)$ in $\mathsf{F}(\varphi)$,

$$v := \sum_{k \in \mathbb{Z}} u^k 1_{n^{-1}(k)} \in C(X) \rtimes_\varphi \mathbb{Z}$$

is a unitary. One can check that $I(\alpha)$ equals the Fredholm index of $p\rho(v)p$.

2 Ample groupoids

Main reference: [13], [14], [15], [17].

2.1 Ample groupoids and topological full groups

Definition 2.1. A topological groupoid $\mathcal{G}$ is always assumed to be locally compact and Hausdorff (LCH). It is said to be étale if the range map $r : \mathcal{G} \to \mathcal{G}^{(0)}$ is a local homeomorphism.

An ample groupoid is an étale groupoid whose unit space is 0-dimensional (totally disconnected).

An element $g \in \mathcal{G}$ can be thought of as an arrow from $s(g)$ to $r(g)$.

For $x \in \mathcal{G}^{(0)}$, the set $r(s^{-1}(x))$ is called the orbit of x. When every orbit is dense in $\mathcal{G}^{(0)}$, $\mathcal{G}$ is said to be minimal.

The isotropy bundle of $\mathcal{G}$ is $\mathrm{Iso}(\mathcal{G}) = \{g \in \mathcal{G} \mid r(g) = s(g)\}$. We say that $\mathcal{G}$ is principal if $\mathrm{Iso}(\mathcal{G}) = \mathcal{G}^{(0)}$. When the interior of $\mathrm{Iso}(\mathcal{G})$ is $\mathcal{G}^{(0)}$, we say that $\mathcal{G}$ is essentially principal.

A subset $U \subset \mathcal{G}$ is called a bisection if $r|U, s|U$ are injective (a "fat arrow"). Any open bisection U induces the homeomorphism $\theta_U := (r|U) \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$.

A (probability) measure μ on $\mathcal{G}^{(0)}$ is said to be $\mathcal{G}$-invariant if $\mu(r(U)) = \mu(s(U))$ holds for every compact open bisection U. The set of all $\mathcal{G}$-invariant probability measures is denoted by $M(\mathcal{G})$.

Example 2.2 (Transformation groupoids)**.** Let $\varphi : \Gamma \curvearrowright X$ be an action of a countable discrete group Γ on an LCH 0-dimensional space X. The transformation groupoid $\mathcal{G} := X \rtimes_\varphi \Gamma$ is $X \times \Gamma$ equipped with the product topology. The unit space of $\mathcal{G}$ is given by $\mathcal{G}^{(0)} = X \times \{1\}$ (where 1 is the identity of Γ), and identified with X. The groupoid operations are as follows:

$$r(x, \gamma) = (x, 1), \quad s(x, \gamma) = (\varphi_\gamma^{-1}(x), 1),$$

$$(x, \gamma) \cdot (x', \gamma') = (x, \gamma\gamma'), \quad (x, \gamma)^{-1} = (\varphi_\gamma^{-1}(x), \gamma^{-1}).$$

The groupoid $\mathcal{G}$ is principal if and only if the action φ is free, that is, φ_γ does not have any fixed points unless $\gamma = 1$. The groupoid $\mathcal{G}$ is essentially principal if and only if the action φ is topologically free, that is, $\{x \in X \mid \varphi_\gamma(x) = x\}$ has no interior points unless $\gamma = 1$. The groupoid $\mathcal{G}$ is minimal if and only if the action φ is minimal, that is, any orbit of φ is dense in X.

A measure μ on $\mathcal{G}^{(0)}$ is $\mathcal{G}$-invariant if and only if it is φ-invariant.

Hereafter, we always assume that $\mathcal{G}$ is essentially principal.

Definition 2.3 (Topological full groups)**.** Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. For a compact open bisection $U \subset \mathcal{G}$ such that $r(U) = \mathcal{G}^{(0)} = s(U)$, $\theta_U = (r|U) \circ (s|U)^{-1}$ is a homeomorphism on $\mathcal{G}^{(0)}$. We let $\mathsf{F}(\mathcal{G}) \subset \mathrm{Homeo}(\mathcal{G}^{(0)})$ be the set of those homeomorphisms, and call it the topological full group (TFG) of $\mathcal{G}$.

For $\mathcal{G} = X \rtimes_\varphi \Gamma$ with X compact,

$$\mathsf{F}(\mathcal{G}) = \left\{\alpha \in \mathrm{Homeo}(X) \mid \exists \text{conti. } n : X \to \Gamma, \text{ s.t. } \alpha(x) = \varphi_{n(x)}(x)\right\}.$$

Example 2.4 (AF groupoids)**.** cf. [20, Definition III.1.1], [4, Definition 3.7], [14, Definition 2.2] $\mathcal{K}$ is said to be elementary if $\mathcal{K}$ is principal and compact. When $\mathcal{K}$ is elementary:

- the topology on $\mathcal{K}$ agrees with the relative topology from $\mathcal{K}^{(0)} \times \mathcal{K}^{(0)}$,
- the equivalence relation $\mathcal{K}$ is uniformly finite, i.e. $\sup_x \# r^{-1}(x) < \infty$.

