Poly-$\mathbb{Z}$ group actions on Kirchberg algebras III

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Goal

Classify outer actions of poly-$\mathbb{Z}$ groups on Kirchberg algebras up to KK-trivial cocycle conjugacy (as much as possible).

- A C^*-algebra A is called a Kirchberg algebra if A is separable, nuclear, simple and purely infinite.
- A poly-$\mathbb{Z}$ group is a group of the form $(((\mathbb{Z} \rtimes \mathbb{Z}) \rtimes \dots) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}$.
- $\alpha, \beta : G \curvearrowright A$ are said to be KK-trivially cocycle conjugate if $\exists$ α-cocycle $(u_g)_g$, $\exists\ \theta \in \mathrm{Aut}(A)$ such that $KK(\theta) = 1$ and $\mathrm{Ad}\, u_g \circ \alpha_g = \theta \circ \beta_g \circ \theta^{-1}$ holds for all $g \in G$.

Overview

✓① Conjecture and partial answers

✓② Equivariant version of Nakamura's theorem

✓③ Uniqueness of outer G-actions on $\mathcal{O}_\infty$

✓④ Absorption of outer G-actions on $\mathcal{O}_\infty$

⑤ Stability

⑥ Classification

Stability

Theorem (poly-$\mathbb{Z}$ stability)

Let $\alpha : G \curvearrowright A$ be an action of a poly-$\mathbb{Z}$ group G on a unital separable C^*-algebra A, which accepts $(\mathcal{O}_\infty, \beta)$.
Let $I \subset A$ be an ideal. Suppose that a family $(x_g)_{g \in G}$ of continuous maps from $[0,1] \times [0,\infty)$ to $U(I)$ satisfies

$$x_g(0,t) = 1, \quad \lim_{t\to\infty} \max_{s\in[0,1]} \|[x_g(s,t), a]\| = 0 \quad \forall g \in G,\ a \in A,$$

$$\lim_{t\to\infty} \max_{s\in[0,1]} \|x_g(s,t)\alpha_g(x_h(s,t)) - x_{gh}(s,t)\| = 0 \quad \forall g, h \in G.$$

Then there exists a continuous map $y : [0,\infty) \to U(I)$ such that

$$\lim_{t\to\infty} \|[y(t), a]\| = 0 \quad \forall a \in A,$$

$$\lim_{t\to\infty} \|x_g(1,t) - y(t)\alpha_g(y(t)^*)\| = 0 \quad \forall g \in G.$$

Cocycle actions (1/2)

Let A be a unital C^*-algebra and let G be a group.

A pair (α, u) of a map $\alpha : G \to \mathrm{Aut}(A)$ and
a map $u : G \times G \to U(A)$ is called a cocycle action of G on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in G$. We write $(\alpha, u) : G \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in G$.

When α_g is not inner for any $g \in G \setminus \{1\}$,
(α, u) is said to be outer.

When $u \equiv 1$, $\alpha : G \curvearrowright A$ is a genuine action.

Cocycle actions (2/2)

Two cocycle actions $(\alpha, u) : G \curvearrowright A$ and $(\beta, v) : G \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g \in G}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \operatorname{Ad} w_g \circ \beta_g$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^*$$

hold for every $g, h \in G$.

Our eventual goal is

- to classify (α, u) up to (KK-trivial) cocycle conjugacy and
- to determine when (α, u) is cocycle conjugate to a genuine action.

Stability

Let G be a poly-$\mathbb{Z}$ group and fix a finite generating set $S \subset G$. Let $\beta : G \curvearrowright \mathcal{O}_\infty$ be an outer action.

Theorem (H^2-stability)

For any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds: Let $(\sigma, w) : G \curvearrowright A$ be a cocycle action which accepts $(\mathcal{O}_\infty, \beta)$. If $\|w(g,h) - 1\| < \delta$ for all $g, h \in S$, then there exist unitaries $(v_g)_g$ in A such that $\|v_g - 1\| < \varepsilon$ for all $g \in S$ and

$$w(g,h) = \sigma_g(v_h^*) v_g^* v_{gh} \quad \forall g, h \in G,$$

that is, w is a coboundary.

