Various examples of topological full groups

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Overview

dynamics on Cantor set X

$\downarrow$

$C_r^*(\mathcal{G})$
$K_i(C_r^*(\mathcal{G}))$

$\longleftarrow$

étale groupoid $\mathcal{G}$
with $\mathcal{G}^{(0)} = X$

$\longrightarrow$

topological
full group
$[[\mathcal{G}]] \subset \mathrm{Homeo}(X)$

$\downarrow$

$K_i(C_r^*(\mathcal{G}))$
vs. $H_n(\mathcal{G})$

homology group of $\mathcal{G}$
$H_n(\mathcal{G})$, $n = 0, 1, 2, \ldots$

index map
$I : [[\mathcal{G}]] \to H_1(\mathcal{G})$
$[[\mathcal{G}]]_{\mathrm{ab}}$ vs. $H_n(\mathcal{G})$

Minimal $\mathbb{Z}$ action

A Cantor set is a compact, metrizable, totally disconnected space with no isolated points.
Any two such spaces are homeomorphic to each other.
The infinite product space $\{0,1\}^{\mathbb{N}}$ is a Cantor set.

Let $\varphi : \mathbb{Z} \curvearrowright X$ be an action on a Cantor set X by homeo.
Assume that φ is minimal, i.e.
$\{\varphi^n(x) \mid n \in \mathbb{Z}\}$ is dense in X for all $x \in X$.

$$[[\mathcal{G}_\varphi]] := \{\gamma \in \mathrm{Homeo}(X) \mid \exists \text{conti. } c : X \to \mathbb{Z},\ \gamma(x) = \varphi^{c(x)}(x)\}$$

is called the topological full group of φ.

TFG of minimal $\mathbb{Z}$ action

Let $\varphi, \psi : \mathbb{Z} \curvearrowright X$ be minimal $\mathbb{Z}$-actions.

Theorem (Giordano-Putnam-Skau 1999)

- $[[\mathcal{G}_\varphi]] \cong [[\mathcal{G}_\psi]]$ *iff* φ *is conjugate to* ψ *or* ψ^{-1}.
- *There exists a surjective homo.* $I : [[\mathcal{G}_\varphi]] \to \mathbb{Z}$ *(index map).*

Theorem (M 2006)

- $D([[\mathcal{G}_\varphi]])$ *is simple.*
- $[[\mathcal{G}_\varphi]]_{\mathrm{ab}} \cong \mathbb{Z} \oplus H_0(\mathbb{Z}, C(X, \mathbb{Z}_2))$.
- $D([[\mathcal{G}_\varphi]])$ *is finitely generated iff* φ *is expansive.*

Theorem (Juschenko-Monod 2013)

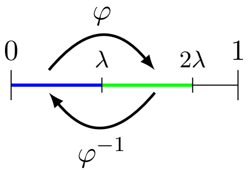
$[[\mathcal{G}_\varphi]]$ *is amenable.*

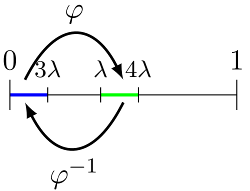
Example of minimal $\mathbb{Z}$ action

Let $0 < \lambda < 1$ be an irrational number and let X be the Cantor set obtained by cutting $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ at the points $n\lambda$, $n \in \mathbb{Z}$.
Let $\varphi : X \to X$ be the translation by λ.

$D([[\mathcal{G}_\varphi]])$ is simple, and $[[\mathcal{G}_\varphi]]_{\mathrm{ab}} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

For simplicity, we assume $1/3 < \lambda < 1/2$.

$\sigma :=$ 

$\tau :=$ 

Then, $[[\mathcal{G}_\varphi]]$ is generated by φ, σ, τ.

$[[\mathcal{G}_\varphi]]$ is amenable.

TFG of minimal $\mathbb{Z}^N$ action

Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a free minimal action and consider $\mathcal{G}_\varphi$.

