On the homology groups of totally disconnected étale groupoids

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Overview

dynamical systems on X

$\downarrow$

étale groupoid $\mathcal{G}$
with unit space $\mathcal{G}^{(0)} = X$

$\longrightarrow$

(when X is 0-dim.)
homology groups of $\mathcal{G}$
$H_n(\mathcal{G})$, $n = 0, 1, 2, \ldots$

$\downarrow$

groupoid C^*-algebra
$C_r^*(\mathcal{G})$

$\longrightarrow$

K-groups
$K_i(C_r^*(\mathcal{G}))$, $i = 0, 1$

Étale groupoid

A groupoid $\mathcal{G}$ is a 'group-like' algebraic object, in which the product may not be defined for all pairs in $\mathcal{G}$.

- $g \in \mathcal{G}$ is thought of as an arrow $\bullet \xleftarrow{g} \bullet$.
- $r : g \mapsto gg^{-1}$ is called the range map.
- $s : g \mapsto g^{-1}g$ is called the source map.
- $\mathcal{G}^{(0)} = r(\mathcal{G}) = s(\mathcal{G}) \subset \mathcal{G}$ is called the unit space.

$\mathcal{G}$ is an étale groupoid if $\mathcal{G}$ is equipped with a locally compact Hausdorff topology compatible with the groupoid structure and the range (or source) map is a local homeomorphism.
An arrow $\bullet \xleftarrow{g} \bullet$ is thought of as a germ at $s(g) = g^{-1}g$.

Example of étale groupoid

Let $\varphi : \Gamma \curvearrowright X$ be an action of a discrete group Γ on a locally compact Hausdorff space X.
$\mathcal{G}_\varphi = \Gamma \times X$ is an étale groupoid with

$$(\gamma', \varphi_\gamma(x)) \cdot (\gamma, x) = (\gamma'\gamma, x), \quad (\gamma, x)^{-1} = (\gamma^{-1}, \varphi_\gamma(x)).$$

$\mathcal{G}_\varphi$ is called the transformation groupoid.
Thus, (γ, x) is $\varphi_\gamma(x) \bullet \longleftarrow \bullet\, x$
The unit space $\mathcal{G}_\varphi^{(0)} = \{1\} \times X$ is identified with X.

The groupoid C^*-algebra $C_r^*(\mathcal{G}_\varphi)$ is canonically isomorphic to the crossed product $C_0(X) \rtimes_{r,\varphi} \Gamma$.

When X is totally disconnected, the homology groups $H_n(\mathcal{G}_\varphi)$ are canonically isomorphic to the group homology $H_n(\Gamma, C_c(X, \mathbb{Z}))$.

Groupoid C^*-algebra

For an étale groupoid $\mathcal{G}$, the space $C_c(\mathcal{G}, \mathbb{C})$ of compactly supported continuous functions becomes a $*$-algebra by

$$(f_1 \cdot f_2)(g) = \sum_{h \in \mathcal{G}} f_1(gh) f_2(h^{-1}),$$

$$f^*(g) = \overline{f(g^{-1})}.$$

As a completion by a suitable norm,
we get a (reduced) groupoid C^*-algebra $C_r^*(\mathcal{G})$.

$C_r^*(\mathcal{G})$ contains the abelian subalgebra $C_0(\mathcal{G}^{(0)})$.
It is maximal, and its unitary normalizers generate $C_r^*(\mathcal{G})$.
Such a subalgebra $C_0(\mathcal{G}^{(0)})$ is called a Cartan subalgebra.

Homology group

Suppose that $\mathcal{G}^{(0)}$ is totally disconnected.
$H_n(\mathcal{G})$ are the homology groups of the chain complex

$$0 \longleftarrow C_c(\mathcal{G}^{(0)}, \mathbb{Z}) \xleftarrow{\delta_1} C_c(\mathcal{G}^{(1)}, \mathbb{Z}) \xleftarrow{\delta_2} C_c(\mathcal{G}^{(2)}, \mathbb{Z}) \xleftarrow{\delta_3} \dots,$$

where $\mathcal{G}^{(n)}$ is the space of composable strings of n elements:

$$\mathcal{G}^{(n)} = \{(g_1, g_2, \dots, g_n) \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \quad \forall i\}.$$

We denote by $\mathcal{C}(\mathcal{G})$ the chain complex above.