We say that $\mathcal{G}$ is an AF groupoid if it can be written as an increasing union of open elementary subgroupoids.

The C^*-algebra associated with an AF groupoid is an AF algebra.

It is known that any AF groupoids are represented by Bratteli diagrams (see [4, Theorem 3.9]). We provide a brief explanation of it. A directed graph $B = (V, E)$ is called a Bratteli diagram when $V = \bigsqcup_{n=0}^{\infty} V_n$ and $E = \bigsqcup_{n=1}^{\infty} E_n$ are disjoint unions of finite sets of vertices and edges with maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$ both of which are surjective. Let

$$X_B := \left\{ e = (e_n)_n \in \prod_n E_n \mid e_n \in E_n,\ t(e_n) = i(e_{n+1}) \quad \forall n \in \mathbb{N} \right\}.$$

The set X_B endowed with the relative topology is called the infinite path space of B. Define an equivalence relation (i.e. principal groupoid) $\mathcal{K}_m$ by

$$\mathcal{K}_m = \{(e, f) \in X_B \times X_B \mid e_n = f_n \quad \forall n \geq m\}.$$

Then, $\mathcal{K}_m$ equipped with the relative topology from $X_B \times X_B$ is a compact principal ample groupoid. Clearly one has $\mathcal{K}_m \subset \mathcal{K}_{m+1}$. Set $\mathcal{G} = \bigcup_m \mathcal{K}_m$. Endowed with the inductive limit topology, $\mathcal{G}$ becomes an AF groupoid. Conversely, Theorem 3.9 of [4] states that any AF groupoid arises in such a way.

The AF groupoid $\mathcal{G}$ is minimal if and only if for any $n \in \mathbb{N}$ there exists $m > n$ such that for any $v \in V_n$ and $w \in V_m$ there exists a path from v to w.

2.2 TFG of AF groupoids

Let $B = (V, E)$ be such as Example 2.4. For paths p, q from V_0 to V_m with $t(p) = t(q)$, we can define $\tau_{p,q} \in \mathrm{Homeo}(X_B)$ as follows: for $(e_n)_n \in X_B$, if its initial segment is either p or q, then exchange it; otherwise, do nothing. Then, $\tau_{p,q}$ is in $\mathsf{F}(\mathcal{G})$. For a fixed $m \in \mathbb{N}$, we let G_m be the subgroup generated by these $\tau_{p,q}$'s. Clearly,

$$G_m \cong \bigoplus_{v \in V_m} \mathfrak{S}_{h(v)},$$

where $h(v)$ denotes the number of paths from V_0 to v, and

$$G_m \subset G_{m+1}, \quad \mathsf{F}(\mathcal{G}) = \bigcup_m G_m.$$

Theorem 2.5 ([13, Proposition 3.2]). *Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. The following are equivalent.*

(1) *$\mathcal{G}$ is an AF groupoid.*

(2) *$\mathsf{F}(\mathcal{G})$ is locally finite.*

Example 2.6. Let $\varphi : \mathbb{Z} \curvearrowright X$ be minimal and let $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$. Pick $y \in X$. Define

$$\mathcal{H} := \mathcal{G} \setminus \{(\varphi^m(y), n) \in \mathcal{G} \mid m \leq 0 < m-n \text{ or } m-n \leq 0 < m\}.$$

Then $\mathcal{H}$ is known to be a minimal AF groupoid, and its TFG $\mathsf{F}(\mathcal{H})$ is $\mathsf{F}(\mathcal{G})_y$ discussed in Section 1.

2.3 Homology groups and index map

We do not give a complete definition of homology groups for ample groupoids. Instead, let us define only H_0 and H_1.

Definition 2.7 ([14, Section 3.1]). Let $\mathcal{G}$ be an ample groupoid.

(1) Define a homomorphism $\delta_1 : C_c(\mathcal{G}, \mathbb{Z}) \to C_c(\mathcal{G}^{(0)}, \mathbb{Z})$ by

$$\delta_1(f)(x) := \sum_{s(g)=x} f(g) - \sum_{r(g)=x} f(g)$$

and let

$$H_0(\mathcal{G}) := C_c(\mathcal{G}^{(0)}, \mathbb{Z}) / \operatorname{Im} \delta_1.$$

(2) Let

$$\mathcal{G}^{(2)} := \{(g_1, g_2) \mid s(g_1) = r(g_2)\}.$$

Define a homomorphism $\delta_2 : C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \to C_c(\mathcal{G}, \mathbb{Z})$ by

$$\delta_2(f)(g) := \sum_{k=g} f(h,k) - \sum_{hk=g} f(h,k) + \sum_{h=g} f(h,k)$$

and let

$$H_1(\mathcal{G}) := \operatorname{Ker} \delta_1 / \operatorname{Im} \delta_2.$$

The homology groups $H_*(\mathcal{G})$ are the homology of a chain complex:

$$0 \longleftarrow C_c(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \longleftarrow \cdots.$$

For later use, we observe the following.

