Recent progress in classification of simple C^*-algebras

Hiroki Matui
matui@math.s.chiba-u.ac.jp

Chiba University / University of Copenhagen

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Classification theory of C^*-algebras

In 1990's, G. A. Elliott initiated a program to classify nuclear C^*-algebras via K-groups.

Goal

For unital separable simple nuclear C^*-algebras A and B,

$$A \cong B \qquad \Longleftrightarrow \qquad K_*(A) \cong K_*(B).$$

But ...
$K_*(A)$ is always isomorphic to $K_*(A \otimes \mathcal{Z})$,
while there exists A such that $A \not\cong A \otimes \mathcal{Z}$.

$\mathcal{Z}$ is the Jiang-Su algebra, which is unital simple separable nuclear, infinite-dimensional, has a unique tracial state and $K_*(\mathcal{Z}) \cong K_*(\mathbb{C})$.

$\mathcal{Z}$ absorbs $\mathcal{Z}$ tensorially, i.e. $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$.

Regularity properties (1/2)

We must assume that A is $\mathcal{Z}$-stable, i.e. $A \cong A \otimes \mathcal{Z}$.

Two other regularity properties are known: strict comparison and finite decomposition rank (for stably finite algebras).

A is said to have strict comparison
if for any positive elements $a, b \in A \otimes M_m$,

$$\lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_m)(a^{1/n}) < \lim_{n\to\infty} (\tau \otimes \mathrm{Tr}_m)(b^{1/n}) \quad \forall \tau \in QT(A)$$

implies $\exists (v_n)_n$ in $A \otimes M_m$ such that $v_n b v_n^* \to a$.

Unlike the case of von-Neumann algebras, we have to consider comparison property not only for projections but also for positive elements.

Regularity properties (2/2)

A has decomposition rank at most d if for any finite subset $F \subset A$ and $\varepsilon > 0$ we can find completely positive contractions

$$A \xrightarrow{\varphi} \bigoplus_{i=0}^{d} E_i \xrightarrow{\psi} A$$

such that $\dim E_i < \infty$, $\|\psi(\varphi(a)) - a\| < \varepsilon$ for $a \in F$ and $\psi|E_i$ is order zero, i.e.

$$x, y \in E_i,\ xy = 0 \quad \Longrightarrow \quad \psi(x)\psi(y) = 0.$$

Decomposition rank is a noncommutative analogue of ordinary covering dimension for topological spaces.
If A has finite decomposition rank, then A is quasidiagonal
(E. Kirchberg and W. Winter 2004).

Toms-Winter conjecture (1/2)

It is then natural to ask whether these regularity properties are equivalent to each other (Toms-Winter conjecture).

For a unital simple separable exact finite C^*-algebra,
$\mathcal{Z}$-stability implies strict comparison (M. Rørdam 2004).

For a unital simple separable nuclear finite C^*-algebra A such that $T(A)$ has a compact extreme boundary with finite covering dimension,
strict comparison implies $\mathcal{Z}$-stability
(Kirchberg-Rørdam, Sato, Toms-White-Winter 2012).

For a unital simple separable nuclear finite C^*-algebra,
finite decomposition rank implies $\mathcal{Z}$-stability (W. Winter 2010).

Toms-Winter conjecture (2/2)

Theorem (M-Sato 2013)

Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state. If A is quasidiagonal and has strict comparison, then A has decomposition rank at most three.

Corollary

Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state. The following are equivalent.

1. *A has finite decomposition rank.*
2. *A is quasidiagonal and $\mathcal{Z}$-stable.*
3. *A is quasidiagonal and has strict comparison.*

Classification theorem (1/2)

Theorem (M-Sato 2013)

Let A be a unital separable simple nuclear C^*-algebra with a unique tracial state. If A is quasidiagonal, then $A \otimes U$ has tracial rank zero for any UHF algebra U.

A unital separable simple C^*-algebra A has tracial rank zero if for any finite subset $F \subset A$ and $\varepsilon > 0$ there exists a finite dimensional subalgebra $B \subset A$ such that

- $\tau(1_A - 1_B) < \varepsilon$ for every $\tau \in T(A)$.
- $\|[a, 1_B]\| < \varepsilon$ for every $a \in F$.
- $\mathrm{dist}(1_B a 1_B, B) < \varepsilon$ for every $a \in F$.

H. Lin proved that nuclear C^*-algebras with tracial rank zero satisfying the UCT are classified by K-groups.

Classification theorem (2/2)

Let $\mathcal{C}$ denote the class of all unital separable simple nuclear C^*-algebras A which has a unique tracial state, is quasidiagonal, and has strict comparison.
Let $\mathcal{C}_{\mathsf{UCT}}$ be the subclass of $\mathcal{C}$ consisting of A which satisfies the UCT.

