Jiang-Su 環への群作用について

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UHF algebras

Let $(r_n)_n \subset \mathbb{N}$ be such that r_n divides r_{n+1} and $r_n \to \infty$, and let $\varphi_n : M_{r_n} \to M_{r_{n+1}}$ be a unital homomorphism. The inductive limit C^*-algebra A arising from $(M_{r_n}, \varphi_n)_n$ is called the UHF algebra of type $(r_n)_n$. For example, when $r_n = k^n$, A is thought of as an infinite tensor product of M_k:

$$A = M_k \otimes M_k \otimes M_k \otimes \cdots .$$

A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (J. Glimm 1960)

UHF algebras are completely classified in terms of $(r_n)_n$.

The Jiang-Su algebra (1/2)

For relatively prime natural numbers p, q, we call

$$I(p,q) = \{f \in C([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes \mathbb{C},\ f(1) \in \mathbb{C} \otimes M_q\}$$

a prime dimension drop algebra. $I(p,q)$ does not contain non-trivial projections, and $K_0(I(p,q)) \cong \mathbb{Z}$, $K_1(I(p,q)) = 0$.

Let $(p_n)_n, (q_n)_n \subset \mathbb{N}$ be sequences such that p_n and q_n are relatively prime, p_n divides p_{n+1} and q_n divides q_{n+1}. We can construct unital homomorphisms $\varphi_n : I(p_n, q_n) \to I(p_{n+1}, q_{n+1})$ so that the inductive limit C^*-algebra arising from $(I(p_n, q_n), \varphi_n)_n$ is simple and has a unique trace.
We call it the Jiang-Su algebra and write $\mathcal{Z}$.
$\mathcal{Z}$ does not depend on the choice of p_n, q_n, φ_n.

The Jiang-Su algebra (2/2)

Theorem (X. Jiang and H. Su 1999)

- $K_0(\mathcal{Z}) \cong \mathbb{Z}$ *and* $K_1(\mathcal{Z}) = 0$.
- $\mathcal{Z}$ *does not contain non-trivial projections.*
- $\mathcal{Z}$ *is unital simple separable and nuclear.*
- *For any UHF algebra* A*, we have* $A \otimes \mathcal{Z} \cong A$.
- *Also,* $\mathcal{Z} \otimes \mathcal{Z} \cong \mathcal{Z}$.
- *Any automorphism of* $\mathcal{Z}$ *is approximately inner.*

Classification theory of C^*-algebras (1/2)

In 1990's, G. A. Elliott initiated a program to classify nuclear C^*-algebras via K-groups.

Original Conjecture

For unital separable simple nuclear C^*-algebras A and B,

$$A \cong B \qquad \Longleftrightarrow \qquad K_*(A) \cong K_*(B).$$

So far, the conjecture is known to hold for several large classes of C^*-algebras (e.g. Kirchberg algebras, certain AH algebras, certain ASH algebras). However, it is already known that there exist A and B for which the conjecture is not true.

Thus, we need to 'revise' the conjecture above.

Classification theory of C^*-algebras (2/2)

Revised Conjecture

For unital separable simple nuclear and regular A and B,

$$A \cong B \qquad \Longleftrightarrow \qquad K_*(A) \cong K_*(B).$$

What does 'regular' mean? Several regularity conditions have been considered, and one of them is $\mathcal{Z}$-stability, where A is said to be $\mathcal{Z}$-stable if $A \cong A \otimes \mathcal{Z}$.

All unital simple separable nuclear C^*-algebras classified by K-theory so far are $\mathcal{Z}$-stable.
Conversely, known counter examples for the original conjecture are not $\mathcal{Z}$-stable.

Cocycle actions

Let Γ be a countable discrete group.

Definition

A pair (α, u) of a map $\alpha : \Gamma \to \mathrm{Aut}(A)$ and a map $u : \Gamma \times \Gamma \to U(A)$ is called a cocycle action of Γ on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in \Gamma$. We write $(\alpha, u) : \Gamma \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in \Gamma$.
When α_g is not inner for any $g \in \Gamma \setminus \{1\}$,
(α, u) is said to be outer.
When $u \equiv 1$, $\alpha : \Gamma \curvearrowright A$ is a genuine action.

Cocycle conjugacy

Definition

Two cocycle actions $(\alpha, u) : \Gamma \curvearrowright A$ and $(\beta, v) : \Gamma \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g\in\Gamma}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, w_g \circ \beta_g$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^*$$

hold for every $g, h \in \Gamma$.

Our eventual goal is

- to classify the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$,
- to classify (α, u) up to cocycle conjugacy and to determine when (α, u) is cocycle conjugate to a genuine action.

Strong outerness

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.

Definition

$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
A cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{1\}$.