Lemma 2.8. (1) *For compact open bisections $U, V \subset \mathcal{G}$ such that $s(U) = r(V)$, let $W := (U \times V) \cap \mathcal{G}^{(2)}$. Then $\delta_2(1_W) = 1_U - 1_{UV} + 1_V$.*

(2) *For any compact open $A \subset \mathcal{G}^{(0)}$, $[1_A] = 0$ in $H_1(\mathcal{G})$.*

(3) *For any compact open bisection $U \subset \mathcal{G}^{(0)}$, $[1_U + 1_{U^{-1}}] = 0$ in $H_1(\mathcal{G})$.*

Example 2.9. When $\mathcal{G}$ is the transformation groupoid of $\varphi : \Gamma \curvearrowright X$, $H_n(\mathcal{G})$ is canonically isomorphic to $H_n(\Gamma, C_c(X, \mathbb{Z}))$.

In particular, when $\Gamma = \mathbb{Z}$,

$$H_n(\mathcal{G}) = \begin{cases} C_c(X,\mathbb{Z})/\{f - f \circ \varphi \mid f \in C_c(X,\mathbb{Z})\} & n = 0 \\ \{f \in C_c(X,\mathbb{Z}) \mid f = f \circ \varphi\} & n = 1 \\ 0 & n \geq 2. \end{cases}$$

If φ is minimal, then $H_1(\mathcal{G}) = \mathbb{Z}$.

Example 2.10. When $\mathcal{G}$ is an AF groupoid, $H_0(\mathcal{G})$ is the dimension group of the Bratteli diagram:

$$\lim_{m\to\infty} \left(\mathbb{Z}^{V_m} \to \mathbb{Z}^{V_{m+1}}\right),$$

and $H_n(\mathcal{G}) = 0$ for $n \geq 1$.

Definition 2.11 (Index map,[14, Definition 7.1])**.** For $\alpha \in \mathsf{F}(\mathcal{G})$, a compact open bisection $U \subset \mathcal{G}$ satisfying $\alpha = \theta_U$ uniquely exists. It is easy to see that 1_U is in $\operatorname{Ker}\delta_1$. We define a map $I : \mathsf{F}(\mathcal{G}) \to H_1(\mathcal{G})$ by $I(\alpha) := [1_U]$ and call it the index map.

By Lemma 2.8 (1), I is a homomorphism. We put $\mathsf{K}(\mathcal{G}) := \operatorname{Ker} I$. Also, we denote by $\mathsf{D}(\mathcal{G})$ the commutator subgroup of $\mathsf{F}(\mathcal{G})$. Thus, we have

$$\mathsf{D}(\mathcal{G}) \triangleleft \mathsf{K}(\mathcal{G}) \triangleleft \mathsf{F}(\mathcal{G}).$$

Example 2.12. When $\mathcal{G}$ arises from a minimal homeomorphism on a Cantor set X,

$$\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = \mathbb{Z}, \quad \mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) = H_0(\mathcal{G}) \otimes \mathbb{Z}/2$$

and $\mathsf{F}(\mathcal{G})_y \subset \mathsf{K}(\mathcal{G})$ for all $y \in X$. See Section 1.

Example 2.13. Suppose that $\mathcal{G}$ is an AF groupoid. One has $\mathsf{K}(\mathcal{G}) = \mathsf{F}(\mathcal{G})$ because $H_1(\mathcal{G}) = 0$. Recall

$$\mathsf{F}(\mathcal{G}) = \bigcup_m G_m, \quad G_m \cong \bigoplus_{v \in V_m} \mathfrak{S}_{h(v)}$$

(see Section 2.2). It follows that

$$\mathsf{D}(\mathcal{G}) \cong \bigcup_m \bigoplus_{v \in V_m} \mathfrak{A}_{h(v)}$$

and

$$\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong \lim_m \left((\mathbb{Z}/2)^{V_m} \to (\mathbb{Z}/2)^{V_{m+1}}\right) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2.$$

Remark 2.14 (comparison maps)**.** Let $\mathcal{G}$ be an ample groupoid.

(1) It is easy to see that there exists a homomorphism $\mu_0 : H_0(\mathcal{G}) \to K_0(C_r^*(\mathcal{G}))$ such that $\mu_0([1_A]) = [1_A]$ for every compact open set $A \subset \mathcal{G}^{(0)}$.

(2) It is known that there exists a homomorphism $\mu_1 : H_1(\mathcal{G}) \to K_1(C_r^*(\mathcal{G}))$ such that $\mu_1([1_U]) = [1_U]$ for every compact open bisection $U \subset \mathcal{G}$ satisfying $r(U) = \mathcal{G}^{(0)} = s(U)$. See [1].