We can prove H^1-stability and H^2-stability by induction on the Hirsch length:
H^2-stability for $G \implies H^1$-stability for G
$\implies H^2$-stability for $G \rtimes \mathbb{Z} \implies H^1$-stability for $G \rtimes \mathbb{Z} \implies \dots$

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Conjecture

Let us recall our conjecture.

Conjecture (Izumi 2010)

Let A be a unital Kirchberg algebra and let G be a poly-$\mathbb{Z}$ group.
Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright A$ be outer actions.
The following are equivalent.

1. α and β are KK-trivially cocycle conjugate.
2. There exists a base point preserving isomorphism between $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$.

The second condition is equivalent to the following:
there exists a continuous map $\Phi : EG \to \mathrm{Aut}(A \otimes \mathbb{K})_0$
such that $\Phi(x_0) = \mathrm{id}$ and $\Phi(g.x) \circ \alpha^s_g = \beta^s_g \circ \Phi(x)$.

Intertwining argument

For a poly-$\mathbb{Z}$ group G, we let $\mu^G : G \curvearrowright \mathcal{O}_\infty$ be an outer action.

Theorem (Intertwining argument)

Let G be a poly-$\mathbb{Z}$ group and let A be a unital separable C^*-algebra. Let $(\alpha, u) : G \curvearrowright A$ and $(\beta, v) : G \curvearrowright A$ be cocycle actions which accept $(\mathcal{O}_\infty, \mu^G)$.
Suppose that there exists a family $(x_g)_g$ of unitaries in $A^\flat$ such that

$$(\mathrm{Ad}\, x_g \circ \alpha_g)(a) = \beta_g(a) \quad \forall a \in A,$$

$$x_g \alpha_g(x_h) u(g,h) x_{gh}^* = v(g,h) \quad \forall g, h \in G.$$

Then (α, u) and (β, v) are cocycle conjugate via an asymptotically inner automorphism.

Asymptotically representable actions

As an immediate corollary, we get the following.

Theorem (Izumi-M)

Let A be a unital Kirchberg algebra and let G be a poly-$\mathbb{Z}$ group. All asymptotically representable outer actions of G on A are mutually KK-trivially cocycle conjugate.

Proof.

Let $\alpha : G \curvearrowright A$ be an asymptotically representable outer action. By definition, there exist $x_g : [0, \infty) \to U(A)$ $(g \in G)$ such that

$$x_g(t)x_h(t) - x_{gh}(t) \to 0, \quad x_g(t)\ a\ x_g(t)^* \to \alpha_g(a).$$

Let $\mu^G : G \curvearrowright \mathcal{O}_\infty$ be an outer action. Then $(x_g \otimes 1)_{g \in G}$ is thought of as a family of unitaries in $(A \otimes \mathcal{O}_\infty)^\flat$. Thanks to the previous theorem, we can conclude that $\alpha \otimes \mu^G$ is KK-trivially cocycle conjugate to $\mathrm{id} \otimes \mu^G$. □

Obstruction (1/4)

Let $\alpha, \beta : G \curvearrowright A$ be outer actions of a poly-$\mathbb{Z}$ group G on a unital Kirchberg algebra A such that $KK(\alpha_g) = KK(\beta_g)$ for all $g \in G$.

For each $g \in G$, there exists $u_g : [0, \infty) \to U(A)$ such that $\mathrm{Ad}\, u_g(t) \circ \alpha_g \to \beta_g$ as $t \to \infty$. Then the unitaries

$$w(g, h) = u_g \alpha_g(u_h) u_{gh}^*$$

belong to $A_\flat = A^\flat \cap A'$. Define $\sigma_g \in \mathrm{Aut}(A_\flat)$ by $\sigma_g = \mathrm{Ad}\, u_g \circ \alpha_g$ for each $g \in G$. Then it is easy to check that (σ, w) is a cocycle action of G on $A_\flat$.

Lemma

If w is a coboundary, α and β are KK-trivially cocycle conjugate.

Proof.