If $T(A) = \{\tau\}$, then

$$(\alpha, u) : \Gamma \curvearrowright A \text{ is strongly outer} \iff (\bar{\alpha}, u) : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Central sequence algebras

For a unital C^*-algebra A, we let

$$c_0(\mathbb{N}, A) = \{(a_n)_n \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\}.$$

The limit algebra of A is $A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$.
A is identified with the subalgebra of A^∞ consisting of equivalence classes of constant sequences.

The central sequence algebra of A is $A_\infty = A^\infty \cap A'$.
We call $(a_n)_n \in \ell^\infty(\mathbb{N}, A)$ a central sequence if $[a_n, x] \to 0$ as $n \to \infty$ for every $x \in A$. A central sequence is a representative of an element in A_∞.

$(\alpha, u) : \Gamma \curvearrowright A$ extends to A^∞ and A_∞ naturally.

$\mathcal{Z}$-stability of crossed products

Theorem (Y. Sato and M)

Let A be a unital, simple, separable, nuclear, stably finite C^*-algebra with finitely many extremal tracial states. Suppose that A is $\mathcal{Z}$-stable. Let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action of an elementary amenable group Γ. Then (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright A \otimes \mathcal{Z}$. In particular, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is $\mathcal{Z}$-stable.

This says that (α, u) absorbs the trivial action on $\mathcal{Z}$ up to cocycle conjugacy.

In order to prove this, it suffices to construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$.

Absorption of non-trivial actions on $\mathcal{Z}$ (1/2)

Let $\mu : \mathbb{Z} \curvearrowright \mathcal{Z}$ be a strongly outer action. Let $\pi_i : \mathbb{Z}^N \to \mathbb{Z}$ be the canonical projection from $\mathbb{Z}^N = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ to the i-th coordinate.
Define a strongly outer action $\gamma : \mathbb{Z}^N \curvearrowright \mathcal{Z} \otimes \mathcal{Z} \otimes \cdots \otimes \mathcal{Z} \cong \mathcal{Z}$ by

$$\gamma_g = \mu_{\pi_1(g)} \otimes \mu_{\pi_2(g)} \otimes \cdots \otimes \mu_{\pi_N(g)} \quad g \in \mathbb{Z}^N$$

When $N = 1, 2$, strongly outer actions of $\mathbb{Z}^N$ on $\mathcal{Z}$ are known to be unique up to cocycle conjugacy.

Absorption of non-trivial actions on $\mathcal{Z}$ (2/2)

Let A be a unital, simple, separable, nuclear, stably finite C^*-algebra with finitely many extremal tracial states.

Theorem (Y. Sato and M)

Suppose that A is $\mathcal{Z}$-stable.
Let $(\alpha, u) : \mathbb{Z}^N \curvearrowright A$ be a strongly outer cocycle action.
Then (α, u) is cocycle conjugate to $(\alpha \otimes \gamma, u \otimes 1) : \mathbb{Z}^N \curvearrowright A \otimes \mathcal{Z}$.

For a UHF algebra B, it is known that $\gamma \otimes \mathrm{id} : \mathbb{Z}^N \curvearrowright \mathcal{Z} \otimes B \cong B$ has the Rohlin property. Hence we obtain the following.

Corollary (Y. Sato and M)

Let $(\alpha, u) : \mathbb{Z}^N \curvearrowright A$ be a strongly outer cocycle action and let B be a UHF algebra. Then $(\alpha \otimes \mathrm{id}, u \otimes 1) : \mathbb{Z}^N \curvearrowright A \otimes B$ has the Rohlin property.

$\mathbb{Z}$-actions on UHF algebras (1/2)

Let $A = M_k \otimes M_k \otimes M_k \otimes \cdots$ be a UHF algebra and let $u \in M_k$ be a unitary such that u^n is not a scalar for any $n \in \mathbb{N}$. Define $\alpha \in \mathrm{Aut}(A)$ by $\alpha = \mathrm{Ad}\, u \otimes \mathrm{Ad}\, u \otimes \mathrm{Ad}\, u \otimes \cdots$. Then $\alpha : \mathbb{Z} \curvearrowright A$ is a strongly outer action.

Another typical example is the Bernoulli shift. Regarding A as the two-sided infinite tensor product

$$A = \cdots \otimes M_k \otimes M_k \otimes M_k \otimes \cdots ,$$

we let $\alpha : \mathbb{Z} \curvearrowright A$ be the bilateral shift of the tensor components. It is well-known that α is strongly outer.

$\mathbb{Z}$-actions on UHF algebras (2/2)

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e. for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1.$$

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on A are cocycle conjugate to each other.

$\mathbb{Z}$-actions on $\mathcal{Z}$

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are cocycle conjugate to each other.