Theorem (M 2012, 2015)

- *The index map* $I : [[\mathcal{G}_\varphi]] \to H_1(\mathcal{G}_\varphi)$ *is surjective.*
- *There exists an exact sequence:*

$$H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2 \longrightarrow [[\mathcal{G}_\varphi]]_{\mathrm{ab}} \xrightarrow{I} H_1(\mathcal{G}_\varphi) \longrightarrow 0$$

- $D([[\mathcal{G}_\varphi]])$ *is simple.*

Theorem (Elek-Monod 2013)

$[[\mathcal{G}_\varphi]]$ *is sometimes amenable and sometimes NOT.*

Theorem (Nekrashevych 2019)

$D([[\mathcal{G}_\varphi]])$ *is finitely generated iff* φ *is expansive.*

Orbit equivalence

$\varphi : \Gamma \curvearrowright X$ and $\psi : \Lambda \curvearrowright Y$ are said to be orbit equivalent,
if $\exists$ homeo. $h : X \to Y$ such that

$$h(\varphi\text{-orbit of } x) = \psi\text{-orbit of } h(x)$$

holds for all $x \in X$.

Theorem (Giordano-M-Putnam-Skau 2010)

Let $\varphi : \mathbb{Z}^M \curvearrowright X$ and $\psi : \mathbb{Z}^N \curvearrowright Y$ be minimal actions on Cantor sets. T.F.A.E.

1. *φ and ψ are orbit equivalent.*
2. *$\exists$ homeo. $h : X \to Y$ such that*
   $h_*(\{\varphi\text{-inv. prob. measure}\}) = \{\psi\text{-inv. prob. measure}\}$.

Remark

$\mathcal{G}_\varphi \cong \mathcal{G}_\psi$ iff φ and ψ are orbit equivalent with continuous cocycles.

Étale groupoid

A groupoid $\mathcal{G}$ is a 'group-like' algebraic object, in which the product may not be defined for all pairs in $\mathcal{G}$.

- $g \in \mathcal{G}$ is thought of as an arrow $\bullet \xleftarrow{g} \bullet$.
- $r : g \mapsto gg^{-1}$ is called the range map.
- $s : g \mapsto g^{-1}g$ is called the source map.
- $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G}) \subset \mathcal{G}$ is called the unit space.

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$\mathcal{G}$ is an étale groupoid if $\mathcal{G}$ is equipped with a locally compact Hausdorff topology compatible with the groupoid structure and the range (or source) map is a local homeomorphism.
An arrow $\bullet \xleftarrow{g} \bullet$ is thought of as a germ at $s(g) = g^{-1}g$.

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In what follows, we assume that $\mathcal{G}^{(0)}$ is a Cantor set.

Example of étale groupoid

Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a Cantor set.
$\mathcal{G}_\varphi = \Gamma \times X$ with the product topology is an étale groupoid with

$$(\gamma', \varphi_\gamma(x)) \cdot (\gamma, x) = (\gamma'\gamma, x), \quad (\gamma, x)^{-1} = (\gamma^{-1}, \varphi_\gamma(x)).$$

$\mathcal{G}_\varphi$ is called the transformation groupoid.

Thus, (γ, x) is $\varphi_\gamma(x) \bullet \longleftarrow \bullet\, x$

The unit space $\mathcal{G}_\varphi^{(0)} = \{1\} \times X$ is identified with X.

Topological full group

A compact open set $U \subset \mathcal{G}$ is called a bisection
if both $r|U$ and $s|U$ are injective.
The topological full group $[[\mathcal{G}]]$ is defined by

$$[[\mathcal{G}]] = \left\{ \gamma \in \mathrm{Homeo}(\mathcal{G}^{(0)}) \mid \exists \text{bisection } U,\ \gamma = (r|U) \circ (s|U)^{-1} \right\}.$$

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When $\varphi : \Gamma \curvearrowright X$ is a group action on a Cantor set X, $\mathcal{G}_\varphi = \Gamma \times X$ becomes an étale groupoid in a natural way.
In this situation,

$$\gamma \in [[\mathcal{G}_\varphi]] \iff \exists \text{conti. map } c : X \to \Gamma,\ \gamma(x) = \varphi_{c(x)}(x).$$

Groupoid C^*-algebra

For an étale groupoid $\mathcal{G}$, the space $C_c(\mathcal{G}, \mathbb{C})$ of compactly supported continuous functions becomes a $*$-algebra by

$$(f_1 \cdot f_2)(g) = \sum_{h \in \mathcal{G}} f_1(gh) f_2(h^{-1}),$$

$$f^*(g) = \overline{f(g^{-1})}.$$

As a completion by a suitable norm,
we get a (reduced) groupoid C^*-algebra $C_r^*(\mathcal{G})$.

