$\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions on simple C^*-algebras

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Contents

Goal

Classify outer $\mathbb{Z}^N$-actions on simple C^*-algebras (AF, AT, AH and Kirchberg algebras) up to cocycle conjugacy.

type	$\mathbb{Z}$-action	$\mathbb{Z}^2$-action
finite	UHF in Kishimoto '95	UHF in Nakamura '99
	AT in Kishimoto '98	UHF in Katsura-M '08
	AH	AF
infinite	Nakamura '00	M '08, Izumi-M '09

In what follows, $\mathbb{Z}$-actions are identified with single automorphisms.

Central sequence algebras

For a unital C^*-algebra A, we let

$$c_0(\mathbb{N}, A) = \{(a_n)_n \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\},$$

$$c_\omega(\mathbb{N}, A) = \{(a_n)_n \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\omega} \|a_n\| = 0\},$$

where $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter.
Limit algebras of A are defined by

$$A^\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A) \text{ and } A^\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(\mathbb{N}, A).$$

Central sequence algebras of A are defined by

$$A_\infty = A^\infty \cap A' \text{ and } A_\omega = A^\omega \cap A'.$$

$\alpha : G \curvearrowright A$ extends to $A^\infty, A^\omega, A_\infty, A_\omega$.

Automorphisms

$\alpha \in \mathrm{Aut}(A)$ is said to be approximately inner, if there exists a sequence of unitaries $\{u_n\}_n$ in A such that

$$\alpha(x) = \lim_{n\to\infty} u_n x u_n^* \quad \forall x \in A.$$

$\overline{\mathrm{Inn}}(A)$ denotes the set of approximately inner automorphisms.
$\alpha \in \mathrm{Aut}(A)$ is said to be asymptotically inner, if there exists a continuous path of unitaries $\{u_t\}_{t\in[0,\infty)}$ in A such that

$$\alpha(x) = \lim_{t\to\infty} u_t x u_t^* \quad \forall x \in A.$$

$\alpha, \beta \in \mathrm{Aut}(A)$ are said to be asymptotically unitarily equivalent, if $\alpha\beta^{-1}$ is asymptotically inner.
When A is a unital Kirchberg algebra, α, β are asymptotically unitarily equivalent if and only if $KK(\alpha) = KK(\beta)$.

Kishimoto's results

Theorem (Kishimoto 1998)

Let A be a unital simple AT algebra of real rank zero with unique tracial state τ and let $\alpha \in \overline{\mathrm{Inn}}(A)$. Then, the following are equivalent.

1. *α has the Rohlin property.*
2. *$\bar{\alpha}$ on $\pi_\tau(A)''$ is aperiodic, i.e. $\bar{\alpha}^n$ is outer for all $n \in \mathbb{N}$.*

Theorem (Kishimoto 1998)

Let A be a unital simple AT algebra of real rank zero. If $\alpha, \beta \in \mathrm{Aut}(A)$ have the Rohlin property and $\alpha\beta^{-1}$ is asymptotically inner, then α is cocycle conjugate to β.

Rohlin property of single automorphisms

Definition (Kishimoto 1995)

$\alpha \in \mathrm{Aut}(A)$ is said to have the Rohlin property, if for any $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \geq M$ $(r = 1, 2, \dots, R)$ satisfying the following: for any $F \Subset A$ and $\varepsilon > 0$

$$\exists \text{projections } e_g^{(r)} \in A \quad (r = 1, 2, \dots, R,\ g \in \mathbb{Z}/m_r\mathbb{Z})$$

such that

$$\sum_{r,g} e_g^{(r)} = 1, \quad \|\alpha(e_g^{(r)}) - e_{g+1}^{(r)}\| < \varepsilon$$

and

$$\|[a, e_g^{(r)}]\| < \varepsilon \quad \forall a \in F.$$

Construction of Rohlin towers (1)

By the outerness of $\bar{\alpha}^n \in \mathrm{Aut}(\pi_\tau(A)'')$,
for any $m \in \mathbb{N}$, one can find a Rohlin tower

$$e, \alpha(e), \dots, \alpha^{m-1}(e) \in A_\infty$$

such that $\alpha^m(e) = e$ and the 'trace' of

$$f = 1 - (e + \alpha(e) + \cdots + \alpha^{m-1}(e))$$

is very small. We have to construct another tower so that the projections form a partition of unity.