Problem 2.15. Does there exist $\mu_* : H_*(\mathcal{G}) \to K_*(C_r^*(\mathcal{G}))$ for $* \geq 2$?

2.4 Reconstruction

The following is a generalization of Theorem 1.1.

Theorem 2.16 ([22],[15, Theorem 3.10])**.** *For $i = 1, 2$, let $\mathcal{G}_i$ be a minimal ample groupoid with compact unit space. The following are equivalent.*

(1) $\mathcal{G}_1 \cong \mathcal{G}_2$.

(2) $\mathsf{F}(\mathcal{G}_1) \cong \mathsf{F}(\mathcal{G}_2)$.

(3) $\mathsf{K}(\mathcal{G}_1) \cong \mathsf{K}(\mathcal{G}_2)$.

(4) $\mathsf{D}(\mathcal{G}_1) \cong \mathsf{D}(\mathcal{G}_2)$.

Remark 2.17. The assumption of minimality can be relaxed (see [22]).

The theorem above ensures TFG's are a rich source of interesting infinite groups.

3 Simplicity

Main reference: [14], [15], [17].

3.1 Two classes

Definition 3.1 ([14, Definition 6.2])**.** Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. We say that $\mathcal{G}$ is almost finite (abbreviated as a.f.) if for any compact subset $C \subset \mathcal{G}$ and $\varepsilon > 0$ there exists an elementary subgroupoid $\mathcal{K} \subset \mathcal{G}$ such that

$$\frac{\#(C\mathcal{K}x \setminus \mathcal{K}x)}{\#(\mathcal{K}x)} < \varepsilon$$

for all $x \in \mathcal{G}^{(0)}$.

This definition says that any compact subset C is almost 'covered' by an elementary subgroupoid $\mathcal{K}$.

This also reminds us of the notion of tracially AF C^*-algebras.

By definition, AF groupoids are almost finite.

Theorem 3.2 ([10, 18])**.** *When $\Gamma \curvearrowright X$ is a free action of an elementary amenable group on a Cantor set, its transformation groupoid $\mathcal{G} = X \rtimes \Gamma$ is almost finite.*

We remark that for the $\mathcal{G}$ above, $C_r^*(\mathcal{G})$ is $\mathcal{Z}$-stable ([9, Theorem 12.4]).

Definition 3.3 ([15, Definition 4.9])**.** Let $\mathcal{G}$ be an ample groupoid with $\mathcal{G}^{(0)}$ compact. We say that $\mathcal{G}$ is purely infinite (p.i.) if for every clopen set $A \subset \mathcal{G}^{(0)}$, there exist compact open bisections $U, V \subset \mathcal{G}$ such that $s(U) = s(V) = A$, $r(U) \sqcup r(V) \subset A$.

When $\mathcal{G}$ is purely infinite, it is easy to see that $\mathsf{F}(\mathcal{G})$ contains the free group $\mathbb{Z} * \mathbb{Z}$ ([15, Proposition 4.10]). Also, $C_r^*(\mathcal{G})$ is purely infinite ([21, Theorem 4.1]).

Example 3.4 (SFT groupoids)**.** Let (V, E) be a finite directed graph and let A be its adjacency matrix of (V, E). We assume that A is irreducible and is not a permutation matrix. Define

$$X := \{(x_k)_{k \in \mathbb{N}} \in E^{\mathbb{N}} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\}.$$

With the product topology, X is a Cantor set. The shift σ on X is called the one-sided irreducible shift of finite type (SFT) associated with the graph (V, E) (or the matrix A).

The SFT groupoid $\mathcal{G}_{(V,E)}$ is the graph groupoid:

$$\mathcal{G}_{(V,E)} := \{(x, n, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ n = k - l,\ \sigma^k(x) = \sigma^l(y)\}$$

with the topology generated by the sets

$$\{(x, k-l, y) \in \mathcal{G}_{(V,E)} \mid x \in P,\ y \in Q,\ \sigma^k(x) = \sigma^l(y)\},$$

where $P, Q \subset X$ are open and $k, l \in \mathbb{N}$. The groupoid structure is given by

$$(x, n, y) \cdot (y, n', y') = (x, n+n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

We identify X with the unit space $\mathcal{G}_{(V,E)}^{(0)}$ via $x \mapsto (x, 0, x)$.

It is easy to see that $\mathcal{G}_{(V,E)}$ is minimal and purely infinite.

When V is a singleton, σ is the full shift. The TFG of $\mathcal{G}_{(V,E)}$ is isomorphic to the Higman-Thompson group.

The homology groups of the SFT groupoid $\mathcal{G}_{(V,E)}$ was computed in [14].

Theorem 3.5 ([14, Theorem 4.14])**.** *One has*

$$H_n(\mathcal{G}_{(V,E)}) \cong \begin{cases} \mathrm{Coker}(I - A^t) & n = 0 \\ \mathrm{Ker}(I - A^t) & n = 1 \\ 0 & n \geq 2, \end{cases}$$

where the matrix A acts on the abelian group $\mathbb{Z}^V$ by multiplication.