$C_r^*(\mathcal{G})$ contains the abelian subalgebra $C(\mathcal{G}^{(0)})$.
It is maximal, and its unitary normalizers generate $C_r^*(\mathcal{G})$.
Such a subalgebra $C(\mathcal{G}^{(0)})$ is called a Cartan subalgebra.

Isomorphism theorem

Theorem

For minimal groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$, the following are equivalent.

1. *$\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$ as an étale groupoid.*
2. *$[[\mathcal{G}_1]]$ is isomorphic to $[[\mathcal{G}_2]]$ as a group.*
3. *$D([[\mathcal{G}_1]])$ is isomorphic to $D([[\mathcal{G}_2]])$ as a group.*
4. *There exists an isomorphism $\pi : C_r^*(\mathcal{G}_1) \to C_r^*(\mathcal{G}_2)$ such that $\pi(C(\mathcal{G}_1^{(0)})) = C(\mathcal{G}_2^{(0)})$.*

Thus, $[[\mathcal{G}]]$ (or $D([[\mathcal{G}]])$) 'remembers' $\mathcal{G}$.

Homology group

$H_n(\mathcal{G})$ are the homology groups of the chain complex

$$0 \longleftarrow C(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots,$$

where $\mathcal{G}^{(n)}$ is the space of composable strings of n elements.

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where $\mathcal{G}^{(n)}$ is the space of composable strings of n elements.

$$\delta_1(f)(x) = \sum_{s(g)=x} f(g) - \sum_{r(g)=x} f(g)$$

So,

$$H_0(\mathcal{G}) = C(\mathcal{G}^{(0)}, \mathbb{Z}) / \langle 1_{s(U)} - 1_{r(U)} \mid U \text{ is a bisection} \rangle.$$

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If U is a bisection such that $s(U) = r(U) = \mathcal{G}^{(0)}$, then $1_U \in \operatorname{Ker} \delta_1$. Hence, one can define the index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$.

When $\mathcal{G} = \mathcal{G}_\varphi$, the homology groups $H_n(\mathcal{G}_\varphi)$ are canonically isomorphic to the group homology $H_n(\Gamma, C(X, \mathbb{Z}))$.

Simplicity of commutator subgroup

Theorem (M 2012, 2015)

Let $\mathcal{G}$ be a minimal étale groupoid which is either almost finite or purely infinite.

1. $D([[\mathcal{G}]])$ *is simple.*
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The transformation groupoid $\mathcal{G}_\varphi$ of $\varphi : \Gamma \curvearrowright X$ is almost finite when Γ is finitely generated and has polynomial growth.
If $\mathcal{G}$ is almost finite, there exists an invariant prob. measure.

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If $\mathcal{G}$ is almost finite, there exists an invariant prob. measure.

$\mathcal{G}$ is said to be purely infinite, if every clopen subset $A \subset \mathcal{G}^{(0)}$ admits a paradoxical decomposition: $\exists$ bisections $U, V \subset \mathcal{G}$ such that $s(U) = s(V) = A$ and $r(U) \sqcup r(V) \subset A$.

One-sided shifts of finite type (1/2)

Let $(\mathcal{V}, \mathcal{E})$ be an irreducible finite directed graph and let A be the adjacency matrix.
Set

$$X = \{(x_n)_n \in \mathcal{E}^{\mathbb{N}} \mid t(x_n) = i(x_{n+1}) \quad \forall n \in \mathbb{N}\},$$

The one-sided shift σ on X is called a shift of finite type (SFT).

