$\mathbb{Z}^N$-actions on UHF algebras of infinite type

Hiroki Matui
matui@math.s.chiba-u.ac.jp

Chiba University

June 26, 2010
Recent Developments in Operator Algebras
- on the occasion of the 77th birthday of Masamichi Takesaki -
University of Tokyo
Goal

Goal (too ambitious)

Classify outer actions of discrete amenable groups on simple classifiable $C^*$-algebras (Kirchberg algebras, UHF, AF, AT, AH algebras, the Jiang-Su alg. etc.) up to cocycle conjugacy.

Outer actions of finite groups on unital simple $C^*$-algebras do not always have the Rohlin property. In fact, if an action $\alpha : \Gamma \curvearrowright A$ of a finite group $\Gamma$ on a unital simple $C^*$-algebra $A$ has the Rohlin property, then $K_0(A)$ and $K_1(A)$ are completely cohomologically trivial as $\Gamma$-modules.

Goal (realistic)

Classify outer actions of $\mathbb{Z}^N$ (or “poly-infinite-cyclic” groups) on simple classifiable $C^*$-algebras up to cocycle conjugacy.
Cocycle conjugacy

**Definition**

Let $\alpha : \Gamma \curvearrowright A$ be an action of a countable discrete group $\Gamma$ on a unital $C^*$-algebra $A$. $(u_g)_{g \in \Gamma} \subset U(A)$ is called an $\alpha$-cocycle if

$$u_g \alpha_g(u_h) = u_{gh} \quad \forall g, h \in \Gamma.$$ 

Two actions $\alpha, \beta : \Gamma \curvearrowright A$ are said to be cocycle conjugate, if

$$\exists \gamma \in \text{Aut}(A), \quad \exists (u_g)_g \alpha \text{-cocycle}$$

$$\text{Ad } u_g \circ \alpha_g = \gamma \circ \beta_g \circ \gamma^{-1} \quad \forall g \in \Gamma.$$ 

**Strong cocycle conjugacy** further requires

$$\exists v_n \in U(A), \quad \|u_g - v_n \alpha_g(v_n^*)\| \to 0.$$
**Strong outerness**

An automorphism of the from $\text{Ad} \, u$ is said to be inner.

An action $\alpha : \Gamma \curvearrowright A$ is said to be **outer** if $\alpha_g$ is not inner for every $g \in \Gamma \setminus \{e\}$.

Let $T(A)$ denote the set of tracial states and let $\pi_\tau$ be the GNS representation by $\tau \in T(A)$.

$\alpha \in \text{Aut}(A)$ is said to be **not weakly inner** if the extension $\bar{\alpha}$ on $\pi_\tau(A)^{\prime\prime}$ is not inner for any $\tau \in T(A)^{\alpha}$, that is, there does not exist a unitary $U \in \pi_\tau(A)^{\prime\prime}$ such that $\bar{\alpha} = \text{Ad} \, U$.

An action $\alpha : \Gamma \curvearrowright A$ is said to be **strongly outer** if $\alpha_g$ is not weakly inner for every $g \in \Gamma \setminus \{e\}$.

If $T(A) = \{\tau\}$, then

\[ \alpha : \Gamma \curvearrowright A \text{ is strongly outer} \iff \bar{\alpha} : \Gamma \curvearrowright \pi_\tau(A)^{\prime\prime} \text{ is outer.} \]
A UHF algebra $A$ is said to be of **infinite type** if $A \otimes A \cong A$.

**Theorem (M)**

Let $A$ be a UHF algebra of infinite type. For $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. $\alpha$ is strongly outer.
2. $\alpha$ has the Rohlin property.

**Theorem (M)**

Let $A$ be a UHF algebra of infinite type. Any strongly outer actions of $\mathbb{Z}^N$ on $A$ are mutually strongly cocycle conjugate to each other.
Group actions on AFD factors

- (C. E. Sutherland and M. Takesaki 1989) Actions of discrete amenable groups on AFD factors of type III\(_\lambda\) with \(\lambda \neq 1\).
- (Y. Kawahigashi, C. E. Sutherland and M. Takesaki 1992) Actions of discrete abelian groups on the AFD factor of type III\(_1\).
- (T. Masuda) A short proof, which is independent of types and is based on Evans-Kishimoto intertwining argument developed in C\(^*\)-algebras.
$\mathbb{Z}^N$-actions on simple $C^*$-algebras