Problem 3.6. Does there exist a minimal ample groupoid which is neither almost finite nor purely infinite?

3.2 Commutator subgroups

Theorem 3.7 ([15, Theorem 4.7, Theorem 4.16])**.** *Let $\mathcal{G}$ be a minimal ample groupoid. If $\mathcal{G}$ is either almost finite or purely infinite, then $\mathsf{D}(\mathcal{G})$ is simple.*

We give a sketchy proof for the purely infinite case. Assume that $\mathcal{G}$ is minimal and purely infinite. Let $N \triangleleft \mathsf{D}(\mathcal{G})$ be a non-trivial normal subgroup.

Lemma 3.8. *For any $\tau \in N$ and $\alpha \in \mathsf{F}(\mathcal{G})$, we have $\alpha\tau\alpha^{-1} \in N$.*

Proof of a part of Theorem 3.9. Let $\tau \in N \setminus \{1\}$. There exists a non-empty clopen set $A \subset \mathcal{G}^{(0)}$ such that $A \cap \tau(A) = \emptyset$.

It suffices to show $[\alpha, \beta] \in N$ for any $\alpha, \beta \in \mathsf{F}(\mathcal{G})$. We can find $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathsf{F}(G)$ such that

$$\alpha = \alpha_1\alpha_2, \quad \beta = \beta_1\beta_2$$
$$\mathrm{supp}(\alpha_i) \neq \mathcal{G}^{(0)}, \quad \mathrm{supp}(\beta_i) \neq \mathcal{G}^{(0)}.$$

Thanks to the lemma above, we may assume $\mathrm{supp}(\alpha) \neq \mathcal{G}^{(0)}$ and $\mathrm{supp}(\beta) \neq \mathcal{G}^{(0)}$ from the start.

Since $\mathcal{G}$ is purely infinite and minimal, there exists $\gamma \in \mathsf{F}(\mathcal{G})$ such that

$$\gamma(\mathrm{supp}(\alpha)) \cap \mathrm{supp}(\beta) = \emptyset \quad \text{and} \quad \mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma) \neq \mathcal{G}^{(0)}.$$

Also, there exists $\sigma \in \mathsf{F}(\mathcal{G})$ such that

$$\sigma(\mathrm{supp}(\alpha) \cup \mathrm{supp}(\gamma)) \subset A.$$

By the lemma above, $\tilde{\tau} := \sigma^{-1}\tau\sigma$ is in N. It follows from $A \cap \tau(A) = \emptyset$ that

$$\mathrm{supp}(\alpha) \cap \tilde{\tau}(\mathrm{supp}(\gamma)) = \emptyset.$$

Hence $\tilde{\gamma} := [\gamma, \tilde{\tau}]$ satisfies

$$\tilde{\gamma}(\mathrm{supp}(\alpha)) \cap \mathrm{supp}(\beta) = \emptyset,$$

that is, $\tilde{\gamma}\alpha\tilde{\gamma}^{-1}$ commutes with β. Again, by the lemma above, $\tilde{\gamma}$ is in N. Therefore

$$[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1} = \alpha(\tilde{\gamma}\alpha^{-1}\tilde{\gamma}^{-1})\beta(\tilde{\gamma}\alpha\tilde{\gamma}^{-1})\alpha^{-1}\beta^{-1} = [[\alpha, \tilde{\gamma}], \beta]$$

is in N. □

3.3 Abelianization

Theorem 3.9 ([14, Theorem 7.5], [15, Theorem 5.2]). *If $\mathcal{G}$ is either almost finite or purely infinite, then the index map $I : \mathsf{F}(\mathcal{G}) \to H_1(\mathcal{G})$ is surjective.*

Proof. We give a sketchy proof for the purely infinite case. Suppose that $\mathcal{G}$ is purely infinite.

Let $f \in \operatorname{Ker} \delta_1$. By Lemma 2.8 (3), we may assume that there exist compact open bisections $C_1, C_2, \ldots, C_n$ such that

$$f = 1_{C_1} + 1_{C_2} + \cdots + 1_{C_n}.$$

By $\delta_1(f) = 0$, we have

$$\sum_{i=1}^{n} 1_{r(C_i)} = \sum_{j=1}^{n} 1_{s(C_j)}.$$

Hence we can find clopen subsets $A_{ij} \subset \mathcal{G}^{(0)}$ for $i, j = 1, 2, \ldots, n$ satisfying

$$\bigsqcup_{i=1}^{n} A_{ij} = r(C_j) \quad \text{and} \quad \bigsqcup_{j=1}^{n} A_{ij} = s(C_i).$$

(How to find A_{ij}: Take a clopen partition $(D_l)_l$ generated by $r(C_i)$'s and $s(C_j)$'s. For each l,

$$\#\{i \mid D_l \subset r(C_i)\} = \#\{j \mid D_l \subset s(C_j)\},$$

and so there exists a bijective correspondence between them. Let us denote it by $i \sim_l j$. Set

$$A_{ij} := \bigsqcup_{i \sim_l j} D_l,$$

which is a desired one.)