Lemma (Kishimoto 1998)

Let A be a unital simple AT algebra with real rank zero. $\forall\ F \Subset A$ and $\varepsilon > 0$, $\exists\ G \Subset A$, $\delta > 0$ and $k \in \mathbb{N}$ satisfying the following: If $p, q \in A$ are projections satisfying $k[q] \leq [p]$, $\|[a, p]\| < \delta$ and $\|[a, q]\| < \delta$ for all $a \in G$, then $\exists$ a partial isometry $v \in A$ such that $v^*v = q$, $vv^* \leq p$ and $\|[a, v]\| < \varepsilon$ for all $a \in F$.

Construction of Rohlin towers (2)

By using the previous lemma, one can construct Rohlin towers $e, \alpha(e), \dots, \alpha^{m-1}(e)$ and $f, \alpha(f), \dots, \alpha^{m}(f)$ in A_∞ such that $\alpha^m(e) = e$, $\alpha^{m+1}(f) = f$ and $\sum \alpha^i(e) + \alpha^j(f) = 1$, and so the proof is completed.

Lemma

The previous lemma also holds for any unital simple classifiable AH algebra.

Proof.

Let A be a unital simple classifiable AH algebra and let Q be the UHF algebra such that $K_0(Q) \cong \mathbb{Q}$. By the classification theorem, $A \otimes Q$ is a unital simple AT algebra with real rank zero, and so we can apply the previous lemma to it. Since A is approximately divisible, one can 'come back' to A. $\square$

Construction of Rohlin towers (3)

Consequently we obtain the following extension of Kishimoto's theorem.

Theorem (in preparation)

Kishimoto's Rohlin type theorem holds for unital simple classifiable AH algebras with unique tracial state and approximately inner automorphisms.

Remark

Some further argument shows the following: Let A be a unital simple classifiable AH algebra with finitely many extremal traces. Suppose that two projections $p, q \in A_\omega$ satisfy $\tau_\omega(p) < \tau_\omega(q)$ for all $\tau \in T(A)$, where $\tau_\omega \in T(A_\omega)$ is a natural extension of τ. Then there exists $v \in A_\omega$ such that $v^*v = p$ and $vv^* \leq q$. In particular, A_ω satisfies Blackadar's second fundamental comparability question.

Homotopy lemma (1)

Kishimoto's classification theorem is derived from the Evans-Kishimoto intertwining argument and the following homotopy lemma.

Lemma (Kishimoto 1998)

Let A be a unital simple AT algebra with real rank zero. For any $F \Subset A$ and $\varepsilon > 0$, there exist $G \Subset A$ and $\delta > 0$ such that the following holds: If $u : [0,1] \to A$ is a path of unitaries satisfying $\|[a, u(t)]\| < \delta$ for all $a \in G$ and $t \in [0,1]$, then there exists a path of unitaries $v : [0,1] \to A$ such that

$$v(0) = u(0), \quad v(1) = u(1), \quad \|[a, v(t)]\| < \varepsilon \quad \forall a \in F,\ t \in [0,1]$$

and $\mathrm{Lip}(v) < 5\pi + 1$.

Homotopy lemma (2)

Lemma

The previous homotopy lemma also holds for any unital simple classifiable AH algebra, the Lipschitz constant being bounded by 11π.

Consequently we obtain the following extension of Kishimoto's theorem.

Theorem (in preparation)

Let A *be a unital simple classifiable AH algebra. If* $\alpha, \beta \in \mathrm{Aut}(A)$ *have the Rohlin property and* $\alpha\beta^{-1}$ *is asymptotically inner, then* α *is cocycle conjugate to* β.

Nakamura's result

Nakamura gave the following classification result for aperiodic automorphisms of unital Kirchberg algebras.

Theorem (Nakamura 2000)

Let A be a unital Kirchberg algebra. For aperiodic automorphisms $\alpha, \beta \in \mathrm{Aut}(A)$, the following are equivalent.

1. $KK(\alpha) = KK(\beta)$.
2. α *is cocycle conjugate to* β *via an automorphism* γ *such that* $KK(\gamma) = 1$.

The proof uses the Evans-Kishimoto intertwining argument and the following theorem.

Theorem (Kirchberg-Phillips 2000)

If A is a unital Kirchberg alg., A_ω is purely infinite and simple.

