Cantor minimal $\mathbb{Z}^d$-systems and C^*-algebras

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We consider minimal dynamical systems on Cantor sets. Let

$$X = \{0, 1\}^{\mathbb{N}}$$

be a Cantor set and let

$$\varphi : \mathbb{Z}^d \to \mathrm{Homeo}(X)$$

be an action of $\mathbb{Z}^d$ on X by homeomorphisms.

φ is said to be free, if $\varphi^n(x) \neq x$ for any $n \in \mathbb{Z}^d \setminus \{0\}$ and $x \in X$.
φ is said to be minimal, if every φ-orbit is dense in X.

When φ is free and minimal, we call (X, φ) a Cantor minimal $\mathbb{Z}^d$-system.

Let $C^*(X, \varphi)$ denote the crossed product C^*-algebra arising from (X, φ).
$C^*(X, \varphi)$ is a unital simple stably finite C^*-algebra.

Crossed product C^*-algebras

Theorem 1 (Putnam 1990)**.** *When* $d = 1$, $C^*(X, \varphi)$ *is a unital simple AT algebra with real rank zero.*

Problem 2. Let $d \geq 2$. Does $C^*(X, \varphi)$ have tracial rank zero ?

The following result provides circumstantial evidence for an affirmative answer to the problem above.

Theorem 3 (N. C. Phillips 2005)**.** *For any* $d \geq 1$, $C^*(X, \varphi)$ *has the following properties.*

(1) *Real rank zero.*

(2) *Stable rank one.*

(3) *The order on the* K_0*-group is determined by traces.*

Moreover, these properties also hold for the C^**-algebras arising from tiling spaces.*

The proof of the theorem above uses Forrest's construction of a 'large' AF subalgebra of $C^*(X, \varphi)$.

Another evidence for tracial rank zero is the AF embeddability.

Theorem 4 (M 2002). *Let X be a compact metrizable space and let $\varphi : \mathbb{Z}^2 \to \mathrm{Homeo}(X)$ be a free minimal action. If there exists $n \in \mathbb{Z}^2 \setminus \{0\}$ satisfying*

$\forall\varphi^n$*-invariant open subset* $U \neq \emptyset$

$\exists\varphi^n$*-invariant open subset* $V \neq \emptyset$

$s.t. \quad \overline{V} \subset U,$

then $C^*(X, \varphi)$ is AF embeddable.

For example, we can apply this theorem to any almost one-to-one extension of a product system of two minimal $\mathbb{Z}$-systems.

Problem 5. Let $d \geq 2$ and let φ be a free minimal action of $\mathbb{Z}^d$ on a compact metrizable space X. Is the crossed product $C^*(X, \varphi)$ always AF embeddable ?

Recently, H. Lin proved the following theorem.

Theorem 6 (H. Lin)**.** *Let X be a compact metrizable space and let $\varphi : \mathbb{Z}^2 \to \mathrm{Homeo}(X)$ be an action which is not necessarily minimal. If there exists a φ-invariant probability measure whose support is X, then $C^*(X, \varphi)$ is quasi-diagonal.*

We note that if φ is minimal, then every φ-invariant measure has full support.

Problem 7. Is it possible to extend the result above to the case of $\mathbb{Z}^d$-actions ?

Problem 8. Let $d = 2$. What is the necessary and sufficient condition so that $C^*(X, \varphi)$ is quasi-diagonal ?

Examples from tiling spaces

Let $\mathcal{P}$ be a finite collection of non-empty polyhedra in $\mathbb{R}^d$.
We call each element of $\mathcal{P}$ a prototile.
For $p \in \mathcal{P}$ and $t \in \mathbb{R}^d$, $p + t$ is called a tile.

A collection T of tiles is called a tiling, if the elements of T cover $\mathbb{R}^d$ with pairwise disjoint interiors.

We equip the set of tilings with a topology as follows:
Two tilings T and T' are close, if there exist a small $\varepsilon \in \mathbb{R}^d$ and a large $R > 0$ such that $T + \varepsilon$ and T' agree on $B(0, R)$.

