Group actions on simple stably finite C^*-algebras

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joint work with Y. Sato

Notations

- A group Γ is always assumed to be countable, discrete and amenable.
- A C^*-algebra A is always assumed to be unital, simple and separable.
- $U(A)$ denotes the group of unitaries of A, and $\mathrm{Aut}(A)$ denotes the group of automorphisms of A.
- For $u \in U(A)$, $\mathrm{Ad}\, u \in \mathrm{Aut}(A)$ is given by $x \mapsto uxu^*$ and is called an inner automorphism.
- $\mathcal{Z}$ denotes the Jiang-Su algebra.

Cocycle actions

Definition

A pair (α, u) of a map $\alpha : \Gamma \to \mathrm{Aut}(A)$ and a map $u : \Gamma \times \Gamma \to U(A)$ is called a cocycle action of Γ on A if

$$\alpha_g \circ \alpha_h = \mathrm{Ad}\, u(g,h) \circ \alpha_{gh}$$

and

$$u(g,h)u(gh,k) = \alpha_g(u(h,k))u(g,hk)$$

hold for any $g, h, k \in \Gamma$. We write $(\alpha, u) : \Gamma \curvearrowright A$.

We always assume $\alpha_1 = \mathrm{id}$, $u(g,1) = u(1,g) = 1$ for all $g \in \Gamma$.
When α_g is not inner for any $g \in \Gamma \setminus \{1\}$,
(α, u) is said to be outer.
When $u \equiv 1$, $\alpha : \Gamma \curvearrowright A$ is a genuine action.

Cocycle conjugacy

Definition

Two cocycle actions $(\alpha, u) : \Gamma \curvearrowright A$ and $(\beta, v) : \Gamma \curvearrowright B$ are said to be cocycle conjugate if there exist a family of unitaries $(w_g)_{g\in\Gamma}$ in B and an isomorphism $\theta : A \to B$ such that

$$\theta \circ \alpha_g \circ \theta^{-1} = \mathrm{Ad}\, w_g \circ \beta_g$$

and

$$\theta(u(g,h)) = w_g \beta_g(w_h) v(g,h) w_{gh}^*$$

hold for every $g, h \in \Gamma$.

Our eventual goal is

- to classify the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$.
- to classify (α, u) up to cocycle conjugacy and to determine when (α, u) is cocycle conjugate to a genuine action.

Twisted crossed product

Definition

For $(\alpha, u) : \Gamma \curvearrowright A$, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is the universal C^*-algebra generated by A and a family of unitaries $(\lambda_g^\alpha)_{g\in\Gamma}$ satisfying

$$\lambda_g^\alpha \lambda_h^\alpha = u(g,h)\lambda_{gh}^\alpha \quad \text{and} \quad \lambda_g^\alpha a \lambda_g^{\alpha *} = \alpha_g(a)$$

for all $g, h \in \Gamma$ and $a \in A$.

If (α, u) and (β, v) are cocycle conjugate via $\theta : A \to B$ and $(w_g)_g$, then $A \rtimes_{(\alpha,u)} \Gamma$ and $B \rtimes_{(\beta,v)} \Gamma$ are canonically isomorphic by

$$a \mapsto \theta(a) \quad \text{and} \quad \lambda_g^\alpha \mapsto w_g \lambda_g^\beta.$$

$\mathcal{Z}$-stability of crossed product

The following is the main theorem.

Theorem (Y. Sato and M)

Let A be a stably finite C^*-algebra with finite nuclear dimension and with finitely many extremal tracial states. Let Γ be an elementary amenable group.
Let $(\alpha, u) : \Gamma \curvearrowright A$ be a strongly outer cocycle action.
Then (α, u) is cocycle conjugate to $(\alpha \otimes \mathrm{id}, u \otimes 1) : \Gamma \curvearrowright A \otimes \mathcal{Z}$.
In particular, the twisted crossed product $A \rtimes_{(\alpha,u)} \Gamma$ is $\mathcal{Z}$-stable.

In order to prove this, it suffices to construct a unital embedding of $\mathcal{Z}$ into the fixed point algebra $(A^\infty \cap A')^\alpha$.

Strong outerness

Let $T(A)$ denote the set of tracial states and let π_τ be the GNS representation by $\tau \in T(A)$.

Definition

$\alpha \in \mathrm{Aut}(A)$ is said to be not weakly inner if the extension $\bar{\alpha}$ on $\pi_\tau(A)''$ is not inner for any $\tau \in T(A)^\alpha$, that is, there does not exist a unitary $U \in \pi_\tau(A)''$ such that $\bar{\alpha} = \mathrm{Ad}\, U$.
A cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ is said to be strongly outer if α_g is not weakly inner for every $g \in \Gamma \setminus \{1\}$.

