Classifying Stein's groups

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Stein's group

Suppose we are given:

- $\Lambda \subset (0, \infty)$, a countable multiplicative subgroup,
- $\Gamma \subset \mathbb{R}$, a countable $\mathbb{Z}\Lambda$-module, dense in $\mathbb{R}$,
- $\ell \in \Gamma \cap (0, \infty)$.

Stein's group $V(\Gamma, \Lambda, \ell)$ (Stein 1992) is the group consisting of piecewise linear bijections f of $[0, \ell)$ satisfying the following:

- f is right continuous,
- f has finitely many discontinuous or nondifferential points, all in Γ,
- f has slopes only in Λ.

Example

$V(\mathbb{Z}[1/n], \langle n \rangle, \ell)$ is the Higman-Thompson group.

Main theorem

Theorem (M 2024)

For $i = 1, 2$*, let* $V(\Gamma_i, \Lambda_i, \ell_i)$ *be Stein's group.*
Suppose that Λ_i *are finitely generated and* $\operatorname{rank} \Gamma_i \geq 2$*.*
The following are equivalent.

1. $V(\Gamma_1, \Lambda_1, \ell_1)$ *is isomorphic to* $V(\Gamma_2, \Lambda_2, \ell_2)$*.*
2. $\Lambda_1 = \Lambda_2$ *and there exists* $s > 0$ *such that* $\Gamma_1 = s\Gamma_2$ *and* $\ell_1 - s\ell_2$ *is zero in* $H_0(\Lambda_1, \Gamma_1)$*.*

Remark

2 $\Longrightarrow$ 1 is obvious.

Outline of proof

$$
\begin{tikzcd}
\text{dynamical system} \arrow[d] & & \Gamma \rtimes \Lambda \curvearrowright \mathbb{R}_\Gamma \\
\text{ample groupoid} \arrow[d] \arrow[u, dotted, bend left, "\text{``cocycle rigidity''}"] & & \mathcal{S}(\Gamma, \Lambda) \\
\text{topological full group} \arrow[u, dotted, bend left, "\text{``reconstruction''}"] & & V(\Gamma, \Lambda, \ell)
\end{tikzcd}
$$

Stein groupoid

Let

$$\mathbb{R}_\Gamma := (\mathbb{R} \setminus \Gamma) \sqcup \{t_+, t_- \mid t \in \Gamma\}$$

with the natural order topology.
$\mathbb{R}_\Gamma$ is a totally disconnected space, on which $\Gamma \rtimes \Lambda$ acts.

Then

$$\mathcal{S}(\Gamma, \Lambda) := \mathbb{R}_\Gamma \rtimes (\Gamma \rtimes \Lambda) = \mathbb{R}_\Gamma \times (\Gamma \times \Lambda)$$

becomes an ample groupoid, that is, a topological groupoid whose unit space is totally disconnected. We call this the Stein groupoid.

It is easy to see that $V(\Gamma, \Lambda, \ell)$ is isomorphic to
the topological full group of the ample groupoid $\mathcal{S}(\Gamma, \Lambda)|[0_+, \ell_-]$.

Reconstruction

Theorem (reconstruction, Rubin 1989, M 2015)

For $i = 1, 2$, let $\mathcal{G}_i$ be an ample groupoid whose unit space is a Cantor set. Suppose that $\mathcal{G}_i$ are essentially principal and minimal. The following are equivalent.

1. *The ample groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.*
2. *The topological full groups $[[\mathcal{G}_1]]$ and $[[\mathcal{G}_2]]$ are isomorphic.*

Corollary

The following are equivalent.

1. $\mathcal{S}(\Gamma_1, \Lambda_1)|[0_+, \ell_{1-}]$ *is isomorphic to* $\mathcal{S}(\Gamma_2, \Lambda_2)|[0_+, \ell_{2-}]$.
2. $V(\Gamma_1, \Lambda_1, \ell_1)$ *is isomorphic to* $V(\Gamma_2, \Lambda_2, \ell_2)$.

Rigidity

Suppose that Λ is finitely generated and $\operatorname{rank}\Gamma \geq 2$.

Theorem (cocycle rigidity, M 2024)

The natural homomorphism $\mathcal{S}(\Gamma, \Lambda) \to \Gamma \rtimes \Lambda$ induces an isomorphism between $H^1(\mathcal{S}(\Gamma, \Lambda))$ and $H^1(\Gamma \rtimes \Lambda)$.

Theorem (orbit equivalence rigidity, M 2024)

The following are equivalent.

1. $\mathcal{S}(\Gamma_1, \Lambda_1)$ *is isomorphic to* $\mathcal{S}(\Gamma_2, \Lambda_2)$.
2. $\Lambda_1 = \Lambda_2$ *and there exists* $s > 0$ *such that* $\Gamma_1 = s\Gamma_2$.