For a transformation groupoid $\mathcal{G}_\varphi$ arising from $\varphi : \Gamma \curvearrowright X$,
we have $H_n(\mathcal{G}_\varphi) = H_n(\Gamma, C_c(X, \mathbb{Z}))$.

References

- I. F. Putnam, An excision theorem for the K-theory of C^*-algebras. J. Operator Theory 38 (1997), 151–171.
- I. F. Putnam, On the K-theory of C^*-algebras of principal groupoids. Rocky Mountain J. Math. 28 (1998), 1483–1518.
- H. Matui, in preparation.

Putnam's Thm for factor groupoid (1/2)

Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids, and let $i_0, i_1 : \mathcal{H} \to \mathcal{G}$ be two continuous injective homomorphisms with disjoint images. We assume:

- $i_j(\mathcal{H}^{(0)}) \subset \mathcal{G}^{(0)}$ is $\mathcal{G}$-invariant and $\mathcal{G}|i_j(\mathcal{H}^{(0)}) = i_j(\mathcal{H})$.
- For any $f \in C_0(\mathcal{G}, \mathbb{R})$, the function $\iota(f) : \mathcal{H} \to \mathbb{R}$ defined by

$$\iota(f)(g) = f(i_0(g)) - f(i_1(g)) \quad g \in \mathcal{H}$$

  belongs to $C_0(\mathcal{H}, \mathbb{R})$.
- $\mathcal{G}' := \mathcal{G}/\langle i_0(g) \sim i_1(g) \mid g \in \mathcal{H} \rangle$ is a locally compact and Hausdorff groupoid.
- The quotient map $\pi : \mathcal{G} \to \mathcal{G}'$ is proper.

One can prove that $\mathcal{G}'$ is an étale groupoid.

Putnam's Thm for factor groupoid (2/2)

Theorem (Putnam 1998)

There exists a six-term exact sequence:

$$\begin{array}{ccccc} K_1(C_r^*(\mathcal{H})) & \longrightarrow & K_0(C_r^*(\mathcal{G}')) & \xrightarrow{K_0(\pi^*)} & K_0(C_r^*(\mathcal{G})) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\mathcal{G})) & \xleftarrow{K_1(\pi^*)} & K_1(C_r^*(\mathcal{G}')) & \longleftarrow & K_0(C_r^*(\mathcal{H})), \end{array}$$

where $\pi^* : C_r^*(\mathcal{G}') \to C_r^*(\mathcal{G})$ *is the inclusion map.*

The vertical arrows are induced by an element in $KK(C_r^*(\mathcal{G}), C_r^*(\mathcal{H}))$. (Hint: $C_r^*(\mathcal{G})$ acts on the Hilbert module $C_r^*(\mathcal{H}) \oplus C_r^*(\mathcal{H})$ by left multiplication.)

Homology group version (1/2)

Assume that $\mathcal{H}, \mathcal{G}, \mathcal{G}'$ are totally disconnected.

Theorem (M)

There exists a long exact sequence of homology groups

$$\begin{aligned} \dots \longrightarrow\ & H_n(\mathcal{G}') \xrightarrow{H_n^*(\pi)} H_n(\mathcal{G}) \xrightarrow{\iota} H_n(\mathcal{H}) \\ \longrightarrow\ & H_{n-1}(\mathcal{G}') \xrightarrow{H_{n-1}^*(\pi)} H_{n-1}(\mathcal{G}) \xrightarrow{\iota} H_{n-1}(\mathcal{H}) \dots \end{aligned}$$