If $T(A) = \{\tau\}$, then

$$(\alpha, u) : \Gamma \curvearrowright A \text{ is strongly outer} \quad \Longleftrightarrow \quad (\bar{\alpha}, u) : \Gamma \curvearrowright \pi_\tau(A)'' \text{ is outer.}$$

Elementary amenable groups

Definition

The class of elementary amenable groups is defined as the smallest family of groups containing all finite groups and all abelian groups, and closed under the processes of taking subgroups, quotients, group extensions and increasing unions.

For instance, all solvable groups are elementary amenable.
There exist amenable groups which are not elementary (R. I. Grigorchuk).

Weak Rohlin property (1/2)

Proposition

Let A be a nuclear stably finite C^*-algebra with finitely many extremal tracial states and let Γ be elementary.
Then any strongly outer cocycle action $(\alpha, u) : \Gamma \curvearrowright A$ has the weak Rohlin property, i.e. for any $F \Subset \Gamma$ and $\varepsilon > 0$, there exist an (F, ε)-invariant $K \Subset \Gamma$ and a sequence $(e_n)_n$ of positive contractions in A such that

$$[e_n, a] \to 0, \quad \alpha_g(e_n)\alpha_h(e_n) \to 0, \quad \tau(e_n) \to |K|^{-1}$$

as $n \to \infty$ for all $a \in A$, $g, h \in K$ with $g \neq h$ and $\tau \in T(A)$.

Weak Rohlin property (2/2)

We sketch the proof in the case A has a unique trace τ.
We regard A as a subalgebra of $M = \pi_\tau(A)''$, which is the AFD II_1-factor. The cocycle action (α, u) extends to $(\bar{\alpha}, u) : \Gamma \curvearrowright M$, and $(\bar{\alpha}, u)$ on M is outer because (α, u) is strongly outer.
Hence $\exists K \Subset \Gamma$ and a sequence of projections $(p_n)_n$ in M such that

$$\|[p_n, x]\|_2 \to 0 \quad \forall x \in M, \qquad \left\| 1 - \sum_{g \in K} \bar{\alpha}_g(p_n) \right\|_2 \to 0.$$

By using Haagerup's theorem and Kishimoto's technique, we can find a sequence $(e_n)_n$ of positive contractions in A such that

$$[e_n, a] \to 0, \quad \alpha_g(e_n)\alpha_h(e_n) \to 0, \quad \|e_n - p_n\|_2 \to 0.$$

for all $a \in A$ and $g, h \in K$ with $g \neq h$.

Property (SI)

A bounded sequence $(x_n)_n$ in A is said to be central if $[x_n, a] \to 0$ as $n \to \infty$ for all $a \in A$.

Proposition

Let A be a stably finite C^*-algebra with finite nuclear dimension. Then A has the property (SI), i.e. for any central sequences $(x_n)_n$ and $(y_n)_n$ in A satisfying $0 \leq x_n, y_n \leq 1$,

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(x_n) = 0 \quad \textit{and} \quad \inf_{m\in\mathbb{N}} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(y_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n\to\infty} \|s_n^* s_n - x_n\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|y_n s_n - s_n\| = 0.$$

Proof of the main theorem

It suffices to find a unital homomorphism from the prime dimension drop algebra $I(k,k+1)$ into $(A^\infty \cap A')^\alpha$. $I(k,k+1)$ is known to be the universal C^*-algebra generated by $c_1, c_2, \dots, c_k, s$ satisfying

$$c_1 \geq 0, \quad c_i c_j^* = \delta_{ij} c_1^2,$$

$$c_1 s = s, \quad c_1^2 + c_2^* c_2 + \cdots + c_k^* c_k + s^* s = 1.$$

Since A is $\mathcal{Z}$-stable (W. Winter), we can embed these generators into $A^\infty \cap A'$. An "averaging argument" using the weak Rohlin property enables us to replace c_i with $\tilde{c}_i \in (A^\infty \cap A')^\alpha$. Thanks to the property (SI) (and also the weak Rohlin property again), we can replace s with $\tilde{s} \in (A^\infty \cap A')^\alpha$.
It follows that there exists a unital homomorphism from $I(k,k+1)$, and hence from $\mathcal{Z}$, into $(A^\infty \cap A')^\alpha$.

Classification up to cocycle conjugacy

Known results

- $\mathbb{Z} \curvearrowright$ UHF (A. Kishimoto)
- $\mathbb{Z}^2 \curvearrowright$ UHF (H. Nakamura, T. Katsura and M)
- $\mathbb{Z} \curvearrowright \mathcal{Z}$ (Y. Sato)
- $\mathbb{Z}^2 \curvearrowright \mathcal{Z}$ (Y. Sato and M)
- $\mathbb{Z}^N \curvearrowright$ UHF$_\infty$ (M)

Work in progress (Y. Sato and M)

$\Gamma \curvearrowright$ UHF and $\Gamma \curvearrowright \mathcal{Z}$,
where $\Gamma = \langle a, b \mid bab^{-1} = a^{-1} \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}$,
called the Klein bottle group.