Corollary

For $A, B \in \mathcal{C}_{\mathsf{UCT}}$, we have $A \cong B \iff K_*(A) \cong K_*(B)$.

Proof.

For $A \in \mathcal{C}$, A is $\mathcal{Z}$-stable, and $A \otimes U$ has tracial rank zero for any UHF algebra U. Then the conclusion follows from the classification theorem of Lin-Niu, Winter 2008. □

Minimal $\mathbb{Z}^N$-actions on Cantor sets (1/2)

Let X be a Cantor set and let $\alpha : \mathbb{Z}^N \curvearrowright X$ be an action of $\mathbb{Z}^N$ on X by homeomorphisms. We say that α is free if α_n has no fixed points for any $n \neq 0$. We say that α is minimal if the orbit $\{\alpha_n(x) \mid n \in \mathbb{Z}^N\}$ of x is dense in X for every $x \in X$.

$\alpha : \mathbb{Z}^N \curvearrowright X$ induces $\bar{\alpha} : \mathbb{Z}^N \curvearrowright C(X)$ by $\bar{\alpha}_n(f) = f \circ \alpha_n^{-1}$.
When α is free and minimal, $C(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^N$ is simple.

Theorem (I. F. Putnam 1990)

When $\alpha : \mathbb{Z} \curvearrowright X$ is free and minimal, the C^*-algebra $C(X) \rtimes_{\bar{\alpha}} \mathbb{Z}$ is a unital simple AT-algebra with real rank zero.
In particular, it is classifiable by K-groups.

Minimal $\mathbb{Z}^N$-actions on Cantor sets (2/2)

Theorem

Let $\alpha : \mathbb{Z}^N \curvearrowright X$ be a free minimal action on a Cantor set X.
If α is uniquely ergodic, then $C(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^N$ is in the class $\mathcal{C}_{\mathrm{UCT}}$.

Proof.

Clearly $A = C(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^N$ is unital, separable, simple, nuclear and satisfies the UCT.
Since α is uniquely ergodic, A has a unique tracial state.
A is AF-embeddable (H. Lin 2008), and hence is quasidiagonal.
A is of real rank zero and has strict comparison for projections (N. C. Phillips 2005), and hence has strict comparison.
Therefore A is in $\mathcal{C}_{\mathrm{UCT}}$. $\square$

Crossed products

Let A be a unital, separable, simple, nuclear C^*-algebra with a unique tracial state τ. The weak closure $\pi_\tau(A)''$ with respect to τ is the hyperfinite II_1-factor.
Let $\alpha : \Gamma \curvearrowright A$ be an action of a discrete group Γ.
$\alpha : \Gamma \curvearrowright A$ gives rise to $\bar{\alpha} : \Gamma \curvearrowright \pi_\tau(A)''$.

We say that α is strongly outer if $\bar{\alpha}$ is outer.
In this case, $A \rtimes_\alpha \Gamma$ has a unique trace.

Theorem (M-Sato 2012)

Suppose that A is $\mathcal{Z}$-stable and that Γ is elementary amenable.
If $\alpha : \Gamma \curvearrowright A$ is strongly outer, then $A \rtimes_\alpha \Gamma$ is $\mathcal{Z}$-stable.

Crossed products by the integers

Theorem

Let $A \in \mathcal{C}_{\mathsf{UCT}}$ and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action.
Then $A \rtimes_\alpha \mathbb{Z}$ is again in $\mathcal{C}_{\mathsf{UCT}}$.

Proof.

Clearly, $A \rtimes_\alpha \mathbb{Z}$ is unital, separable, simple, nuclear and satisfies the UCT.
As mentioned before, it has a unique trace, and is $\mathcal{Z}$-stable (and hence has strict comparison).
Let Q be the universal UHF algebra. Then $A \otimes Q$ is a unital simple AT algebra with real rank zero. It follows that
$(A \otimes Q) \rtimes_{\alpha \otimes \mathrm{id}} \mathbb{Z} \cong (A \rtimes_\alpha \mathbb{Z}) \otimes Q$ is AF-embeddable (M 2002).
Thus $A \rtimes_\alpha \mathbb{Z}$ is AF-embeddable, and hence is quasidiagonal.
Therefore $A \rtimes_\alpha \mathbb{Z}$ is in $\mathcal{C}_{\mathsf{UCT}}$. $\square$

Crossed products by finite groups

Let Γ be a finite group.

Theorem

Let $A \in \mathcal{C}$ and let $\alpha : \Gamma \curvearrowright A$ be a strongly outer action. Then $A \rtimes_\alpha \Gamma$ is again in $\mathcal{C}$.