Equivariant Kirchberg-Phillips' theorem

An action $\alpha : G \curvearrowright A$ is said to be approximately representable, if there exists a family of unitaries $\{v_g\}_{g\in G}$ in $U(A^\infty)$ such that

$$v_g v_h = v_{gh}, \quad \alpha_g(v_h) = v_{ghg^{-1}} \text{ and } v_g a v_g^* = \alpha_g(a)$$

hold for all $g, h \in G$ and $a \in A$.

Theorem (Izumi-M 2009)

Let A be a unital Kirchberg algebra and let $\alpha : G \curvearrowright A$ be an approximately representable outer action of a discrete amenable group G. Then, $(A_\omega)^\alpha$ is purely infinite and simple.

$\alpha : G \curvearrowright A$ is said to be asymptotically representable, when we can take each v_g as a continuous path $v_g : [0, \infty) \to U(A)$ rather than a sequence. Nakamura's theorem shows that $\alpha \in \mathrm{Aut}(A)$ is asymptotically representable, if α is aperiodic and $KK(\alpha) = 1$.

Equivariant Nakamura's theorem

Let $\alpha : G \curvearrowright A$ be an asymptotically representable outer action of G on a unital Kirchberg algebra A.
Suppose that $\beta_i \in \mathrm{Aut}(A)$ $(i = 1, 2)$ commutes with α_g and that $\beta_i^n \alpha_g$ is outer for all $(n, g) \neq (0, e)$,
i.e. $\alpha \times \beta_i$ gives an outer action of $G \times \mathbb{Z}$.
By using the pure infiniteness of $(A_\omega)^\alpha$, we can carry out the G-equivariant Evans-Kishimoto intertwining argument.

Theorem (Izumi-M 2009)

If $KK(\tilde{\beta}_1) = KK(\tilde{\beta}_2)$ in $KK(A \rtimes_\alpha G, A \rtimes_\alpha G)$, then the actions $\alpha \times \beta_1$ and $\alpha \times \beta_2$ of $G \times \mathbb{Z}$ on A are cocycle conjugate via an automorphism $\gamma \in \mathrm{Aut}(A)$ with $KK(\gamma) = 1$.

Locally KK-trivial $\mathbb{Z}^2$-actions

An action $\alpha : G \curvearrowright A$ is said to be locally KK-trivial, if $KK(\alpha_g) = 1$ for all $g \in G$.
Two actions $\alpha, \beta : G \curvearrowright A$ are KK-trivially cocycle conjugate, if they are cocycle conjugate via $\gamma \in \mathrm{Aut}(A)$ such that $KK(\gamma) = 1$.

Theorem (Izumi-M 2009)

Let A be a unital Kirchberg algebra. There exists a bijective correspondence between the following two sets.

1. *KK-trivially cocycle conjugacy classes of locally KK-trivial outer $\mathbb{Z}^2$-actions on A.*
2. $\{x \in KK(A, S \otimes A) \mid K_0(x)([1]) = 0\}$.

For example, the Cuntz algebra $\mathcal{O}_n$ has $n-1$ cocycle conjugacy classes of outer $\mathbb{Z}^2$-actions.

Uniqueness of $\mathbb{Z}^N$-actions

By using the equivariant Nakamura's theorem inductively, one can show the uniqueness of $\mathbb{Z}^N$-actions for algebras with sufficiently simple K-theory.

Theorem (Izumi-M 2009)

Suppose that A is $\mathcal{O}_2$, $\mathcal{O}_\infty$ or $\mathcal{O}_\infty \otimes C$, where C is a UHF algebra of infinite type. Then any outer $\mathbb{Z}^N$-actions on A are mutually cocycle conjugate.

Remark

There exist infinitely many cocycle conjugacy classes of outer $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty \otimes \mathbb{K}$ for $N \geq 3$.
This means that for $N \geq 3$, there exist a lot of outer cocycle $\mathbb{Z}^N$-actions on $\mathcal{O}_\infty$ that are not equivalent to genuine actions.

Rohlin property of $\mathbb{Z}^2$-actions

Definition (Nakamura 1999)

$\alpha : \mathbb{Z}^2 \curvearrowright A$ is said to have the Rohlin property, if for any $M \in \mathbb{N}$, there exist $R \in \mathbb{N}$, $m_r \in \mathbb{N}^2$ $(r = 1, 2, \ldots, R)$ satisfying the following: for any $F \Subset A$ and $\varepsilon > 0$

$$\exists\text{projections } e_g^{(r)} \in A \quad (r = 1, 2, \ldots, R,\ g \in \mathbb{Z}^2/m_r\mathbb{Z}^2)$$

such that

$$m_r \geq M, \quad \sum_{r,g} e_g^{(r)} = 1, \quad \|\alpha_{\xi_i}(e_g^{(r)}) - e_{g+\xi_i}^{(r)}\| < \varepsilon$$

and $\|[a, e_g^{(r)}]\| < \varepsilon \ \forall a \in F$, where $\xi_1 = (1, 0)$ and $\xi_2 = (0, 1)$.