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The SFT groupoid $\mathcal{G}_A$ of (X, σ) (or of A) is

$$\mathcal{G}_A = \left\{(x, k{-}l, y) \in X \times \mathbb{Z} \times X \mid \exists k, l \in \mathbb{N},\ \sigma^k(x){=}\sigma^l(y)\right\}$$

with the product $(x, m, y) \cdot (y, n, z) := (x, m{+}n, z)$.
It is known $H_0(\mathcal{G}_A) \cong \mathrm{Coker}(\mathrm{id} - A^t)$, $H_1(\mathcal{G}_A) \cong \mathrm{Ker}(\mathrm{id} - A^t)$ and $H_n(\mathcal{G}_A) = 0$ for $n \geq 2$. (M 2012)

Higman-Thompson groups

In 1965 R. Thompson gave the first example of a finitely presented infinite simple group. G. Higman and K. S. Brown later generalized it to infinite families $F_n \subset T_n \subset V_n$ for $n \in \mathbb{N} \setminus \{1\}$.

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The group V_n consists of PL right continuous bijections $f : [0,1) \to [0,1)$ with finitely many singularities, all in $\mathbb{Z}[1/n]$, slopes lying in powers of n, and mapping $\mathbb{Z}[1/n] \cap [0,1)$ to itself. V_n is called the Higman-Thompson group.

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F_n is a subgroup of V_n consisting of continuous maps f.
F_n is also finitely presented.
It is not yet known if F_n is amenable or not.

Nekrashevych's observation

Theorem (Nekrashevych 2004)

When (X, σ) is the full shift over n symbols,
the topological full group $[[\mathcal{G}_n]]$ is isomorphic to V_n.

Let $\mathcal{V} = \{*\}$ and $\mathcal{E} = \{0, 1, \ldots, n-1\}$.
The continuous map $\rho : \mathcal{E}^{\mathbb{N}} \to [0, 1]$ defined by

$$\rho((x_k)_k) = \sum_{k=1}^{\infty} \frac{x_k}{n^k}$$

induces the isomorphism $[[\mathcal{G}_n]] \cong V_n$.

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$[[\mathcal{G}_A]]$ for general SFT groupoids $\mathcal{G}_A$ may be thought of as a generalization of the Higman-Thompson group V_n.

One-sided shifts of finite type (2/2)

Let A be the adjacency matrix of
an irreducible finite directed graph $(\mathcal{V}, \mathcal{E})$.

Theorem (Matsumoto-M 2014)

The triple $(\mathrm{Coker}(\mathrm{id} - A^t), [u_A], \det(\mathrm{id} - A))$ *is a complete invariant for the isomorphism class of* $\mathcal{G}_A$ *within SFT groupoids.*

Theorem (M 2015)

- $D([[\mathcal{G}_A]])$ *is simple.*
- $[[\mathcal{G}_A]]_{\mathrm{ab}} \cong H_1(\mathcal{G}_A) \oplus (H_0(\mathcal{G}_A) \otimes \mathbb{Z}_2)$.
- $[[\mathcal{G}_A]]$ *is of type* F_∞ *(in particular, finitely presented).*
- $[[\mathcal{G}_A]]$ *has the Haagerup property.*

Boundary action of the free group

Let $F_2 := \langle a, b \rangle$ be the free group and let $S := \{a, b, a^{-1}, b^{-1}\}$.
The hyperbolic boundary of F_2 is

$$\partial F_2 := \Big\{ (x_n)_n \in S^{\mathbb{N}} \mid \{x_n, x_{n+1}\} \neq \{a, a^{-1}\},\ \{b, b^{-1}\} \quad \forall n \in \mathbb{N} \Big\}.$$

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Let $\varphi : F_2 \curvearrowright \partial F_2$ be the boundary action.

Then, $\mathcal{G}_\varphi$ is canonically isomorphic to the SFT groupoid $\mathcal{G}_A$ with

$$A := \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

In particular, $H_0(\mathcal{G}_\varphi) \cong H_1(\mathcal{G}_\varphi) \cong \mathbb{Z}^2$ and $H_n(\mathcal{G}_\varphi) = 0$ for $n \geq 2$.