We obtain a topological space consisting of tilings and an action of $\mathbb{R}^d$ on it by translation.

Let T_0 be an aperiodic and repetitive tiling which satisfies the finite pattern condition.
Let Ω be the orbit closure of T_0, namely

$$\Omega = \overline{\{T_0 + t \mid t \in \mathbb{R}^d\}}.$$

Then, it is known that Ω is compact and metrizable.
In addition, the natural $\mathbb{R}^d$ action φ on Ω is free and minimal.

Theorem 9 (Sadun-Williams 2003, etc)**.** *For any (Ω, φ) as above, there exists a Cantor minimal $\mathbb{Z}^d$-system (Y, ψ) such that $C^*(Y, \psi)$ is strong Morita equivalent to $C(\Omega) \times_\varphi \mathbb{R}^d$.*

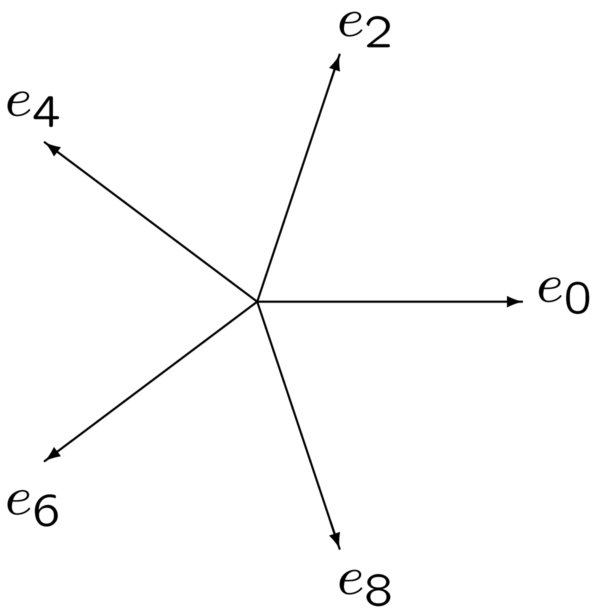
We would like to describe (Y, ψ) of the theorem above for the tiling space (Ω, φ) arising from the Penrose tiling.

For $n = 0, 1, \ldots, 9$, we put

$$e_n = \left(\cos \frac{2\pi n}{10}, \sin \frac{2\pi n}{10}\right) \in \mathbb{R}^2.$$

Let Γ be the $\mathbb{Z}$-span of $e_0, e_1, \ldots, e_9$.

$$\begin{aligned}\Gamma &= \langle e_0, e_1, \ldots, e_9 \rangle \\ &= \langle e_0, e_2, e_4, e_6, e_8 \rangle \\ &= \langle e_0, e_2, e_4, e_6 \rangle \\ &\cong \mathbb{Z}^4\end{aligned}$$

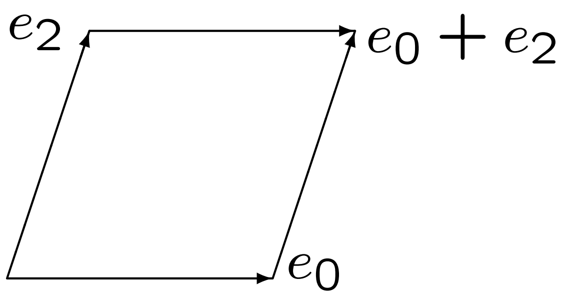


Let $\mathbb{R}e_n$ denote the line which is parallel to e_n and passes through the origin. Then

$$\mathcal{C} = \{\gamma + \mathbb{R}e_n \mid \gamma \in \Gamma, n = 0, 1, \ldots, 9\}$$

is a countable family of lines in $\mathbb{R}^2$.
By 'cutting' the plane along the lines ℓ in $\mathcal{C}$, we obtain a totally disconnected (non-compact) space X. Γ acts on X naturally by translation.