Observation

$\alpha : \mathbb{Z}^2 \curvearrowright A$ has the Rohlin property if and only if

> $\exists$ *Rohlin projections for the single automorphism* α_{ξ_1} *in the fixed point subalgebra* $(A_\infty)^{\alpha_{\xi_2}}$

and

> $\exists$ *Rohlin projections for the single automorphism* α_{ξ_2} *in the fixed point subalgebra* $(A_\infty)^{\alpha_{\xi_1}}$.

Unlike the case of Kirchberg algebras, we know little about A_∞ and $(A_\infty)^\alpha$ when A is stably finite. This makes the classification of actions hard.

A Rohlin type theorem for $\mathbb{Z}^2 \curvearrowright$ AF

Theorem (in preparation)

Let $\alpha : \mathbb{Z}^2 \curvearrowright A$ be an action on a unital simple classifiable AH algebra with unique trace τ such that α^n is in $\overline{\mathrm{Inn}}(A)$ for all $n \in \mathbb{Z}^2$. If A is AF or α is approximately representable, then the following are equivalent.

1. *α has the Rohlin property.*
2. *$\bar{\alpha}^n$ on $\pi_\tau(A)''$ is outer for all $n \in \mathbb{Z}^2 \setminus \{0\}$.*

A key step of the proof is the following:
For any projection $e \in (A_\infty)^{\alpha_{(1,0)}}$, under the assumptions above, e and $\alpha_{(0,1)}(e)$ are Murray-von Neumann equivalent in $(A_\infty)^{\alpha_{(1,0)}}$.

An equivariant homotopy lemma

The other ingredient for the Evans-Kishimoto intertwining argument is a $\mathbb{Z}$-equivariant version of the homotopy lemma.

Lemma

Let A be a unital simple AF algebra with unique trace τ and let $\alpha \in \overline{\mathrm{Inn}}(A)$ be an automorphism such that $\bar{\alpha}^n$ on $\pi_\tau(A)''$ is outer for all $n \in \mathbb{N}$.
For any $F \Subset A \rtimes_\alpha \mathbb{Z}$ and $\varepsilon > 0$, there exist $G \Subset A \rtimes_\alpha \mathbb{Z}$ and $\delta > 0$ such that the following holds: If $u : [0,1] \to A$ is a path of unitaries satisfying $\|[a, u(t)]\| < \delta$ for all $a \in G$ and $t \in [0,1]$, then there exists a path of unitaries $v : [0,1] \to A$ such that

$$v(0) = u(0), \quad v(1) = u(1), \quad \|[a, v(t)]\| < \varepsilon \quad \forall a \in F, \ t \in [0,1]$$

and $\mathrm{Lip}(v)$ is bounded by a universal constant.

Equivariant Evans-Kishimoto intertwining argument

Let A and α be as in the previous slide, that is, A is a unital simple AF algebra with unique trace τ and $\alpha \in \overline{\mathrm{Inn}}(A)$ has the Rohlin property.
Suppose that $\beta_i \in \overline{\mathrm{Inn}}(A)$ $(i = 1, 2)$ commutes with α and that $\bar{\beta_i}^n \bar{\alpha}^m$ on $\pi_\tau(A)''$ is outer for all $(n, m) \neq (0, 0)$.
By using the Rohlin property of $\alpha \times \beta_i : \mathbb{Z}^2 \curvearrowright A$ and the α-equivariant homotopy lemma, we can prove the following.

Theorem (in preparation)

If $\tilde{\beta}_1$ and $\tilde{\beta}_2$ are asymptotically unitarily equivalent in $\mathrm{Aut}(A \rtimes_\alpha \mathbb{Z})$, then the actions $\alpha \times \beta_1$ and $\alpha \times \beta_2$ of $\mathbb{Z}^2$ on A are cocycle conjugate via an automorphism $\gamma \in \overline{\mathrm{Inn}}(A)$.

This theorem yields a new proof of classification of $\mathbb{Z}^2$-actions on UHF algebras which was obtained by Katsura-Matui.