Classifying Stein's groups

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@ RIMS Kyto
H. Matui

Def (Stein's group)

$\Lambda \subset (0,\infty)$ countable, multiplicative subgroup

$\Gamma \subset \mathbb{R}$ countable, $\mathbb{Z}[\Lambda]$-module, dense

$\ell \in \Gamma$, positive

$$V(\Lambda,\Gamma,\ell) := \left\{ \begin{array}{l} \text{right conti. bijections on } [0,\ell), \\ \text{piecewise linear, slopes} \in \Lambda \\ \text{singular points} \in \Gamma \end{array} \right\}$$

For positive $\lambda \neq 1$,

set $V_\lambda := V(\langle\lambda\rangle, \mathbb{Z}[\lambda,\lambda^{-1}], 1)$.

Example

(1) $n \in \mathbb{N} \setminus \{1\}$, V_n is the Higman-Thompson group.

$V_n \cong V_m \iff n = m$

Brown '87 V_n is of type F_∞

(2) Stein '92 the same for $V(\langle \underbrace{n_1,\cdots,n_k}_{\text{integers}} \rangle, \mathbb{Z}[\frac{1}{n_1\cdots n_k}], \ell)$.

(3) Cleary '95, '00 the same for V_λ

when $\lambda = \sqrt{2}+1, \frac{\sqrt{5}+1}{2}$

Thm. B (M)

$i=1,2$ $\quad \Lambda_i$ is f.g. $\quad$ rank $\Gamma_i \geq 2$ $\quad$ dim of $\Gamma_i \otimes \mathbb{Q}$

T.F.A.E.

(1) $V(\Lambda_1, \Gamma_1, \ell_1) \cong V(\Lambda_2, \Gamma_2, \ell_2)$

(2) $\Lambda_1 = \Lambda_2$,

$\exists s>0$ $\quad \Gamma_1 = s\Gamma_2$, $\quad \ell_1 - s\ell_2$ is zero in $H_0(\Lambda_1, \Gamma_1)$

Rem $\quad$ (2) $\Rightarrow$ (1) is obvious

$$\mathbb{R}_\Gamma := \text{spectrum of } C^*(\{ 1_{[t,s)} \mid t,s \in \Gamma, t<s \})$$

$$\mathbb{R}_\Gamma = (\mathbb{R} \setminus \Gamma) \sqcup \{ t_-, t_+ \mid t \in \Gamma \}$$

Def (Stein groupoid)

$$\mathcal{S} = \mathcal{S}(\Lambda, \Gamma) := \mathbb{R}_\Gamma \rtimes \Gamma \rtimes \Lambda$$

translation $\quad$ multiplication

$V(\Lambda, \Gamma, \ell)$ is isomorphic to the topologicall full group of $\mathcal{S}(\Lambda, \Gamma)|[0_+, \ell_-]$

Thm (Rubin, M)

$i=1,2$ $\quad \mathcal{G}_i$; minimal ample groupoids with $\mathcal{G}_i^{(0)}$ compact

$$[\![\mathcal{G}_1]\!] \cong [\![\mathcal{G}_2]\!] \iff \mathcal{G}_1 \cong \mathcal{G}_2$$

Goal Classify the Stein groupoids $\mathcal{S}$

Homology groups $H_*(\mathcal{S})$

For $\lambda \neq 1$, $\mathcal{S}_\lambda := \mathcal{S}(\langle\lambda\rangle, \mathbb{Z}[\lambda, \lambda^{-1}])$

$\lambda = n \in \mathbb{N}$
$$H_*(\mathcal{S}_n) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & *=0 \\ 0 & *\geq 1 \end{cases}$$

$\lambda = \sqrt{2}+1$

$\lambda^2 - 2\lambda - 1 = 0$
$$H_*(\mathcal{S}_\lambda) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & *=0,1 \\ 0 & *\geq 2 \end{cases}$$

$\lambda = \frac{\sqrt{5}+1}{2}$

$\lambda^2 - \lambda - 1 = 0$
$$H_*(\mathcal{S}_\lambda) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & *=1 \\ 0 & *=0, \geq 2 \end{cases}$$

$\lambda = \sqrt{n}$ ($n \neq$ square)

$\lambda^2 - n = 0$
$$H_*(\mathcal{S}_{\sqrt{n}}) = \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & *=0 \\ \mathbb{Z}/(n+1)\mathbb{Z} & *=1 \\ 0 & *\geq 2 \end{cases}$$