The theorem above is a direct consequence of the following exact sequence of chain complexes:

$$0 \longrightarrow \mathcal{C}(\mathcal{G}') \xrightarrow{\pi^*} \mathcal{C}(\mathcal{G}) \xrightarrow{\iota^*} \mathcal{C}(\mathcal{H}) \longrightarrow 0.$$

Homology group version (2/2)

$$0 \longrightarrow \mathcal{C}(\mathcal{G}') \xrightarrow{\pi^*} \mathcal{C}(\mathcal{G}) \xrightarrow{\iota^*} \mathcal{C}(\mathcal{H}) \longrightarrow 0$$

We assumed $\pi : \mathcal{G} \to \mathcal{G}'$ is proper, and so $\pi^{(n)} : \mathcal{G}^{(n)} \to \mathcal{G}'^{(n)}$ is also proper. It follows that

$$(\pi^{(n)})^* : C_c(\mathcal{G}'^{(n)}, \mathbb{Z}) \to C_c(\mathcal{G}^{(n)}, \mathbb{Z})$$

is well-defined, and we get the chain map π^*.

The homomorphisms $i_j : \mathcal{H} \to \mathcal{G} \quad (j = 0, 1)$ naturally induce $i_j^{(n)} : \mathcal{H}^{(n)} \to \mathcal{G}^{(n)}$. We can prove that

$$\iota^{(n)}(f)(\xi) := f(i_0^{(n)}(\xi)) - f(i_1^{(n)}(\xi)) \quad \forall \xi \in \mathcal{H}^{(n)}.$$

gives rise to a well-defined homomorphism

$$\iota^{(n)} : C_c(\mathcal{G}^{(n)}, \mathbb{Z}) \to C_c(\mathcal{H}^{(n)}, \mathbb{Z}).$$

Clearly $\iota^* = (\iota^{(n)})_n$ is a chain map.

Putnam's Thm for subgroupoid (1/2)

Let $\mathcal{G}$ be an étale groupoid and let $\mathcal{G}' \subset \mathcal{G}$ be an open subgroupoid with $\mathcal{G}'^{(0)} = \mathcal{G}^{(0)}$. Assume that a closed subset $L \subset \mathcal{G}$ satisfies the following.

- $\mathcal{G}$ is the disjoint union of $\mathcal{G}'$, L and L^{-1}.
- $r(L)$ and $s(L)$ are disjoint.
- $L\mathcal{G}' \subset L$ and $\mathcal{G}'L \subset L$.

Define a groupoid $\mathcal{H}$ by

$$\mathcal{H} := \mathcal{G}|(r(L) \cup s(L)) = (\mathcal{G}|r(L)) \cup (\mathcal{G}|s(L)) \cup L \cup L^{-1}.$$

With a suitable new topology, $\mathcal{H}$ becomes an étale groupoid.

Putnam's Thm for subgroupoid (2/2)

Theorem (Putnam 1998)

There exists a six-term exact sequence:

$$\begin{array}{ccccc} K_0(C_r^*(\mathcal{H})) & \longrightarrow & K_0(C_r^*(\mathcal{G}')) & \xrightarrow{K_0(\alpha)} & K_0(C_r^*(\mathcal{G})) \\ \uparrow & & & & \downarrow \\ K_1(C_r^*(\mathcal{G})) & \xleftarrow{K_1(\alpha)} & K_1(C_r^*(\mathcal{G}')) & \longleftarrow & K_1(C_r^*(\mathcal{H})), \end{array}$$

where $\alpha : C_r^*(\mathcal{G}') \to C_r^*(\mathcal{G})$ *is the inclusion map.*

The vertical arrows are induced by an element in $KK^1(C_r^*(\mathcal{G}), C_r^*(\mathcal{H}))$. (Hint: $C_r^*(\mathcal{G})$ acts on the Hilbert module $C_r^*(\mathcal{H})$ by left multiplication, and there exists a suitable self-adjoint unitary $z \in \mathcal{L}(C_r^*(\mathcal{H}))$.)

