Group actions on simple stably finite C^*-algebras

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Cocycle actions

Let Γ be a countable discrete group.

Definition

A pair (α, u) of a map $\alpha : \Gamma \to \mathrm{Aut}(A)$ and a map $u : \Gamma \times \Gamma \to U(A)$ is called a cocycle action of Γ on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in \Gamma$. We write $(\alpha, u) : \Gamma \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in \Gamma$.
When α_g is not inner for any $g \in \Gamma \setminus \{1\}$,
(α, u) is said to be outer.
When $u \equiv 1$, $\alpha : \Gamma \curvearrowright A$ is a genuine action.

Cocycle conjugacy

Definition

Two cocycle actions $(\alpha, u) : \Gamma \curvearrowright A$ and $(\beta, v) : \Gamma \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g\in\Gamma}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, w_g \circ \beta_g$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^*$$

hold for every $g, h \in \Gamma$.

Our eventual goal is

- to classify the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$,
- to classify (α, u) up to cocycle conjugacy and to determine when (α, u) is cocycle conjugate to a genuine action.

Twisted crossed product

Definition

For $(\alpha, u) : \Gamma \curvearrowright A$, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is the universal C^*-algebra generated by A and a family of unitaries $(\lambda_g^\alpha)_{g \in \Gamma}$ satisfying

$$\lambda_g^\alpha \lambda_h^\alpha = u(g,h)\lambda_{gh}^\alpha \quad \text{and} \quad \lambda_g^\alpha a \lambda_g^{\alpha *} = \alpha_g(a)$$

for all $g, h \in \Gamma$ and $a \in A$.

If (α, u) and (β, v) are cocycle conjugate via $\theta : A \to B$ and $(w_g)_g$, then $A \rtimes_{(\alpha,u)} \Gamma$ and $B \rtimes_{(\beta,v)} \Gamma$ are canonically isomorphic by

$$a \mapsto \theta(a) \quad \text{and} \quad \lambda_g^\alpha \mapsto w_g \lambda_g^\beta.$$

$\mathcal{Z}$-stability of crossed product

Theorem (Y. Sato and M)

Let A be a unital, simple, separable, nuclear, stably finite C^*-algebra with finitely many extremal tracial states. Suppose that A has the property (SI). Let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action of an elementary amenable group Γ.
Then (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright A \otimes \mathcal{Z}$.
In particular, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is $\mathcal{Z}$-stable.

This says that (α, u) absorbs the trivial action on $\mathcal{Z}$ up to cocycle conjugacy.

In order to prove this, it suffices to construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$.

Strong outerness

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.

Definition

$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
A cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{1\}$.

If $T(A) = \{\tau\}$, then

$$(\alpha, u) : \Gamma \curvearrowright A \text{ is strongly outer}$$
$$\iff \quad (\bar{\alpha}, u) : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Elementary amenable groups

Definition

The class of elementary amenable groups is defined as the smallest family of groups containing all finite groups and all abelian groups, and closed under the processes of taking subgroups, quotients, group extensions and increasing unions.

For instance, all solvable groups are elementary amenable.
There exist amenable groups which are not elementary (R. I. Grigorchuk).

Property (SI)

Definition

We say that A has the property (SI) if for any central sequences $(x_n)_n$ and $(y_n)_n$ in A satisfying $0 \leq x_n, y_n \leq 1$,

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(x_n) = 0 \quad \text{and} \quad \inf_{m\in\mathbb{N}} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(y_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n\to\infty} \|s_n^* s_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n s_n - s_n\| = 0.$$

'Many' C^*-algebras have the property (SI):
we have recently obtained that for any unital simple separable nuclear and stably finite C^*-algebra with finitely many extremal tracial states, $\mathcal{Z}$-stability implies the property (SI).

Proof of the $\mathcal{Z}$-stability theorem

For simplicity, we assume that A has a unique trace τ.
Since A is nuclear, $R = \pi_\tau(A)''$ is the hyperfinite II_1-factor.
Let $\mathcal{R}$ be the central sequence algebra of R in the sense of von Neumann algebras. It is well-known that $\mathcal{R}$ is again a II_1-factor.

The following is the main ingredient of the proof of the theorem.

Proposition

The natural homomorphism

$$\pi : (A^\omega \cap A')^\alpha \to \mathcal{R}^\alpha$$

is surjective.

Once this is done, it is easy to find $C_0((0,1]) \otimes M_k \to (A^\omega \cap A')^\alpha$.
Thanks to the property (SI), we can find $I(k, k+1) \to (A^\omega \cap A')^\alpha$.