Cleary's group (1/2)

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K-groups of $C^*_r(\mathcal{G}_\lambda)$ were computed for many values of λ
by Carey-Phillips-Putnam-Rennie (2011).

Cleary's group (2/2)

Theorem

1. *When* $\lambda > 0$ *satisfies* $\lambda^2 + n\lambda - 1 = 0$, $n \in \mathbb{N}$,

$$H_0(\mathcal{G}_\lambda) = \mathbb{Z}_n, \quad H_1(\mathcal{G}_\lambda) = \mathbb{Z}_2, \quad H_k(\mathcal{G}_\lambda) = 0 \text{ for } k \geq 2.$$

2. *When* $\lambda > 0$ *satisfies* $\lambda^2 - n\lambda + 1 = 0$, $n \in \mathbb{N} \setminus \{1, 2\}$,

$$H_0(\mathcal{G}_\lambda) = \mathbb{Z}_{n-2}, \quad H_1(\mathcal{G}_\lambda) = \mathbb{Z}, \quad H_2(\mathcal{G}_\lambda) = \mathbb{Z},$$
$$H_k(\mathcal{G}_\lambda) = 0 \text{ for } k \geq 3.$$

In both cases, we have

$$\bigoplus_n H_{2n+i}(\mathcal{G}_\lambda) \cong K_i(C_r^*(\mathcal{G}_\lambda)) \quad i = 0, 1$$

and

$$[[\mathcal{G}_\lambda]]_{\mathrm{ab}} \cong (H_0(\mathcal{G}_\lambda) \otimes \mathbb{Z}_2) \oplus H_1(\mathcal{G}_\lambda).$$

Simple periodic group (1/2)

Let $\tau \in \mathrm{Homeo}(X)$ be an involution, i.e. $\tau^2 = \mathrm{id}$.
A finite subgroup $A \subset \mathrm{Homeo}(X)$ is called a fragmentation of τ if the following hold:

- $\forall x \in X$, $\forall h \in A$, one has $h(x) = x$ or $h(x) = \tau(x)$,
- $\forall x \in X$, $\exists h \in A$ such that $h(x) = \tau(x)$.

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Let $\lambda := (\sqrt{5} - 1)/2$ and consider the Cantor set X obtained by cutting $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ at the points $n\lambda$, $n \in \mathbb{Z}$.
Define involutions $\tau, \sigma \in \mathrm{Homeo}(X)$ by

$$\tau(x) := \lambda - x, \quad \sigma(x) := 1 - x.$$

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Define involutions $\tau, \sigma \in \mathrm{Homeo}(X)$ by

$$\tau(x) := \lambda - x, \quad \sigma(x) := 1 - x.$$

Then, $\tau \circ \sigma$ is the translation by λ, which induces a minimal $\mathbb{Z}$-action on X.

Simple periodic group (2/2)

Theorem (Nekrashevych 2018)

There exist fragmentations A and B of τ and σ, respectively, such that $F := \langle A \cup B \rangle$ satisfies the following.

- *There exists a non-Hausdorff étale groupoid $\mathcal{G}$ on X such that $F = [[\mathcal{G}]]$.*
- *F is periodic and has subexponential growth.*
- *$D(F)$ is simple and $F_{\mathrm{ab}} \cong (\mathbb{Z}_2)^9$.*

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The involution σ has one fixed point x_0 corresponding to $1/2 \in [0,1]$. The fragmentation B above has the following property: for every $h \in B$, the closure of the interior of $\mathrm{Fix}(h)$ contains x_0.
The fragmentation A of τ also has the same property.

HK conjecture

In many cases, we have

$$\bigoplus_n H_{2n+i}(\mathcal{G}) \cong K_i(C_r^*(\mathcal{G})) \quad i = 0, 1.$$

Does this always hold true?

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Theorem (Proietti-Yamashita, arXiv:2020)

Let $\mathcal{G}$ be an étale groupoid with torsion-free stabilizers satisfying the strong Baum-Connes conjecture.
Then there exists a convergent spectral sequence

$$E^2_{p,q} = H_p(\mathcal{G}, K_q(\mathbb{C})) \implies K_{p+q}(C_r^*(\mathcal{G})).$$