Homology group version (1/2)

Assume that $\mathcal{H}, \mathcal{G}, \mathcal{G}'$ are totally disconnected.

Theorem (M)

There exists a long exact sequence of homology groups

$$\begin{array}{ccccccc} \ldots \longrightarrow & H_n(\mathcal{H}) & \longrightarrow & H_n(\mathcal{G}') & \longrightarrow & H_n(\mathcal{G}) & \\ \longrightarrow & H_{n-1}(\mathcal{H}) & \longrightarrow & H_{n-1}(\mathcal{G}') & \longrightarrow & H_{n-1}(\mathcal{G}) & \ldots \end{array}$$

In order to prove the theorem above, we set

$$\mathcal{H}' := \mathcal{H} \cap \mathcal{G}' = \mathcal{H} \setminus (L \cup L^{-1}) = (\mathcal{G}|r(L)) \cup (\mathcal{G}|s(L)),$$

which is an open subgroupoid of

$$\mathcal{H} = \mathcal{G}|(r(L) \cup s(L)) = (\mathcal{G}|r(L)) \cup (\mathcal{G}|s(L)) \cup L \cup L^{-1}.$$

Homology group version (2/2)

As sets,

$$\mathcal{G}^{(n)} \setminus \mathcal{G}'^{(n)} = \{(g_1, \ldots, g_n) \in \mathcal{G}^{(n)} \mid \exists k,\ g_k \in L \cup L^{-1}\}$$
$$= \{(g_1, \ldots, g_n) \in \mathcal{H}^{(n)} \mid \exists k,\ g_k \in L \cup L^{-1}\} = \mathcal{H}^{(n)} \setminus \mathcal{H}'^{(n)}.$$

Lemma

$\mathcal{G}^{(n)} \setminus \mathcal{G}'^{(n)}$ is homeomorphic to $\mathcal{H}^{(n)} \setminus \mathcal{H}'^{(n)}$. In particular,

$$C_c(\mathcal{G}^{(n)}, \mathbb{Z})/C_c(\mathcal{G}'^{(n)}, \mathbb{Z}) \cong C_c(\mathcal{H}^{(n)}, \mathbb{Z})/C_c(\mathcal{H}'^{(n)}, \mathbb{Z}).$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{C}(\mathcal{G}') & \longrightarrow & \mathcal{C}(\mathcal{G}) & \longrightarrow & \mathcal{C}(\mathcal{G})/\mathcal{C}(\mathcal{G}') & \longrightarrow & 0 \\ & & & & & & \| & & \\ 0 & \longrightarrow & \mathcal{C}(\mathcal{H}') & \longrightarrow & \mathcal{C}(\mathcal{H}) & \longrightarrow & \mathcal{C}(\mathcal{H})/\mathcal{C}(\mathcal{H}') & \longrightarrow & 0 \end{array}$$

Since $\mathcal{H}'$ is similar to $\mathcal{H} \oplus \mathcal{H}$, we get the conclusion.

Example I (factor groupoid)

Let X be a Cantor set and let $\varphi : \Gamma \curvearrowright X$ be a free action.
Suppose that two points $x_0, x_1 \in X$ satisfy

$$\lim_{\gamma\to\infty} \mathrm{dist}(\varphi_\gamma(x_0), \varphi_\gamma(x_1)) = 0.$$

Let $\mathcal{H} = \Gamma \times \Gamma$ be the groupoid of the left translation $\Gamma \curvearrowright \Gamma$.
One has $C_r^*(\mathcal{H}) \cong \mathcal{K}(\ell^2(\Gamma))$.
Define two homomorphisms $i_j : \mathcal{H} \to \mathcal{G}_\varphi \quad (j = 0, 1)$ by

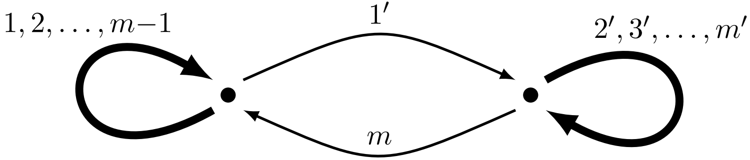
$$i_j(\gamma, \gamma') := (\gamma, \varphi_{\gamma'}(x_j)).$$

Then $\mathcal{G}' := \mathcal{G}_\varphi / \langle i_0(g) \sim i_1(g) \mid g \in \mathcal{H} \rangle$ is an étale groupoid.