λ : transcendental
$$H_*(\mathcal{S}_\lambda) = \bigoplus_{i=0}^{\infty} \mathbb{Z} \quad \forall *$$

$\exists$ canonical homomorphism

$$\mathcal{S} = \mathbb{R}_\Gamma \rtimes \Gamma \rtimes \Lambda \longrightarrow \Gamma \rtimes \Lambda \longrightarrow \Lambda$$

Lem

isotropy group $r^{-1}(x) \cap s^{-1}(x)$

(1) $\forall x \in \mathbb{R}_\Gamma$ $\quad \mathcal{S}_x \to \Lambda$ is injective

(2) $\forall t \in \Gamma$ $\quad \mathcal{S}_{t_\pm} \to \Lambda$ is isomorphism

Proof

(1) $(x, t, \lambda) \in \mathrm{Ker} \Rightarrow \lambda = 1$

$(x, t, 1) \in \mathcal{S}_x \Rightarrow t = 0$

(2) $\forall \lambda \in \Lambda$ $\quad (t_\pm, \underset{\Gamma}{\underset{\cap}{(1-\lambda)t}}, \lambda) \in \mathcal{S}_{t_\pm}$ $\quad \square$

From now on,

we always assume $\Lambda \cong \mathbb{Z}^N$ and $\operatorname{rank} \Gamma \geq 2$.

Def

X ; totally disconnected, locally cpt, Hausdorff

$G \curvearrowright X$

$X \rtimes G$ is H^1-rigid if $X \rtimes G \to G$ induces

isomorphism $H^1(G) \cong H^1(X \rtimes G)$

<u>Prop</u> $\mathcal{S} = \mathbb{R}_\Gamma \rtimes (\Gamma \times \Lambda)$ is H^1-rigid.

$$\mathcal{H} := \mathbb{R}_\Gamma \rtimes \Gamma$$

Let (Γ_n) be an increasing seq. of f.g. subgrps of Γ such that $\Gamma = \bigcup \Gamma_n$.

$$\mathcal{H}_n := \mathbb{R}_\Gamma \rtimes \Gamma_n$$

$$\mathcal{H} = \bigcup \mathcal{H}_n$$

<u>Lem</u>

(1) $\operatorname{rank} \Gamma_n \geq 3 \Rightarrow \mathcal{H}_n$ is H^1-rigid

(2) $\operatorname{rank} \Gamma_n = 2$

$$\Rightarrow H^1(\mathcal{H}_n) = H^1(\Gamma_n) \oplus \operatorname{Ker}\Big(\bigoplus_{\Gamma/\Gamma_n} \mathbb{Z} \to \mathbb{Z}\Big)$$

<u>Proof</u>

Suppose $\Gamma_n \cong \mathbb{Z}^k$.

$$F := \mathbb{Z} \text{ span of } 1_{[t_+,\infty)} \quad \forall t \in \Gamma$$

$$0 \to C_c(\Gamma_n, \mathbb{Z}) \to F \to \mathbb{Z} \to 0$$

is an exact seq. of $\mathbb{Z}\Gamma_n$-modules.

Since F is free,

$$H_*(\Gamma_n, F) = \begin{cases} \bigoplus_{\Gamma/\Gamma_n} \mathbb{Z} & * = 0 \\ 0 & * \geq 1 . \end{cases}$$

Hence

$$H_*(\Gamma_n, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$$

$$= \begin{cases} H_1(\Gamma_n) \oplus \operatorname{Ker}\left(\bigoplus_{\Gamma/\Gamma_n} \mathbb{Z} \to \mathbb{Z}\right) & * = 0 \\ H_{*+1}(\Gamma_n) & * \geq 1 . \end{cases}$$

Then

$$H^1(\mathcal{H}_n) = H^1(\mathcal{H}_n|[0,t_-]) \qquad \text{some } t \in \Gamma_n$$

$$= H^1(\Gamma_n, C_c(\mathbb{R}_\Gamma, \mathbb{Z}))$$

$$\underset{☆}{=} H_{k-2}(\Gamma_n, C_c(\mathbb{R}_\Gamma, \mathbb{Z})) .$$

☆ Poincaré duality

When $k \geq 3$,

$$H^1(\mathcal{H}_n) = H_{k-1}(\Gamma_n) \overset{☆}{=} H^1(\Gamma_n) .$$

The case $k=2$ is done similarly. $\square$

Lem $\mathcal{H} = \mathbb{R}_\Gamma \rtimes \Gamma$ is H^1-rigid.