If $\exists\ v_g \in U(A_\flat)$ such that $w(g, h) = v_g \sigma_g(v_h) v_{g,h}^*$, then $(v_g u_g)_g$ is an α-cocycle satisfying $(\mathrm{Ad}\, v_g u_g \circ \alpha_g)(a) = \beta_g(a)$ for $a \in A$. $\square$

Obstruction (2/4)

For outer actions $\alpha, \beta : G \curvearrowright A$ satisfying $KK(\alpha_g) = KK(\beta_g)$, we have chosen $u_g \in A^\flat$ so that $(\mathrm{Ad}\, u_g \circ \alpha_g)|A = \beta_g|A$ and defined a cocycle action $(\sigma, w) : G \curvearrowright A_\flat$ by

$$\sigma_g = (\mathrm{Ad}\, u_g \circ \alpha_g)|A_\flat, \quad w(g,h) = u_g \alpha_g(u_h) u_{gh}^*.$$

We denote the cohomology class of the 2-cocycle $(g,h) \mapsto K_1(w(g,h))$ by

$$o^2(\alpha, \beta) \in H^2(G, K_1(A_\flat)) \cong H^2(G, KK^1(A,A)),$$

which does not depend on the choice of $(u_g)_g$.

Obstruction (3/4)

Suppose $o^2(\alpha, \beta) = 0$.
By replacing $(u_g)_g$ if necessary, we may assume $K_1(w(g,h)) = 0$ for all $g, h \in G$.
Choose a continuous path $\tilde{w}(g,h) : [0,1] \to U(A_\flat)$ from 1 to $w(g,h)$. Then

$$\sigma_g(\tilde{w}(h,k))\tilde{w}(g,hk)\left(\tilde{w}(g,h)\tilde{w}(gh,k)\right)^* \quad g,h,k \in G$$

are unitaries in $S(A_\flat)$, and their K_1-classes form a 3-cocycle in $K_1(S(A_\flat)) = K_0(A_\flat)$.
We denote its cohomology class by

$$o^3(\alpha, \beta, u) \in H^3(G, K_0(A_\flat)) \cong H^3(G, KK(A,A)),$$

which does not depend on the choice of the path $(\tilde{w}(g,h))_{g,h}$, but may (a priori) depend on the choice of $(u_g)_g \subset U(A^\flat)$.

Obstruction (4/4)

For $\alpha, \beta : G \curvearrowright A$,

- if $KK(\alpha_g) = KK(\beta_g)$, then $o^2(\alpha, \beta) \in H^2(G, KK^1(A, A))$ is defined,
- if $o^2(\alpha, \beta) = 0$, then $o^3(\alpha, \beta, u) \in H^3(G, KK(A, A))$ is defined.

Lemma

If there exists a base point preserving isomorphism between $\mathcal{P}^s_\alpha$ and $\mathcal{P}^s_\beta$, then $KK(\alpha_g) = KK(\beta_g)$, $o^2(\alpha, \beta) = 0$ and $o^3(\alpha, \beta, u) = 0$ for some $(u_g)_g$.

Hirsch length two (1/2)

Let G be a poly-$\mathbb{Z}$ group with Hirsch length two and
let $\alpha, \beta : G \curvearrowright A$ be outer actions on a unital Kirchberg algebra.
Assume $KK(\alpha_g) = KK(\beta_g)$ and $o^2(\alpha, \beta) = 0$.
We want to show that α and β are KK-trivially cocycle conjugate.

Let $\sigma_g = \mathrm{Ad}\, u_g \circ \alpha_g \in \mathrm{Aut}(A_\flat)$ and $w(g,h) \in U(A_\flat)$ be as before.
By replacing $(u_g)_g$, we may assume $K_1(w(g,h)) = 0\ \forall g, h \in G$.
Let $G = N \rtimes \langle \xi \rangle$. By assumption, N is $\mathbb{Z}$, and so we may assume that $\sigma|N$ is a genuine action. Then, there exists a $\sigma|N$-cocycle $(v_g)_g$ in $A_\flat$ such that

$$\sigma_\xi \circ \sigma_{\xi^{-1} g \xi} \circ \sigma_\xi^{-1} = \mathrm{Ad}\, v_g \circ \sigma_g \quad \forall g \in N,$$

and $K_1(v_g) = 0$. Since $N = \mathbb{Z}$, the $\sigma|N$-cocycle $(v_g)_g$ can be approximated by coboundaries. By the H^2-stability, we can conclude that the 2-cocycle w is a coboundary.
Then, by the lemma mentioned before, α and β are KK-trivially cocycle conjugate.