- (A. Kishimoto 1995)
  Uniqueness of strongly outer $\mathbb{Z}$-actions on UHF algebras.
- (A. Kishimoto 1998, M 2010)
  Strongly outer $\mathbb{Z}$-actions on AT or AH algebras.
- (H. Nakamura 2000)
  Outer $\mathbb{Z}$-actions on Kirchberg algebras.
- (T. Katsura and M 2008)
  Strongly outer $\mathbb{Z}^2$-actions on UHF algebras.
- (M. Izumi and M 2010)
  Outer and locally $KK$-trivial $\mathbb{Z}^2$-actions on Kirchberg alg.
  Uniqueness of outer $\mathbb{Z}^N$-actions on $\mathcal{O}_2$, $\mathcal{O}_\infty$ etc.
- (Y. Sato, Y. Sato and M)
  Uniqueness of strongly outer $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on the Jiang-Su algebra $\mathcal{Z}$. 
Rohlin property for $\mathbb{Z}^N$-actions (1)

Let $\xi_1, \xi_2, \ldots, \xi_N \in \mathbb{Z}^N$ be the canonical basis. We let $A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$ and $A^\infty = A^\infty \cap A'$.

Definition (Nakamura 1999)

$\alpha : \mathbb{Z}^N \curvearrowright A$ is said to have the **Rohlin property**, if for any $M \in \mathbb{N}$, there exist $R \in \mathbb{N}, \ m_r \in \mathbb{N}^N \ (r = 1, 2, \ldots, R)$ and projections $e^{(r)}_g \in A^\infty \ (r = 1, 2, \ldots, R, \ g \in \mathbb{Z}^N / m_r \mathbb{Z}^N)$ such that

$$\sum_{r, g} e^{(r)}_g = 1, \quad \alpha_{\xi_i}(e^{(r)}_g) = e^{(r)}_{g + \xi_i} \quad (i = 1, 2, \ldots, N)$$

and each coordinate of $m_r$ is not less than $M$.
Rohlin property for $\mathbb{Z}^N$-actions (2)

We can restate the definition of the Rohlin property as follows:

For any $i = 1, 2, \ldots, N$ and $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \in \mathbb{N}$ ($r = 1, 2, \ldots, R$) and projections $e^{(r)}_g \in A_\infty$ ($r = 1, 2, \ldots, R$, $g \in \mathbb{Z}/m_r\mathbb{Z}$)

such that

$$m_r \geq M, \quad \sum_{r, g} e^{(r)}_g = 1$$

and

$$\alpha_{\xi_i}(e^{(r)}_g) = e^{(r)}_{g+\xi_i}, \quad \alpha_{\xi_j}(e^{(r)}_g) = e^{(r)}_g \quad \forall j \neq i.$$ 

Thus,

$$\alpha : \mathbb{Z}^N \curvearrowright A \text{ has the Rohlin property}$$

$$\iff \exists \text{ Rohlin towers for } \alpha_{\xi_i} \text{ in } (A_\infty)^{\alpha_i^t} \text{ for any } i = 1, 2, \ldots, N.$$
Asymptotically representable actions

An action \( \alpha : \Gamma \bowtie A \) of a discrete group \( \Gamma \) is said to be **approximately representable** if there exists a family of unitaries \((v_g)_{g \in \Gamma}\) in \( A^\infty \) such that

\[
v_g v_h = v_{gh}, \quad \alpha_g(v_h) = v_{ghg^{-1}}, \quad v_g a v_g^* = \alpha_g(a)
\]

for any \( g, h \in \Gamma \) and \( a \in A \). Notice that

\[
a \mapsto a, \quad \lambda_g \mapsto v_g
\]

induce a homomorphism from \( A \times_\alpha \Gamma \) to \( A^\infty \).

An action \( \alpha : \Gamma \bowtie A \) of a discrete group \( \Gamma \) is said to be **asymptotically representable** if there exists a family of continuous paths of unitaries \((v_g(t))_{g \in \Gamma, t \in [0, \infty)}\) in \( A \) satisfying analogous properties.
Uniqueness of actions with the Rohlin property (1)

Proposition

Let $A$ be a UHF algebra of infinite type and let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be two actions with the Rohlin property. Then $\alpha$ and $\beta$ are cocycle conjugate.

The proof is by induction on $N$.

Let $\alpha' : \mathbb{Z}^{N-1} \curvearrowright A$ and $\beta' : \mathbb{Z}^{N-1} \curvearrowright A$ be the actions of $\mathbb{Z}^{N-1}$ generated by $\xi_1, \xi_2, \ldots, \xi_{N-1}$.

By induction hypothesis, $\alpha'$ and $\beta'$ are cocycle conjugate. In particular, the crossed products $A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ and $A \rtimes_{\beta'} \mathbb{Z}^{N-1}$ are identified.

