Minimal dynamical systems and simple C^*-algebras

Hiroki Matui
matui@math.s.chiba-u.ac.jp

Chiba University

July 2, 2013
The Asian Mathematical Conference 2013
Busan, Korea

Crossed product C^*-algebras

Throughout this talk, X is a compact, Hausdorff, metrizable, infinite space. Let $\alpha \in \mathrm{Homeo}(X)$. (X, α) is a topological dynamical system.

Consider the automorphism of $C(X)$ defined by $f \mapsto f \circ \alpha^{-1}$.
We let $C^*(X, \alpha)$ denote the crossed product C^*-algebra of $C(X)$ by this automorphism. $C^*(X, \alpha)$ is the universal C^*-algebra generated by $C(X)$ and a unitary λ subject to the relation

$$\lambda f \lambda^* = f \circ \alpha^{-1} \quad \forall f \in C(X).$$

$C^*(X, \alpha)$ contains a dense subalgebra

$$\left\{ \sum_{i=-N}^{N} f_i \lambda^i \ \middle|\ N \in \mathbb{N},\ f_i \in C(X) \right\}$$

consisting of 'Laurent polynomials' with coefficients in $C(X)$.

Minimal dynamical systems

α is said to be minimal if $\{\alpha^n(x) \in X \mid n \in \mathbb{Z}\}$ is dense for any $x \in X$, or equivalently if α has no non-trivial closed invariant sets.

Theorem

C^*-algebra $C^*(X, \alpha)$ is simple if and only if α is minimal.

It is an important (and difficult) problem to classify those C^*-algebras.

Invariant measures and tracial states

For an α-invariant probability measure μ on X,

$$C^*(X,\alpha) \ni \sum f_i \lambda^i \;\mapsto\; \int_X f_0 \, d\mu \in \mathbb{C}$$

gives rise to a tracial state $\tau_\mu : C^*(X,\alpha) \to \mathbb{C}$, i.e. $\tau_\mu(xy) = \tau_\mu(yx)$.

The correspondence $\mu \mapsto \tau_\mu$ gives a bijection between the space $M(X,\alpha)$ of α-invariant probability measures on X and the space $T(C^*(X,\alpha))$ of tracial states on $C^*(X,\alpha)$.

We say that (X,α) is uniquely ergodic if there exists a unique α-invariant probability measure on X.

Minimal systems on Cantor sets

Let X be a Cantor set, i.e. X is a compact, metrizable, totally disconnected (clopen sets generate the topology) space with no isolated points.
Let $\alpha \in \mathrm{Homeo}(X)$ be a minimal homeomorphism.

Theorem (I. F. Putnam 1990)

$C^*(X, \alpha)$ is a unital simple AT algebra with real rank zero.

Fix $y \in X$. Let A_y be the C^*-subalgebra of $C^*(X, \alpha)$ generated by $C(X)$ and $\{\lambda f \mid f \in C(X),\ f(y) = 0\}$.

Theorem (I. F. Putnam 1989)

A_y is a unital simple AF algebra.

Classification of AF algebras

A finite dimensional C^*-algebra F is a direct sum of matrix algebras:

$$F = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}.$$

An inductive limit of finite dimensional C^*-algebras

$$\varinjlim F_j$$

is called an AF algebra. The UHF algebra

$$M_{n^\infty} = M_n \otimes M_n \otimes M_n \otimes \ldots$$

is a typical AF algebra.

Theorem (G. Elliott 1976)

The class of unital AF algebras is completely classified by the K-theory invariant $(K_0(A), K_0(A)^+, [1_A])$.

K_0 of M_{n^∞} is isomorphic to $\mathbb{Z}[1/n] = \{k/n^l \in \mathbb{Q} \mid k \in \mathbb{Z},\ l \in \mathbb{N}\}$.

Classification of AT algebras

A circle algebra means a C^*-algebra of the form $F \otimes C(\mathbb{T})$, where F is a finite dimensional C^*-algebra.
An inductive limit of circle algebras

$$\varinjlim F_j \otimes C(\mathbb{T})$$

is called an AT algebra.

Theorem (G. Elliott 1993)

The class of unital simple AT algebras with real rank zero is completely classified by the K-theory invariant $(K_0(A), K_0(A)^+, [1_A], K_1(A))$.

A C^*-algebra is of real rank zero if every hereditary subalgebra has an approximate unit consisting of projections.