Since $\mathcal{G}$ is purely infinite, there exist compact open bisections $U_1, U_2, \ldots, U_n$ such that $s(U_i) = r(C_i)$ and the sets $r(U_i)$ are mutually disjoint. Put

$$V_{ij} := U_i C_i A_{ij} U_j^{-1}.$$

We can check that V_{ij} are compact open bisections such that

$$r(V_{ij}) = r(U_i C_i A_{ij}) \quad \text{and} \quad s(V_{ij}) = s(A_{ij} U_j^{-1}).$$

Therefore, $V := \bigcup_{i,j} V_{ij}$ is also a compact open bisection and

$$r(V) = \bigsqcup_i r(U_i C_i) = \bigsqcup_i r(U_i) = \bigsqcup_j s(U_j^{-1}) = s(V).$$

On one hand, by Lemma 2.8 (1)(3),

$$\begin{aligned}[1_V] &= \left[\sum_{i,j} 1_{V_{ij}}\right] = \left[\sum_{i,j} 1_{U_iC_iA_{ij}U_j^{-1}}\right] = \left[\sum_{i,j} \left(1_{U_iC_iA_{ij}} + 1_{A_{ij}U_j^{-1}}\right)\right] \\ &= \left[\sum_i 1_{U_iC_i} + \sum_j 1_{U_j^{-1}}\right] = \left[\sum_i (1_{U_i} + 1_{C_i}) + \sum_j 1_{U_j^{-1}}\right] = [f].\end{aligned}$$

Set $W := V \sqcup (\mathcal{G}^{(0)} \setminus s(V))$. Then one has $I(\theta_W) = [f]$ as desired. $\square$

Let us recall

$$\mathsf{D}(\mathcal{G}) \lhd \mathsf{K}(\mathcal{G}) \lhd \mathsf{F}(\mathcal{G}).$$

Example 3.10. (1) When $\mathcal{G}$ is an AF groupoid, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = 0$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ (see Example 2.13).

(2) When $\mathcal{G}$ arises from a minimal action $\mathbb{Z} \curvearrowright X$, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = \mathbb{Z}$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ ([13, Theorem 4.8]).

(3) When $\mathcal{G}$ is an SFT groupoid, $\mathsf{F}(\mathcal{G})/\mathsf{K}(\mathcal{G}) = H_1(\mathcal{G})$ and $\mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G}) \cong H_0(\mathcal{G}) \otimes \mathbb{Z}/2$ ([15, Corollary 6.24]).

When $\mathcal{G}$ is minimal, a homomorphism

$$\zeta : H_0(\mathcal{G}) \otimes \mathbb{Z}/2 \to \mathsf{K}(\mathcal{G})/\mathsf{D}(\mathcal{G})$$

was constructed by V. Nekrashevych [19, Section 7].

Theorem 3.11 ([12, Corollary E])**.** *Let $\mathcal{G}$ be a minimal ample groupoid. If $\mathcal{G}$ is either almost finite or purely infinite, then there exists an exact sequence:*

$$H_2(\mathsf{D}(\mathcal{G})) \longrightarrow H_2(\mathcal{G}) \longrightarrow H_0(\mathcal{G}) \otimes \mathbb{Z}/2 \xrightarrow{\zeta} H_1(\mathsf{F}(\mathcal{G})) \xrightarrow{I} H_1(\mathcal{G}) \longrightarrow 0$$

Notice that $H_1(\mathsf{F}(\mathcal{G}))$ is isomorphic to the abelianization $\mathsf{F}(\mathcal{G})/\mathsf{D}(\mathcal{G})$.

Li [12] discovered a close connection between homology of TFG and groupoid homology. The theorem above is one consequence from his deep analysis.

Example 3.12 ([16])**.** Fix $m \in \mathbb{N}$. Let $\mathcal{G}$ be the SFT groupoid of the full shift over $m{+}1$ symbols. By Theorem 3.5,

$$H_k(\mathcal{G}) = \begin{cases} \mathbb{Z}/m & k = 0 \\ 0 & k \geq 1. \end{cases}$$

Hence, the Künneth theorem implies

$$H_k(\mathcal{G} \times \mathcal{G}) = \begin{cases} \mathbb{Z}/m & k = 0, 1 \\ 0 & k \geq 2 \end{cases}$$

and

$$H_k(\mathcal{G}\times\mathcal{G}\times\mathcal{G}) = \begin{cases} \mathbb{Z}/m & k=0 \\ (\mathbb{Z}/m)^2 & k=1 \\ \mathbb{Z}/m & k=2 \\ 0 & k\geq 3. \end{cases}$$

Let us describe the generator of $H_1(\mathcal{G}\times\mathcal{G}) \cong \mathbb{Z}/m$. Define $\beta \in \mathrm{Homeo}(\mathcal{G}^{(0)}\times\mathcal{G}^{(0)})$ (the baker's map) as follows:

$$\beta\left((x_n)_n,(y_n)_n\right) := \left((x_2x_3\dots),(x_1y_1y_2\dots)\right).$$

Then β is in $\mathsf{F}(\mathcal{G}\times\mathcal{G})$ and $I(\beta)$ generates $H_1(\mathcal{G}\times\mathcal{G})\cong\mathbb{Z}/m$.