By the theorem, we get

$$H_n(\mathcal{G}_\varphi) = H_n(\mathcal{G}') \quad n \geq 1, \quad H_0(\mathcal{G}_\varphi) = H_0(\mathcal{G}') \oplus \mathbb{Z}.$$

Example II (factor groupoid) (1/3)



Consider the above graph whose adjacency matrix is

$$A := \begin{bmatrix} m-1 & 1 \\ 1 & m-1 \end{bmatrix}.$$

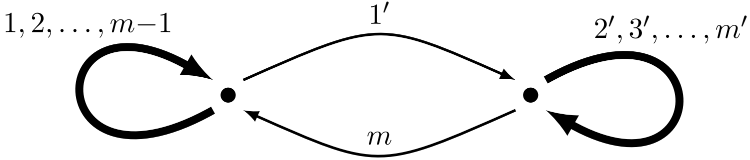
Let X be the one-sided infinite path space.

$$\mathcal{G} := \{((x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}) \in X \times X \mid x_k = y_k \text{ eventually}\}$$

becomes an AF groupoid. We have $H_n(\mathcal{G}) = 0$ for $n \geq 1$ and

$$H_0(\mathcal{G}) = \varinjlim \left(A : \mathbb{Z}^2 \to \mathbb{Z}^2\right).$$

Example II (factor groupoid) (2/3)



Define $Y, Y' \subset X$ by

$$Y := \{(x_k)_k \in X \mid x_k \in \{2, 3, m-1\} \text{ eventually}\},$$

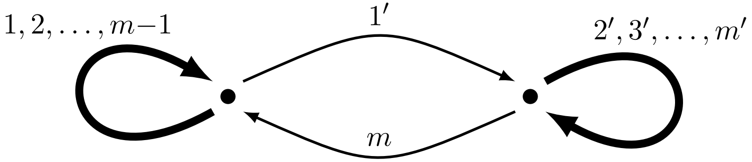
$$Y' := \left\{(x_k)_k \in X \mid x_k \in \{2', 3', (m-1)'\} \text{ eventually}\right\}.$$

There exist an AF groupoid $\mathcal{H}$ with $H_0(\mathcal{H}) \cong \mathbb{Z}[1/(m-2)]$ and injective homomorphisms $i_0 : \mathcal{H} \to \mathcal{G}|Y$ and $i_1 : \mathcal{H} \to \mathcal{G}|Y'$. Then

$$\mathcal{G}' := \mathcal{G}/\langle i_0(g) \sim i_1(g) \mid g \in \mathcal{H}\rangle$$

becomes an AF groupoid such that $H_0(\mathcal{G}') \cong \mathbb{Z}[1/m]$.

Example II (factor groupoid) (3/3)



The long exact sequence

$$\ldots \longrightarrow H_n(\mathcal{G}') \xrightarrow{H_n^*(\pi)} H_n(\mathcal{G}) \xrightarrow{\iota} H_n(\mathcal{H}) \longrightarrow \ldots$$

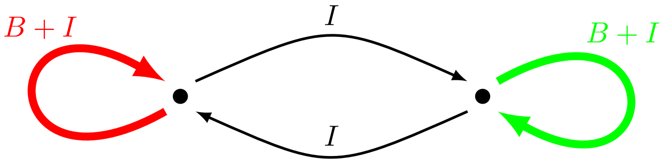
implies

$$0 \longrightarrow \mathbb{Z}\left[\tfrac{1}{m}\right] \longrightarrow H_0(\mathcal{G}) \longrightarrow \mathbb{Z}\left[\frac{1}{m-2}\right] \longrightarrow 0$$

is exact.