The automorphisms $\alpha_{\xi_N}, \beta_{\xi_N}$ of $A$ extend to automorphisms $\tilde{\alpha}_{\xi_N}, \tilde{\beta}_{\xi_N}$ of the crossed products. We would like to compare $\tilde{\alpha}_{\xi_N}$ and $\tilde{\beta}_{\xi_N}$.
Uniqueness of actions with the Rohlin property (2)

(1) First, we observe that the crossed product $A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$ is a unital simple AT algebra with real rank zero.

(2) By using Kishimoto-Kumjian’s theorem, we can show $\tilde{\alpha}_{\xi_N}$ is asymptotically inner, i.e.

$$ \exists (u_t)_{t \in [0, \infty)} \text{ in } U(A \rtimes_{\alpha'} \mathbb{Z}^{N-1}) \text{ such that } \tilde{\alpha}_{\xi_N} = \lim \text{Ad } u_t. $$

(3) By induction hypothesis, $\alpha'$ is asymptotically representable. Hence we may assume that $(u_t)_{t \in [0, \infty)}$ is in $U(A)$.

(4) The same is true for $\tilde{\beta}_{\xi_N}$, and so

$$ \exists (v_t)_{t \in [0, \infty)} \text{ in } U(A) \text{ such that } \tilde{\alpha}_{\xi_N} = \lim \text{Ad } v_t \circ \tilde{\beta}_{\xi_N}. $$

(5) $\alpha$ and $\beta$ have the Rohlin property. Therefore $\tilde{\alpha}_{\xi_N}$ and $\tilde{\beta}_{\xi_N}$ have the Rohlin property (as single automorphisms) and the Rohlin projections can be taken from $A$.

(6) These items (together with a homotopy lemma) enables us to carry out the $\mathbb{Z}^{N-1}$-equivariant Evans-Kishimoto intertwining argument, which completes the proof of the proposition.
Let $A$ be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. Then $\alpha$ has the Rohlin property.

The proof is by induction on $N$. As before, let $\alpha' : \mathbb{Z}^{N-1} \curvearrowright A$ be the action of $\mathbb{Z}^{N-1}$ generated by $\xi_1, \xi_2, \ldots, \xi_{N-1}$.

(1) Since $\bar{\alpha} : \mathbb{Z}^N \curvearrowright \pi_\tau(A)'$ is outer, for any $m \in \mathbb{N}$, one can find a Rohlin tower

$$e, \alpha_{\xi_N}(e), \ldots, \alpha_{\xi_N}^{m-1}(e) \in (A_\infty)^{\alpha'}$$

such that the ‘trace’ of

$$1 - (e + \alpha_{\xi_N}(e) + \cdots + \alpha_{\xi_N}^{m-1}(e))$$

is very small.
(2) We would like to replace $e$ to achieve $\alpha_{\xi_N}^m(e) = e$.
To this end it suffices to find a partial isometry $v \in (A_\infty)^{\alpha'}$
such that $v^*v = e$ and $vv^* = \alpha_{\xi_N}(e)$.
(3) By induction hypothesis, $\alpha'$ has the Rohlin property. Actions
with the Rohlin property are already shown to be unique up to
cocycle conjugacy. It follows that $\alpha'$ is approximately representable.
(4) In the same way as before, we can find a sequence $(u_n)_n$ of
unitaries of $A$ such that $\tilde{\alpha}_{\xi_N}(x) = \lim u_n x u_n^* \forall x \in A \rtimes_{\alpha'} \mathbb{Z}^{N-1}$.
(5) Then we obtain $v$ and achieve $\alpha_{\xi_N}^m(e) = e$.
(6) In the same way as the single automorphism case, we can
construct two Rohlin towers in $(A_\infty)^{\alpha'}$ whose sum is equal to 1.
The same is true for other generators, and so we can conclude that
$\alpha : \mathbb{Z}^N \curvearrowright A$ has the Rohlin property.
Let $A$ be a UHF algebra of infinite type and let $\alpha : \mathbb{Z}^N \curvearrowright A$ be a strongly outer action. For any $\alpha$-cocycle $(u_g)_{g \in \mathbb{Z}^N}$ in $A$ and $\varepsilon > 0$, there exists a unitary $v \in A$ such that

$$\|u_{\xi_i} - v\alpha_{\xi_i}(v^*)\| < \varepsilon \quad \forall i = 1, 2, \ldots, N.$$ 

(1) Consider two homomorphisms $\varphi, \psi : C^*(\mathbb{Z}^N) \to A \rtimes_\alpha \mathbb{Z}^N$ defined by

$$\varphi(\lambda_g) = \lambda_g^\alpha, \quad \psi(\lambda_g) = u_g \lambda_g^\alpha.$$ 

(2) Show $KK(\varphi) = KK(\psi)$ and $\tau \circ \varphi = \tau \circ \psi$.