$\mathbb{Z}$-actions on UHF algebras

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra and let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property, i.e. for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\sum_{i=0}^{m-1} \alpha^i(e_n) + \sum_{j=0}^{m} \alpha^j(f_n) \to 1.$$

Theorem (A. Kishimoto 1995)

Let A be a UHF algebra. All strongly outer $\mathbb{Z}$-actions on A are cocycle conjugate to each other.

$\mathbb{Z}$-actions on AH algebras

We can generalize Kishimoto's results for UHF algebras to certain AH algebras.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth, real rank zero and finitely many extremal tracial states. Let $\alpha : \mathbb{Z} \curvearrowright A$ be a strongly outer action. If α_k is approximately inner for some $k \in \mathbb{N}$, then α has the Rohlin property.

Theorem (M 2010)

Let A be a unital simple AH algebra with slow dimension growth and real rank zero. If two actions $\alpha, \beta : \mathbb{Z} \curvearrowright A$ have the Rohlin property and $\alpha_1 \circ \beta_{-1}$ is asymptotically inner, then α and β are cocycle conjugate.

$\mathbb{Z}$-actions on $\mathcal{Z}$

Theorem (Y. Sato 2010)

All strongly outer $\mathbb{Z}$-actions on $\mathcal{Z}$ are cocycle conjugate to each other.

We sketch the proof. Let $\alpha, \beta : \mathbb{Z} \curvearrowright \mathcal{Z}$ be strongly outer.
(1) By the theorem mentioned before, we may replace α, β with $\alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id} : \mathbb{Z} \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.
(2) $Z = \{f : [0,1] \to M_{2^\infty} \otimes M_{3^\infty} \mid f(0) \in M_{2^\infty},\ f(1) \in M_{3^\infty}\}$ is a unital subalgebra of $\mathcal{Z}$ (M. Rørdam and W. Winter 2010).
(3) By Kishimoto's result, $\alpha \otimes \mathrm{id}$ and $\beta \otimes \mathrm{id}$ are cocycle conjugate as actions on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
(4) With some extra effort we get cocycle conjugacy on $\mathcal{Z} \otimes \mathcal{Z}$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (1/2)

We write $\mathbb{Z}^2 = \langle a, b \mid bab^{-1} = a \rangle$.

Theorem (H. Nakamura 1999, M 2010, Y. Sato and M)

Let A be a unital simple AF algebra with finitely many extremal tracial states and let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ be a strongly outer cocycle action. Suppose that α_a^n and α_b^n are approximately inner for some $n \in \mathbb{N}$. Then (α, u) has the Rohlin property.

For a cocycle action $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$, we have

$$\begin{aligned}\alpha_b \circ \alpha_a &= \operatorname{Ad} u(b,a) \circ \alpha_{ba} \\ &= \operatorname{Ad} u(b,a) \circ \alpha_{ab} = \operatorname{Ad}(u(b,a)u(a,b)^*) \circ \alpha_a \circ \alpha_b\end{aligned}$$

Conversely, two single automorphisms commuting up to an inner automorphism give rise to a cocycle action of $\mathbb{Z}^2$.

Cocycle actions of $\mathbb{Z}^2$ on AF algebras (2/2)

For $(\alpha, u) : \mathbb{Z}^2 \curvearrowright A$ satisfying $\alpha_g \in \overline{\mathrm{Inn}}(A)$ $\forall g \in \mathbb{Z}^2$, we introduce an invariant $c(\alpha, u) \in \mathrm{OrderExt}(K_0(A), K_0(A))$ as follows:
Consider the crossed product $B = A \rtimes_{\alpha_a} \mathbb{Z}$ by the first generator α_a. The second generator $\alpha_b \in \mathrm{Aut}(A)$ extends to $\tilde{\alpha}_b \in \mathrm{Aut}(B)$ by letting $\tilde{\alpha}_b(\lambda^{\alpha_a}) = \check{u}\lambda^{\alpha_a}$, where $\check{u} = u(b, a)u(a, b)^*$.
Let $\tilde{\eta}_0 : \overline{\mathrm{Inn}}(B) \to \mathrm{OrderExt}(K_1(B), K_0(B))$ be the homomorphism introduced by A. Kishimoto and A. Kumjian.
Define $c(\alpha, u) = \tilde{\eta}_0(\tilde{\alpha}_b) \in \mathrm{OrderExt}(K_1(B), K_0(B))$, which can be identified with $\mathrm{OrderExt}(K_0(A), K_0(A))$.