Pimsner-Voiculescu exact sequence

For an automorphism α of a C^*-algebra A we have:

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\mathrm{id} - K_0(\alpha)} & K_0(A) & \longrightarrow & K_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_\alpha \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow[\mathrm{id} - K_1(\alpha)]{} & K_1(A) \end{array}$$

When (X, α) is a Cantor minimal $\mathbb{Z}$-system, $K_0(C(X)) \cong C(X, \mathbb{Z})$ and $K_1(C(X)) = 0$, and so

$$\begin{aligned} K_0(C^*(X, \alpha)) &= \mathrm{Coker}(\mathrm{id} - K_0(\alpha)) \\ &\cong C(X, \mathbb{Z}) / \{f - f \circ \alpha \mid f \in C(X, \mathbb{Z})\} \\ K_1(C^*(X, \alpha)) &= \mathrm{Ker}(\mathrm{id} - K_0(\alpha)) \cong \mathbb{Z}. \end{aligned}$$

Orbit equivalence for Cantor systems (1/2)

(X, α) and (Y, β) are said to be orbit equivalent if there exists a homeomorphism $h : X \to Y$ such that $h(\mathrm{Orb}_\alpha(x)) = \mathrm{Orb}_\beta(h(x))$ holds for all $x \in X$.

Theorem (T. Giordano, I. F. Putnam and C. F. Skau 1995)

For Cantor minimal $\mathbb{Z}$-systems (X, α) and (Y, β), T.F.A.E.

1. $C^*(X, \alpha)$ *is isomorphic to* $C^*(Y, \beta)$.
2. $K_\bullet(C^*(X, \alpha))$ *is isomorphic to* $K_\bullet(C^*(Y, \beta))$.
3. (X, α) *and* (Y, β) *are strongly orbit equivalent.*

Theorem (T. Giordano, I. F. Putnam and C. F. Skau 1995)

For Cantor minimal $\mathbb{Z}$-systems (X, α) and (Y, β), T.F.A.E.

1. (X, α) *and* (Y, β) *are orbit equivalent.*
2. $\exists$ *homeomorphism* $h : X \to Y$ *such that* $h_*(M(X, \alpha)) = M(Y, \beta)$.

Orbit equivalence for Cantor systems (2/2)

Let $\alpha : \mathbb{Z}^N \curvearrowright X$ be a free minimal action of $\mathbb{Z}^N$ on a Cantor set X.
Classification of $C^*(X, \alpha) = C(X) \rtimes_\alpha \mathbb{Z}^N$ is not yet obtained so far.

But, classification up to orbit equivalence is known.

Theorem (T. Giordano, M, I. F. Putnam and C. F. Skau 2010)

For minimal actions $\alpha : \mathbb{Z}^N \curvearrowright X$ and $\beta : \mathbb{Z}^M \curvearrowright Y$ on Cantor sets, the following are equivalent.

1. (X, α) *and* (Y, β) *are orbit equivalent.*
2. $\exists$ *homeomorphism* $h : X \to Y$ *such that* $h_*(M(X, \alpha)) = M(Y, \beta)$.

Irrational rotation algebras

Let $X = \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Take an irrational number $\theta \in (0, 1)$.
Let $\alpha \in \mathrm{Homeo}(X)$ be the translation by θ, i.e. $\alpha(t) = t + \theta$.
It is easy to see that α is minimal, thus $C^*(X, \alpha)$ is simple.
$C^*(X, \alpha)$ is called the irrational rotation algebra.

Theorem (G. Elliott and D. Evans 1993)

$C^*(X, \alpha)$ is a unital simple AT algebra with real rank zero.

K-groups of $C^*(X, \alpha)$ are $K_0(C^*(X, \alpha)) \cong \mathbb{Z} \oplus \mathbb{Z}$,

$$K_0(C^*(X, \alpha))^+ \cong \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid a + b\theta \geq 0\}$$

and $K_1(C^*(X, \alpha)) \cong \mathbb{Z} \oplus \mathbb{Z}$.

C^*-algebras with tracial rank zero

A unital separable simple C^*-algebra A has tracial rank zero
if for every finite subset $F \subset A$, $\varepsilon > 0$ and every nonzero positive $c \in A$, there exists a finite dimensional subalgebra $B \subset A$ such that

- $1_A - 1_B$ is equivalent to a projection in $\overline{cAc}$.
- $\|[a, 1_B]\| < \varepsilon$ for every $a \in F$.
- $\mathrm{dist}(1_B a 1_B, B) < \varepsilon$ for every $a \in F$.