We sketch the proof. Let $\alpha, \beta : \mathbb{Z} \curvearrowright \mathcal{Z}$ be strongly outer.
(1) By the theorem mentioned before, we may replace α, β with $\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id} : \mathbb{Z} \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.
(2) $Z = \{f : [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in M_{2^\infty},\ f(1) \in M_{3^\infty}\}$ is a unital subalgebra of $\mathcal{Z}$ (M. Rørdam and W. Winter 2010).
(3) By Kishimoto's result, $\alpha \otimes \mathrm{id}$ and $\beta \otimes \mathrm{id}$ are cocycle conjugate as actions on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
(4) With some extra effort we get cocycle conjugacy on $\mathcal{Z} \otimes \mathcal{Z}$.

Cocycle actions of $\mathbb{Z}^2$ on UHF algebras (1/2)

We write $\mathbb{Z}^2 = \langle a, b \mid ba = ab \rangle$.

Theorem (H. Nakamura 1999, M 2010, Y. Sato and M)

Let A be a unital simple AF algebra with finitely many extremal tracial states and let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer cocycle action. Suppose that α_a^n and α_b^n are approximately inner for some $n \in \mathbb{N}$. Then (α, u) has the Rohlin property.

For a cocycle action $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$, we have

$$\begin{aligned} \alpha_b \circ \alpha_a &= \mathrm{Ad}\, u(b,a) \circ \alpha_{ba} \\ &= \mathrm{Ad}\, u(b,a) \circ \alpha_{ab} = \mathrm{Ad}(u(b,a)u(a,b)^*) \circ \alpha_a \circ \alpha_b \end{aligned}$$

Conversely, two single automorphisms commuting up to an inner automorphism give rise to a cocycle action of $\mathbb{Z}^2$.

Cocycle actions of $\mathbb{Z}^2$ on UHF algebras (2/2)

For a UHF algebra A, we let $\Delta_\tau : U(A) \to \mathbb{R}/K_0(A)$ be the de la Harpe-Skandalis determinant.

Theorem (T. Katsura and M 2008, Y. Sato and M)

Let A be a UHF algebra. There exists a natural bijective correspondence between the following two sets.

1. *Cocycle conjugacy classes of strongly outer cocycle actions of $\mathbb{Z}^2$ on A.*
2. $\mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A))$.

Moreover, $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ is cocycle conjugate to a genuine action if and only if $\Delta_\tau(u(b,a)u(a,b)^*) = 0$ in $\mathbb{R}/K_0(A)$.

Cocycle actions of $\mathbb{Z}^2$ on $\mathcal{Z}$

Let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be a cocycle action.
Put $\check{u} = u(b,a)u(a,b)^*$.
The following theorem says that the de la Harpe-Skandalis determinant $\Delta_\tau(\check{u}) \in \mathbb{R}/\mathbb{Z}$ is the complete invariant of (α, u).

Theorem (Y. Sato and M)

Let $(\alpha, u), (\beta, v) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be strongly outer cocycle actions. Then they are cocycle conjugate if and only if $\Delta_\tau(\check{u}) = \Delta_\tau(\check{v})$.

The proof uses the same idea as $\mathbb{Z}$-actions:

- (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes \mathcal{Z}$.
- We have already classified $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
- Some extra effort gives the conclusion.

Asymptotic representability

An action $\alpha : \Gamma \curvearrowright A$ is said to be asymptotically representable if there exist continuous paths of unitaries $(v_g(t))_{g\in\Gamma, t\in[0,\infty)}$ in A such that

$$\|v_g(t)v_h(t) - v_{gh}(t)\| \to 0 \quad \forall g, h \in \Gamma,$$

$$\|\alpha_g(v_h(t)) - v_{ghg^{-1}}(t)\| \to 0 \quad \forall g, h \in \Gamma,$$

$$\|v_g(t)av_g(t)^* - \alpha_g(a)\| \to 0 \quad \forall g \in \Gamma, a \in A.$$

$\mathbb{Z}^N$-actions on UHF algebras of infinite type

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (M 2011)

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

Theorem (M 2011)

Let A be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on A are mutually cocycle conjugate to each other.

Open problems

- Classify strongly outer actions of $\mathbb{Z}^N$ on a general UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.

Cocycle actions of the Klein bottle group on UHF algebras

We call $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle$ the Klein bottle group.

Theorem (Y. Sato and M)

Let A be a UHF algebra and let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action. Then for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\alpha_a(e_n) - e_n \to 0, \quad \alpha_a(f_n) - f_n \to 0,$$
$$\sum_{i=0}^{m-1} \alpha_b^i(e_n) + \sum_{j=0}^{m} \alpha_b^j(f_n) \to 1.$$

Theorem (Y. Sato and M)

All strongly outer cocycle actions of Γ on a UHF algebra are mutually cocycle conjugate.

Cocycle actions of the Klein bottle group on $\mathcal{Z}$

Theorem (Y. Sato and M)

All strongly outer cocycle actions of Γ on the Jiang-Su algebra $\mathcal{Z}$ are mutually cocycle conjugate.

The proof uses the same idea as $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions: we know that $(\alpha, u) : \Gamma \curvearrowright \mathcal{Z}$ is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.

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