Now we assume m is even, and let $t\in H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}))$ be the image of the generator of $H_0\otimes\mathbb{Z}/2\cong\mathbb{Z}/2$ by the map ζ.

Define $\tau\in\mathsf{F}(\mathcal{G}\times\mathcal{G})$ by

$$\tau\left((x_n)_n,(y_n)_n\right) := \left((y_1x_2x_3\dots),(x_1y_2y_3\dots)\right).$$

Thus, τ is a transposition whose support is $\{(x,y)\mid x_1\neq y_1\}$. It follows that the equivalence class $[\tau]$ of τ in $H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}))$ equals

$$\frac{m(m+1)}{2}t.$$

Let us consider $\tau\beta$. This sends (x,y) to

$$\left((x_1x_3x_4\dots),(x_2y_1y_2\dots)\right).$$

Therefore $\tau\beta$ is a product of $m+1$ elements with mutually disjoint support and each of them is conjugate to β. Hence $[\tau\beta]=[\beta^{m+1}]$ (i.e. $\tau\beta^{-m}$ belongs to $\mathsf{D}(\mathcal{G}\times\mathcal{G})$), and so we have

$$m[\beta]=\frac{m(m+1)}{2}t\in H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G})).$$

Now we consider $\mathcal{G}\times\mathcal{G}\times\mathcal{G}$. We let

$$\beta_{12} := \beta\times\mathrm{id}\in\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G})$$

and

$$\beta_{23} := \mathrm{id}\times\beta\in\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}).$$

Similarly, β_{13} is defined. Thinking of t as an element in $H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}))$, we have

$$m[\beta_{12}]=m[\beta_{23}]=m[\beta_{31}]=\frac{m(m+1)}{2}t.$$

When $m\in 2\mathbb{Z}\setminus 4\mathbb{Z}$, this is equal to t. On the other hand, one has $\beta_{23}\beta_{12}=\beta_{13}$. Consequently we obtain $t=0$. Thus, the map

$$\zeta : H_0(\mathcal{G}\times\mathcal{G}\times\mathcal{G})\otimes\mathbb{Z}/2\to H_1(\mathsf{F}(\mathcal{G}\times\mathcal{G}\times\mathcal{G}))$$

is zero.

4 Amenability

Main reference: [7], [6].

K. Juschenko and N. Monod obtained the following remarkable result.

Theorem 4.1 ([7, Theorem A]). *When $\varphi : \mathbb{Z} \curvearrowright X$ is minimal, the TFG of $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$ is amenable.*

In the proof of this theorem, the notion of extensive amenability plays the central role. This property was first introduced (without a name) in [7], and studied further in [8, 6].

We recall the definition of extensive amenability from [6, Definition 1.1]. Let $\alpha : G \curvearrowright Z$ be an action of a discrete group G on a set Z. Set

$$P(Z) := \bigoplus_Z \mathbb{Z}/2 = \{f : Z \to \mathbb{Z}_2 \mid \mathrm{supp}(f) \text{ is finite}\}.$$

The action α naturally extends to $\alpha : G \curvearrowright P(Z)$. We say that $\alpha : G \curvearrowright Z$ is extensively amenable if there exists a G-invariant mean (i.e. finitely additive probability measure) m on $P(Z)$ such that $m(\{1_F \in P(Z) \mid E \subset F\}) = 1$ for any finite subset $E \subset Z$. In [7, Lemma 3.1], it was shown that $\alpha : G \curvearrowright Z$ is extensively amenable if and only if the action of $P(Z) \rtimes G$ on $P(Z)$ admits an invariant mean.

We denote by $W(\mathbb{Z})$ the group of all permutations g of $\mathbb{Z}$ for which the quantity $\sup\{|g(j) - j| \mid j \in \mathbb{Z}\}$ is finite. In [7, Theorem C], it was shown that the natural action $W(\mathbb{Z}) \curvearrowright \mathbb{Z}$ is extensively amenable. (This part is technically quite hard.) It follows that the action of $P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $P(\mathbb{Z})$ admits an invariant mean.