Proof.

Clearly, $A \rtimes_\alpha \Gamma$ is unital, separable, simple, nuclear. As mentioned before, it has a unique trace, and is $\mathcal{Z}$-stable (and hence has strict comparison). Since Γ is finite, it is easy to see that $A \rtimes_\alpha \Gamma$ is embeddable into $A \otimes M_{|\Gamma|}$, and hence is quasidiagonal.
Therefore $A \rtimes_\alpha \Gamma$ is in $\mathcal{C}$. $\square$

The Powers-Sakai conjecture (1/2)

We consider an action $\alpha : \mathbb{R} \curvearrowright A$ which is strongly continuous, i.e. $t \mapsto \alpha_t(a)$ is continuous for every $a \in A$. $\alpha : \mathbb{R} \curvearrowright A$ is said to be approximately inner if there exists a sequence $(h_n)_n$ of self-adjoint elements in A such that

$$\max_{|t|\leq 1} \|e^{ith_n} a e^{-ith_n} - \alpha_t(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$.

Conjecture (R. Powers and S. Sakai 1975)

Every strongly continuous action $\alpha : \mathbb{R} \curvearrowright A$ on a UHF algebra A is approximately inner.

The Powers-Sakai conjecture (2/2)

Why is it important?
See "Operator Algebras and Dynamical Systems" by S. Sakai.

- C^*-dynamical systems provide a framework for understanding quantum statistical mechanics.
- For quantum lattice systems, the dynamics is given by approximately inner actions $\alpha : \mathbb{R} \curvearrowright A$ on UHF algebras.
- An approximately inner action always has a ground state, and also has a KMS state at each inverse temperatures provided that A has a tracial state (R. Powers and S. Sakai 1975).

Here we shall present a counter-example to the conjecture by using our main result. To this end, it suffices to construct an action $\alpha : \mathbb{T} \curvearrowright A$ on a UHF algebra A such that the crossed product $A \rtimes_\alpha \mathbb{T}$ is simple.

Kishimoto's construction

A. Kishimoto obtained a counter-example to the AF version of the Powers-Sakai conjecture in 2003.

Take a unital simple AF algebra A with a unique trace τ and two projections $p, q \in A$ such that $\tau(p) = \tau(q)$ but $[p] \neq [q]$ in $K_0(A)$. Let B be the inductive limit of $A^{\otimes 2n}$ with the connecting map

$$A^{\otimes 2n} \ni \ x \ \mapsto \ p \otimes x \otimes q \ \in A^{\otimes(2n+2)}.$$

Then B is a non-unital simple AF algebra.

Let $\sigma : B \to B$ be the shift automorphism. Kishimoto proved that $B \rtimes_\sigma \mathbb{Z}$ is a non-unital simple AF algebra by using Lin's classification result.

Take a projection $e \in B$ and consider $\hat{\sigma} : \mathbb{T} \curvearrowright e(B \rtimes_\sigma \mathbb{Z})e$. Then the crossed product is a hereditary subalgebra of $B \rtimes_\sigma \mathbb{Z} \rtimes_{\hat{\sigma}} \mathbb{T} \cong B \otimes \mathbb{K}$, which is simple.

A counter-example

Let A be a UHF algebra. Define (D, D^+) by $D = \bigoplus_{\mathbb{Z}} K_0(A)$ and

$$D^+ = \left\{ (x_n)_n \in D \mid \sum_n x_n \in K_0(A)^+ \setminus \{0\} \right\} \cup \{0\}.$$

Find a stable simple AF algebra B whose K_0 is (D, D^+).

There exists $\sigma \in \mathrm{Aut}(B)$ such that $K_0(\sigma)$ is the shift on D.
Clearly $K_0(B \rtimes_\sigma \mathbb{Z}) \cong K_0(A)$ and $K_1(B \rtimes_\sigma \mathbb{Z}) = 0$.
Take a projection $e \in B$. By replacing σ with $\sigma \otimes \gamma \in \mathrm{Aut}(B \otimes \mathcal{Z})$ if necessary, we may assume that $e(B \rtimes_\sigma \mathbb{Z})e$ has a unique tracial state and is $\mathcal{Z}$-stable. It is known that $B \rtimes_\sigma \mathbb{Z}$ is AF-embeddable (N. Brown 1998), and hence is quasidiagonal.

Therefore $e(B \rtimes_\sigma \mathbb{Z})e$ is in $\mathcal{C}_{\mathrm{UCT}}$, and so $e(B \rtimes_\sigma \mathbb{Z})e$ is a UHF algebra.
The dual action $\hat{\sigma} : \mathbb{T} \curvearrowright e(B \rtimes_\sigma \mathbb{Z})e$ is a counter-example.