Example III (factor groupoid) (1/3)

Modifying the previous example, we consider the graph:



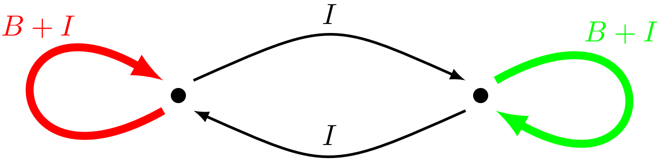
whose adjacency matrix is $A = \begin{bmatrix} B+I & I \\ I & B+I \end{bmatrix}$.
As before, let X be the one-sided infinite path space.

$$\mathcal{G} := \{((x_k)_k, l, (y_k)_k) \in X \times \mathbb{Z} \times X \mid x_{k+l} = y_k \text{ eventually}\}$$

becomes an étale groupoid, called SFT groupoid.
Its homology groups are

$$H_0(\mathcal{G}) \cong \mathrm{Coker}(\mathrm{id} - A),\ H_1(\mathcal{G}) \cong \mathrm{Ker}(\mathrm{id} - A),\ H_n(\mathcal{G}) = 0\ (n \geq 2).$$

Example III (factor groupoid) (2/3)



As before, we define $Y, Y' \subset X$ by

$$Y := \{(x_k)_k \in X \mid x_k \text{ is in } B \text{ eventually}\},$$

$$Y' := \{(x_k)_k \in X \mid x_k \text{ is in } B \text{ eventually}\},$$

We can introduce injective homomorphisms $i_0 : \mathcal{H} \to \mathcal{G}|Y$ and $i_1 : \mathcal{H} \to \mathcal{G}|Y'$, and define

$$\mathcal{G}' := \mathcal{G}/\langle i_0(g) \sim i_1(g) \mid g \in \mathcal{H}\rangle$$

which is the SFT groupoid of the graph corresponding to $B + 2I$.

Example III (factor groupoid) (3/3)

The long exact sequence gives us

$$0 \to \mathrm{Ker}(B+I) \to \mathrm{Ker}\begin{bmatrix} B & I \\ I & B \end{bmatrix} \to \mathrm{Ker}(B-I)$$

$$\to \mathrm{Coker}(B+I) \to \mathrm{Coker}\begin{bmatrix} B & I \\ I & B \end{bmatrix} \to \mathrm{Coker}(B-I) \to 0.$$

For example, when $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, we have

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \\ \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_8 & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}_8 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z}_2 & \longrightarrow 0. \end{array}$$

Example IV (subgroupoid) (1/2)

Let $\varphi : \mathbb{Z} \curvearrowright X$ be a minimal action on a Cantor set X and let $\mathcal{G}_\varphi = \mathbb{Z} \times X$ be the transformation groupoid.
Let $Y \subset X$ be a closed subset such that $Y \cap \varphi^n(Y) = \emptyset$ for all $n \in \mathbb{N}$. Define

$$L := \{(m, \varphi^n(y)) \in \mathcal{G}_\varphi \mid y \in Y,\ n \leq 0 < m+n\},$$

and set $\mathcal{G}' := \mathcal{G}_\varphi \setminus (L \cup L^{-1})$, which is an open subgroupoid of $\mathcal{G}$.

It is known that $\mathcal{G}'$ is an AF groupoid.

Then,

$$\mathcal{H} := \mathcal{G}|(r(L) \cup s(L))$$

is isomorphic to $Y \times \mathbb{Z} \times \mathbb{Z}$, because $Y \cap \varphi^n(Y) = \emptyset$.