Let $\varphi : \mathbb{Z} \curvearrowright X$ be minimal and let $\mathcal{G} := X \rtimes_\varphi \mathbb{Z}$. We would like to show that $\mathsf{F}(\mathcal{G})$ is amenable. Fix a point $y \in X$. For $\alpha \in \mathsf{F}(\mathcal{G})$, we can define $\tilde{\alpha} \in W(\mathbb{Z})$ so that $\alpha(\varphi^j(y)) = \varphi^{\tilde{\alpha}(j)}(y)$. Define a map $\pi : \mathsf{F}(\mathcal{G}) \to P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ by $\pi(\alpha) = (1_\mathbb{N} + 1_{\tilde{\alpha}(\mathbb{N})}, \tilde{\alpha})$ for $\alpha \in \mathsf{F}(\mathcal{G})$. It is routine to check that π is an injective homomorphism. Since the action of $P(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $P(\mathbb{Z})$ admits an invariant mean, in order to show the amenability of $\mathsf{F}(\mathcal{G})$, it suffices to prove that the stabiliser in $\pi(\mathsf{F}(\mathcal{G}))$ of 1_E is amenable for any finite subset $E \subset \mathbb{Z}$.

Lemma 4.2 ([7, Lemma 4.1]). *In the setting above, for any finite subset $E \subset \mathbb{Z}$, the stabiliser*

$$S := \{\alpha \in \mathsf{F}(\mathcal{G}) \mid \pi(\alpha)(1_E) = 1_E\}$$

is locally finite, and hence amenable.

Proof. By definition, $\pi(\alpha)(1_E) = 1_\mathbb{N} + 1_{\tilde{\alpha}(\mathbb{N})} + 1_{\tilde{\alpha}(E)}$, which implies

$$\alpha \in S \iff \tilde{\alpha}(\mathbb{N}\Delta E) = \mathbb{N}\Delta E,$$

where Δ means the symmetric difference. Put

$$k := \#(E \cap \mathbb{N}) - \#(E \setminus \mathbb{N}) \in \mathbb{Z}.$$

We can find a transposition $\tau \in \mathsf{F}(\mathcal{G})$ satisfying

$$\{\tau(\varphi^j(y)) \mid j > k\} = \{\varphi^j(y) \mid j \in \mathbb{N}\Delta E\}.$$

Then, one has

$$(\tau\varphi^k)(\mathrm{Orb}^+_\varphi(y)) = \{\varphi^j(y) \mid j \in \mathbb{N}\Delta E\}.$$

Hence

$$\alpha \in S \iff ((\tau\varphi^k)^{-1}\alpha(\tau\varphi^k))(\mathrm{Orb}^+_\varphi(y)),$$

which means that S is conjugate to $\mathsf{F}(\mathcal{G})_y$ discussed in Section 1. Therefore, S is locally finite, and hence amenable. □

In such a way, Theorem 4.1 is proved.

In [8, 6], the notion of extensive amenability is used to prove amenability of various kinds of groups. Among others, it was shown that all subgroups of the group of interval exchange transformations that have angular components of rational rank ≤ 2 are amenable ([6, Theorem 5.1]). In particular, when $\varphi : \mathbb{Z}^2 \curvearrowright X$ is a free minimal action arising from two irrational rotations on the circle (see [5, Example 30]), the TFG of $X \rtimes_\varphi \mathbb{Z}^2$ is amenable. On the other hand, it is known that there exists a free minimal action $\varphi : \mathbb{Z}^2 \curvearrowright X$ on a Cantor set such that its TFG contains the non-abelian free group ([2]). It may be a rather complicated problem to determine when the TFG is amenable for $\varphi : \mathbb{Z}^2 \curvearrowright X$.

As a generalization of Theorem 4.1, Szőke obtained the following result.

Theorem 4.3 ([23])**.** *Let Γ be a finitely generated group and let X be the Cantor set.*

(1) *If Γ is virtually cyclic, then for any minimal action $\varphi : \Gamma \curvearrowright X$, its TFG is amenable.*

(2) *If Γ is not virtually cyclic, then there exists a free minimal action $\varphi : \Gamma \curvearrowright X$ whose TFG contains the free group.*

Problem 4.4. When is $\mathsf{F}(\mathcal{G})$ amenable?

5 Finiteness

Theorem 5.1 ([19])**.** *Suppose that $\mathcal{G}$ is minimal and either almost finite or purely infinite. If $\mathcal{G}$ is expansive, then $\mathsf{D}(\mathcal{G})$ is finitely generated.*

When $\mathcal{G}$ arises from $\varphi : \Gamma \curvearrowright X$, $\mathcal{G}$ is expansive if and only if φ is expansive.

Example 5.2 ([13, 15])**.** (1) When $\varphi : \mathbb{Z} \curvearrowright X$ is minimal and expansive, its $\mathsf{D}(\mathcal{G})$ is simple, amenable and finitely generated.

(2) AF groupoids never be expansive.

(3) An SFT groupoid is expansive, and so its $\mathsf{D}(\mathcal{G})$ (and also $\mathsf{F}(\mathcal{G})$) is finitely generated. Moreover, it is finitely presented.

X. Li [11] proved that $\mathsf{F}(\mathcal{G})$ is of type F_∞ (in particular, finitely presented) when $\mathcal{G}$ is a product of SFT groupoids.

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