Hirsch length two (2/2)

Theorem (Izumi-M)

The conjecture is true for any poly-$\mathbb{Z}$ group with Hirsch length two.

Example

Let A be the Cuntz algebra $\mathcal{O}_n$. Then $KK^1(A, A) \cong \mathbb{Z}_{n-1}$.

- When $G = \mathbb{Z}^2$, $H^2(\mathbb{Z}^2, \mathbb{Z}_{n-1}) \cong \mathbb{Z}_{n-1}$, and so there exist $n-1$ outer actions $\mathbb{Z}^2 \curvearrowright \mathcal{O}_n$ up to KK-trivial cocycle conjugacy.
- When $G = \langle a, b \mid bab = a^{-1} \rangle$, $H^2(G, \mathbb{Z}_{n-1}) \cong \mathbb{Z}_{n-1} \otimes \mathbb{Z}_2$, and so

$$\#(\{\text{outer actions } G \curvearrowright \mathcal{O}_n\}/\sim) = \begin{cases} 1 & n \in 2\mathbb{N} \\ 2 & n \in 2\mathbb{N}+1. \end{cases}$$

Hirsch length three (1/2)

Now, let us consider a poly-$\mathbb{Z}$ group G with Hirsch length three.
Let $\alpha, \beta : G \curvearrowright A$ be outer actions on a unital Kirchberg algebra.
Assume $KK(\alpha_g) = KK(\beta_g)$, $o^2(\alpha, \beta) = 0$ and
$o^3(\alpha, \beta, u) = 0$ for some $(u_g)_g$.
We want to show that α and β are KK-trivially cocycle conjugate.

Let $\sigma_g = \mathrm{Ad}\, u_g \circ \alpha_g \in \mathrm{Aut}(A_\flat)$ and $w(g,h) \in U(A_\flat)$ be as before.
By replacing $(u_g)_g$, we may assume $K_1(w(g,h)) = 0\ \forall g, h \in G$.
Let $\tilde{w}(g,h) : [0,1] \to U(A_\flat)$ be a path from 1 to $w(g,h)$. Since
$o^3(\alpha, \beta, u) = 0$, we may assume that the loop

$$\sigma_g(\tilde{w}(h,k))\tilde{w}(g,hk)\left(\tilde{w}(g,h)\tilde{w}(gh,k)\right)^*$$

in $U(A_\flat)$ is homotopic to 1 for every $g, h, k \in G$. Let $G = N \rtimes \langle \xi \rangle$. By assumption, N is a poly-$\mathbb{Z}$ group with Hirsch length two. As $K_1(w(g,h)) = 0$, we may assume that $\sigma|N$ is a genuine action. Moreover, ... So, w is a coboundary !

Hirsch length three (2/2)

Theorem (Izumi-M)

The conjecture is true for any poly-$\mathbb{Z}$ group with Hirsch length three.

Let G be a poly-$\mathbb{Z}$ group with Hirsch length three.

Example

Let A be $\mathcal{O}_n$. Then $KK^i(A, A) \cong \mathbb{Z}_{n-1}$ for $i = 0, 1$, and

$$\{\text{outer actions } G \curvearrowright A\}/\sim \quad \cong \quad H^2(G, KK^1(A, A)).$$

Example

Let A be the Cuntz standard form of $\mathcal{O}_n$. Then, there exists a natural bijection between the set of KK-trivial cocycle conjugacy classes of outer actions $\alpha : G \curvearrowright A$ such that $KK(\alpha_g) = 1$ and

$$H^2(G, KK^1(A, A)) \oplus H^3(G, KK(A, A)).$$

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