(3) From these data, one can conclude that $\varphi$ and $\psi$ are approximately unitarily equivalent.
(4) $\alpha$ is already known to be approximately representable, and so there exists $(v_n)_n$ in $U(A)$ such that $\psi = \lim \text{Ad} v_n \circ \varphi$.

(5) This implies $\|u_g - v_n \alpha_g(v_n^*)\| \to 0$ for any $g \in \mathbb{Z}^N$.

Combining the three propositions above, we obtain the main result, i.e. any strongly outer actions of $\mathbb{Z}^N$ on a UHF algebra of infinite type are strongly cocycle conjugate to each other.

**Remark**

In general, strong cocycle conjugacy is strictly stronger than cocycle conjugacy.
Uniqueness of asymptotically representable actions

Let $A$ be a unital simple AH algebra with real rank zero, slow dimension growth and finitely many extremal tracial states.

**Theorem (M)**

For an approximately representable action $\alpha : \mathbb{Z}^N \curvearrowright A$, the following are equivalent.

1. $\alpha$ is strongly outer.
2. $\alpha$ has the Rohlin property.

**Theorem (M)**

Let $\alpha, \beta : \mathbb{Z}^N \curvearrowright A$ be strongly outer, asymptotically representable actions. Then they are cocycle conjugate.
Let $A$ be a UHF algebra. For each prime number $p$, we put

$$
\zeta(p) = \sup\{k \mid [1] \text{ is divisible by } p^k \text{ in } K_0(A)\}.
$$

For a $\mathbb{Z}^2$-action $\alpha$ on $A$, an invariant $[\alpha]$ is defined as an element in

$$
\prod_{p \in P(A)} \mathbb{Z}/p^{\zeta(p)} \mathbb{Z} \cong \pi_1(\text{Aut}(A)),
$$

where $P(A) = \{p \mid 1 \leq \zeta(p) < \infty\}$.

**Theorem (T. Katsura and M 2008)**

Let $A$ be a UHF algebra and let $\alpha, \beta : \mathbb{Z}^2 \curvearrowright A$ be strongly outer actions. Then, $\alpha$ and $\beta$ are strongly cocycle conjugate if and only if $[\alpha] = [\beta]$.
Strongly self-absorbing $C^*$-algebras

A $C^*$-algebra $A \neq \mathbb{C}$ is said to be strongly self-absorbing if there exists an isomorphism $\mu : A \rightarrow A \otimes A$ such that $\mu$ is approximately unitarily equivalent to $a \mapsto a \otimes 1$.

The known examples are $\text{UHF}_\infty$, the Jiang-Su algebra $\mathcal{Z}$, $\mathcal{O}_2$, $\mathcal{O}_\infty$ and $\mathcal{O}_\infty \otimes \text{UHF}_\infty$. Uniqueness of $\mathbb{Z}^N$-actions on these algebras has been obtained except for $\mathbb{Z}_N \curvearrowright \mathcal{Z}$ with $N \geq 3$.

<table>
<thead>
<tr>
<th>algebras</th>
<th>$\mathbb{Z}$</th>
<th>$\mathbb{Z}^2$</th>
<th>$\mathbb{Z}^N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{UHF}_\infty$</td>
<td>Kishimoto ’95</td>
<td>Katsura-M ’08</td>
<td>M</td>
</tr>
<tr>
<td>$\mathcal{Z}$</td>
<td>Sato</td>
<td>Sato-M</td>
<td>?</td>
</tr>
<tr>
<td>Kirchberg</td>
<td>Nakamura ’00</td>
<td></td>
<td>Izumi-M ’10</td>
</tr>
</tbody>
</table>

Why unique? — the key is $\pi_n(\text{Aut}(A)) = 0$ for every $n \geq 0$. 
Open problems

- Show that strongly outer actions of $\mathbb{Z}^N$ on a (general) UHF algebra have the Rohlin property when $N \geq 3$.
- Classify strongly outer actions of $\mathbb{Z}^N$ on a (general) UHF algebra when $N \geq 3$.
- Show the uniqueness of strongly outer $\mathbb{Z}^N$-actions on the Jiang-Su algebra $\mathcal{Z}$ when $N \geq 3$.
- Classify strongly outer $\mathbb{Z}^2$-actions on a unital simple AF algebra (as much as possible).
- Classify strongly outer $\mathbb{Z}$-actions on a unital simple $\mathcal{Z}$-stable $C^*$-algebra $A$ (as much as possible).