Example IV (subgroupoid) (2/2)

In this setting, the long exact sequence

$$\begin{aligned} \dots \longrightarrow H_1(\mathcal{H}) &\longrightarrow H_1(\mathcal{G}') \longrightarrow H_1(\mathcal{G}_\varphi) \\ \longrightarrow H_0(\mathcal{H}) &\longrightarrow H_0(\mathcal{G}') \longrightarrow H_0(\mathcal{G}_\varphi) \longrightarrow 0 \end{aligned}$$

becomes

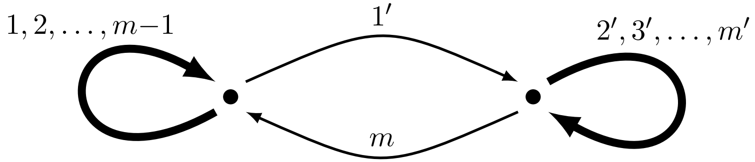
$$\begin{aligned} \dots \longrightarrow \quad 0 \quad &\longrightarrow \quad 0 \quad \longrightarrow \quad \mathbb{Z} \\ \longrightarrow C(Y, \mathbb{Z}) &\longrightarrow H_0(\mathcal{G}') \longrightarrow H_0(\mathcal{G}_\varphi) \longrightarrow 0. \end{aligned}$$

Example V (subgroupoid) (1/2)

Let $X := \{a, b\} \times \{1, 2, \ldots, m\}^{\mathbb{N}}$ and consider the AF groupoid

$$\mathcal{G} := \{((x_k)_k, (y_k)_k) \in X \times X \mid x_k = y_k \text{ eventually}\}.$$

We have $H_0(\mathcal{G}) \cong \mathbb{Z}[1/m]$.



Let $\mathcal{G}'$ be the AF groupoid associated with the graph above, which was discussed in Example II.
By "forgetting the prime symbol", we can obtain a homomorphism from $\mathcal{G}'$ to $\mathcal{G}$ and identify $\mathcal{G}'$ as an open subgroupoid of $\mathcal{G}$.

Example V (subgroupoid) (2/2)

In this setting, the long exact sequence

$$\dots \longrightarrow H_n(\mathcal{H}) \longrightarrow H_n(\mathcal{G}') \longrightarrow H_n(\mathcal{G}) \longrightarrow \dots$$

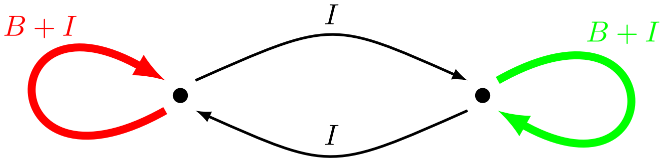
implies

$$0 \longrightarrow \mathbb{Z}\left[\frac{1}{m-2}\right] \longrightarrow H_0(\mathcal{G}') \longrightarrow \mathbb{Z}\left[\frac{1}{m}\right] \longrightarrow 0$$

is exact.

Example VI (subgroupoid) (1/2)

In the same way as in the factor groupoid example, one can generalize the graph of Example V to the graph



and consider the SFT groupoids instead of AF groupoids.

Thus,

$$\mathcal{G} \text{ ``}=\text{''} \text{ SFT groupoid of } B+2I$$

$$\mathcal{G}' = \text{ SFT groupoid of } \begin{bmatrix} B+I & I \\ I & B+I \end{bmatrix}$$

$$\mathcal{H} \text{ ``}=\text{''} \text{ SFT groupoid of } B.$$

Example VI (subgroupoid) (2/2)

The long exact sequence gives us

$$0 \to \mathrm{Ker}(B-I) \to \mathrm{Ker}\begin{bmatrix} B & I \\ I & B \end{bmatrix} \to \mathrm{Ker}(B+I)$$

$$\to \mathrm{Coker}(B-I) \to \mathrm{Coker}\begin{bmatrix} B & I \\ I & B \end{bmatrix} \to \mathrm{Coker}(B+I) \to 0.$$

For example, when $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, we have

$$\begin{aligned} 0 \longrightarrow \quad \mathbb{Z} \quad &\longrightarrow \quad \mathbb{Z}^2 \quad \longrightarrow \quad \mathbb{Z} \\ \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 &\longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}_8 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_8 \longrightarrow 0. \end{aligned}$$