Theorem (M 2010, Y. Sato and M)

Let A be as before. Let (α, u) and (β, v) be strongly outer cocycle actions of $\mathbb{Z}^2$ such that $\alpha_g, \beta_g \in \overline{\mathrm{Inn}}(A)$.
Then $c(\alpha, u) = c(\beta, v)$ if and only if (α, u) and (β, v) are cocycle conjugate via an approximately inner automorphism.

Cocycle actions of $\mathbb{Z}^2$ on UHF algebras

When A is a UHF algebra, we have

$$\mathrm{OrderExt}(K_0(A), K_0(A)) \cong \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)).$$

Corollary (T. Katsura and M 2008, Y. Sato and M)

Let A be a UHF algebra. There exists a natural bijective correspondence between the following two sets.

1. *Cocycle conjugacy classes of strongly outer cocycle actions of $\mathbb{Z}^2$ on A.*
2. $\mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A))$.

Moreover, genuine actions correspond to

$$\{r \in \mathrm{Hom}(K_0(A), \mathbb{R}/K_0(A)) \mid r([1]) = 0\}.$$

Cocycle actions of $\mathbb{Z}^2$ on $\mathcal{Z}$

Let $(\alpha, u) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be a cocycle action.
As before, put $\check{u} = u(b, a)u(a, b)^*$.
The following theorem says that the de la Harpe-Skandalis determinant $\Delta_\tau(\check{u}) \in \mathbb{R}/\mathbb{Z}$ is the complete invariant of (α, u).

Theorem (Y. Sato and M)

Let $(\alpha, u), (\beta, v) : \mathbb{Z}^2 \curvearrowright \mathcal{Z}$ be strongly outer cocycle actions. Then they are cocycle conjugate if and only if $\Delta_\tau(\check{u}) = \Delta_\tau(\check{v})$.

The proof uses the same idea as $\mathbb{Z}$-actions:

- (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes \mathcal{Z}$.
- We have already classified $(\alpha \otimes \mathrm{id}, u \otimes 1)$ on $\mathcal{Z} \otimes B$ with B being a UHF algebra.
- Some extra effort gives the conclusion.

$\mathbb{Z}^N$-actions on UHF algebras of infinite type

A UHF algebra A is said to be of infinite type if $A \otimes A \cong A$.

Theorem (M)

Let A be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then α has the Rohlin property.

Theorem (M)

Let A be a UHF algebra of infinite type. Then, all strongly outer actions of $\mathbb{Z}^N$ on A are mutually cocycle conjugate to each other.

Open problems

- Classify strongly outer actions of $\mathbb{Z}^N$ on a general UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.

Cocycle actions of the Klein bottle group on UHF algebras

We call $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle$ the Klein bottle group.

Theorem (Y. Sato and M)

Let A be a UHF algebra and let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action. Then for any $m \in \mathbb{N}$, there exist central sequences of projections $(e_n)_n, (f_n)_n$ in A such that

$$\alpha_a(e_n) - e_n \to 0, \quad \alpha_a(f_n) - f_n \to 0,$$
$$\sum_{i=0}^{m-1} \alpha_b^i(e_n) + \sum_{j=0}^{m} \alpha_b^j(f_n) \to 1.$$

Theorem (Y. Sato and M)

All strongly outer cocycle actions of Γ on a UHF algebra are mutually cocycle conjugate.

Cocycle actions of the Klein bottle group on $\mathcal{Z}$

Theorem (Y. Sato and M)

All strongly outer cocycle actions of Γ on the Jiang-Su algebra $\mathcal{Z}$ are mutually cocycle conjugate.

The proof uses the same idea as $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions: we know that $(\alpha, u) : \Gamma \curvearrowright \mathcal{Z}$ is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright \mathcal{Z} \otimes \mathcal{Z}$.

Concluding remarks

Informal definition (suggested by T. Katsura)

An amenable group Γ is said to be *too elementary* if a complete classification of strongly outer actions $\alpha : \Gamma \curvearrowright \mathcal{Z}$ is possible.

- $\mathbb{Z}$, $\mathbb{Z}^2$ and $\langle a, b \mid bab^{-1} = a^{-1} \rangle$ are *too elementary*.
- Probably, $\mathbb{Z}^N$ is also *too elementary*.
- Probably, finite groups are not *too elementary*.

Remark

It is not so elementary to prove that a given Γ is *too elementary* (or not).

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