Proof

- $\mathcal{H} = \bigcup_n \mathcal{H}_n$, $\mathcal{H}_n$ is H^1-rigid when $\operatorname{rank}\Gamma \geq 3$.
- $\Gamma_n \curvearrowright \mathbb{R}_\Gamma$ is minimal ($\because$ $\operatorname{rank}\Gamma_n \geq 2$).
- extra effort when rank $= 2$.

$\square$

Proof of Prop.

Cohomology long exact seq. implies:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & H^1(\Gamma \ltimes_\lambda \mathbb{Z}) \arrow[r] \arrow[d, "H^1(\pi')"] & H^1(\Gamma) \arrow[r, "\mathrm{id}-H^1(\lambda)"] \arrow[d, "H^1(\pi)"', "\cong"] & H^1(\Gamma) \arrow[d, "H^1(\pi)"', "\cong"] \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & H^1(\mathcal{H} \ltimes_\lambda \mathbb{Z}) \arrow[r] & H^1(\mathcal{H}) \arrow[r, "\mathrm{id}-H^1(\lambda)"] & H^1(\mathcal{H})
\end{tikzcd}$$

Hence $H^1(\pi')$ is also isomorphism.

As $\Lambda \cong \mathbb{Z}^N$, we can repeat this argument.

$\square$

Remark

$H^1(\mathcal{S}) \cong H^1(\Gamma \rtimes \Lambda) = \operatorname{Hom}(\Gamma \rtimes \Lambda, \mathbb{Z})$ is isomorphic to $\underbrace{H^1(\Lambda)}_{\cong \mathbb{Z}^N}$ and $\underbrace{H^1(\Gamma)^\Lambda}_{\text{may be non-trivial}}$.

Lem

Suppose a homo. $\xi : \mathcal{S} \to \mathbb{Z}^N$ satisfies :

(1) $\operatorname{Ker} \xi$ admits a unique inv. measure ν
up to scalar multiplication

(2) The essential range of ξ is $\mathbb{Z}^N$

Then $R(\mathcal{S}, \nu) = \Lambda$.

ratio set $:= \{ \lambda \in (0, \infty) \mid \exists g \in \mathcal{S} \ \ g_* d\nu = \lambda \, d\nu \}$

Proof

By Prop. $\exists \, \zeta \in \operatorname{Hom}(\Gamma \rtimes \Lambda, \mathbb{Z}^N)$ s.t.

ξ is cohomologous to $\zeta \circ \pi$. $\quad \pi : \mathcal{S} \to \Gamma \rtimes \Lambda$

By (2), ζ is surjective.

By (1), ζ factors through Λ and $\zeta|\Lambda$ is injective.

Let $\omega : \mathbb{Z}^N \to \Lambda$ be the inverse of $\zeta|\Lambda$.

$\exists f \in C(\mathcal{R}_\Gamma, \mathbb{Z}^N) \quad \xi = \zeta \circ \pi + (f \circ r - f \circ s)$

When μ is an $\mathcal{H}$- inv. measure on $\mathcal{R}_\Gamma$,

$$d\nu(x) := \omega(f(x))^{-1} \, d\mu(x)$$

gives a $\operatorname{Ker} \xi$ - invariant measure. $\square$

Thm. A (M)

f.g. rank ≥ 2

$i = 1, 2 \qquad \mathcal{S}_i = \mathcal{S}(\Gamma_i, \Lambda_i)$

T.F.A.E.

(1) $\mathcal{S}_1 \cong \mathcal{S}_2$

(2) $\Lambda_1 = \Lambda_2$, $\exists s > 0 \quad \Gamma_1 = s\Gamma_2$

Proof

Let $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ be an iso.

May assume $\Lambda_1 \cong \Lambda_2 \cong \mathbb{Z}^N$.

Let $\xi : \mathcal{S}_1 \to \Gamma_1 \rtimes \Lambda_1 \to \Lambda_1 \xrightarrow{\cong} \mathbb{Z}^N$.

Apply the lemma to $\xi \circ \Phi^{-1}$.

$\square$

Thm B follows from Thm A.