Theorem (H. Lin 2004)

The class of unital separable simple nuclear C^*-algebras with tracial rank zero satisfying the UCT is completely classified by the K-theory invariant $(K_0(A), K_0(A)^+, [1_A], K_1(A))$.

Lin-Phillips' theorem

Theorem (H. Lin and N. C. Phillips 2010)

Suppose that X has finite covering dimension and α is minimal. If the image of $K_0(C^*(X,\alpha))$ is dense in $\mathrm{Aff}(T(C^*(X,\alpha)))$, then $C^*(X,\alpha)$ has tracial rank zero.

In general, there exists a homomorphism $D_A : K_0(A) \to \mathrm{Aff}(T(A))$ defined by

$$D_A([p])(\tau) = (\tau \otimes \mathrm{Tr})(p)$$

for a projection $p \in A \otimes M_n$ and $\tau \in T(A)$.

For a large class of C^*-algebras, the image of D_A is dense in $\mathrm{Aff}(T(A))$ if and only if A has real rank zero.

Connes' example

Theorem (A. Connes 1981)

Let X be a compact smooth manifold with $H^1(X, \mathbb{Z}) = 0$ and let α be a minimal diffeomorphism of X. Then $C^*(X, \alpha)$ has no non-trivial projections. In particular, $C^*(X, \alpha)$ does not have real rank zero.

A sphere S^n admits a minimal (uniquely ergodic) diffeomorphism $\alpha \in \mathrm{Homeo}(S^n)$ if and only if n is odd (A. Fathi and M. Herman 1977).
$H^1(S^n, \mathbb{Z}) = 0$ for $n \geq 2$.
So, when n is odd and greater than 2, $C^*(S^n, \alpha)$ is not covered by Lin-Phillips' theorem.

Further classification theorems (1/2)

We let $\mathcal{Z}$ denote the Jiang-Su algebra, which is unital simple separable nuclear, infinite-dimensional, has a unique trace and $K_*(\mathcal{Z}) \cong K_*(\mathbb{C})$. $\mathcal{Z}$ has no non-trivial projections.

Let $\mathcal{C}$ denote the class of unital separable simple nuclear C^*-algebras A such that

- A satisfies the UCT,
- A is $\mathcal{Z}$-stable, i.e. $A \cong A \otimes \mathcal{Z}$,
- $A \otimes U$ has tracial rank zero for a UHF algebra U.

Note that A may not be of real rank zero.

Theorem (W. Winter, H. Lin and Z. Niu 2008)

The class $\mathcal{C}$ is completely classified by the K-theory invariant $(K_0(A), K_0(A)^+, [1_A], K_1(A))$.

Further classification theorems (2/2)

Theorem (K. Strung and W. Winter 2010)

Suppose that projections in $C^*(X, \alpha)$ separate tracial states (i.e. if $\tau_1 \neq \tau_2$, then $\exists$ projection p such that $\tau_1(p) \neq \tau_2(p)$). Then $C^*(X, \alpha) \otimes U$ has tracial rank zero for any UHF algebra U.

Theorem (A. Toms and W. Winter 2013)

If X has finite covering dimension, then $C^*(X, \alpha)$ is $\mathcal{Z}$-stable.

In particular, when $\alpha \in \mathrm{Homeo}(S^n)$ is minimal and uniquely ergodic, $C^*(S^n, \alpha)$ belongs to $\mathcal{C}$.

Recent progress

Theorem (M and Y. Sato 2013)

Let A be a unital separate simple nuclear C^*-algebra. Suppose that A has a unique trace. If A has strict comparison and is quasidiagonal, then A is $\mathcal{Z}$-stable and $A \otimes U$ has tracial rank zero for any UHF algebra U.
In particular, if A satisfies the UCT, then A is in $\mathcal{C}$.

Let $\alpha : \mathbb{Z}^N \curvearrowright X$ be a free minimal action of $\mathbb{Z}^N$ on a Cantor set X.
Consider $A = C^*(X, \alpha) = C(X) \rtimes_\alpha \mathbb{Z}^N$.
Assume that α is uniquely ergodic (hence A has a unique trace).
A has strict comparion (N. C. Phillips 2005).
A is AF embeddable (H. Lin 2008), and hence is quasidiagonal.

So, $A = C^*(X, \alpha)$ belongs to $\mathcal{C}$.