Let $\Gamma_0 \subset \Gamma$ be a subgroup generated by e_0, e_2 and let Y be the quotient space of X by the action of Γ_0. We can identify Y with the parallelogram spanned by e_0, e_2.



Clearly Y is a (compact) Cantor set and the translations by e_4 and e_6 induce a minimal $\mathbb{Z}^2$ action ψ on Y.
Then (Y, ψ) is 'equivalent' to the Penrose tiling space.

It is easy to see

$$e_4 = -e_0 + \lambda e_2$$

and

$$e_4 + e_6 = -(1 + \lambda)e_0,$$

where $\lambda = (\sqrt{5} - 1)/2$.
Let R_λ be the irrational rotation on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $x \mapsto x + \lambda$.
It follows that there exists a factor map from (Y, ψ) to the product of two copies of $(\mathbb{T}, R_\lambda)$.
In addition, this factor map is almost one-to-one.

Consequently, by the AF embedding theorem, we can conclude that $C^*(Y, \psi)$ is AF embeddable.

K-theory of $C^*(X,\varphi)$

Next, we would like to consider K-groups of $C^*(X,\varphi)$ for a Cantor minimal $\mathbb{Z}^d$-system and relate it to a group cohomology.

Let Ω be the suspension space of (X,φ), that is, Ω is the quotient space of $X \times \mathbb{R}^d$ by the equivalence relation

$$\{((x,t),(\varphi^n(x),t+n)) \mid x \in X, t \in \mathbb{R}^d, n \in \mathbb{Z}^d\}.$$

There exists a natural $\mathbb{R}^d$ action on Ω induced by the translation $(x,t) \mapsto (x,t+s)$ in $X \times \mathbb{R}^d$. We denote this action by $\tilde{\varphi}$.

We have

$$
\begin{aligned}
& K_*(C^*(X,\varphi)) \\
& \cong K_*(C(\Omega) \times_{\tilde{\varphi}} \mathbb{R}^d) \text{ ($\because$ strong Morita equiv.)} \\
& \cong K_*(C(\Omega)) \text{ ($\because$ Thom Isomorphism)} \\
& \cong K^*(\Omega)
\end{aligned}
$$

$C(X,\mathbb{Z})$ is a $\mathbb{Z}^d$-module in an obvious way. Let $H^*(X,\varphi)$ be the group cohomology of $\mathbb{Z}^d$ with coefficients $C(X,\mathbb{Z})$.
By definition,

$$H^*(X,\varphi) \cong H^*(\Omega;\mathbb{Z}),$$

where the right hand side denotes the Čech cohomology of Ω with coefficients in $\mathbb{Z}$.

Consequently, we have

$$\bigoplus_{n-d\in 2\mathbb{Z}} H^n(X,\varphi)\otimes\mathbb{Q} \cong K_0(C^*(X,\varphi))\otimes\mathbb{Q}$$

$$\bigoplus_{n-d\notin 2\mathbb{Z}} H^n(X,\varphi)\otimes\mathbb{Q} \cong K_1(C^*(X,\varphi))\otimes\mathbb{Q}$$

by the topological Chern character.

Conjecture 10. For any $d\geq 1$, we have

$$\bigoplus_{n-d\in 2\mathbb{Z}} H^n(X,\varphi) \cong K_0(C^*(X,\varphi))$$

and

$$\bigoplus_{n-d\notin 2\mathbb{Z}} H^n(X,\varphi) \cong K_1(C^*(X,\varphi)).$$

If cohomology groups were torsion free, then this conjecture would easily follow. But, in general, cohomology groups and K-groups may contain torsion.
When $d = 1, 2$, we can check that this conjecture holds by direct computation.

Anderson-Putnam (1998) observed that tiling spaces can be viewed as inverse limits and computed the cohomology of several tiling spaces. For example, the cohomology groups of Penrose tilings are $H^0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^5$ and $H^2 = \mathbb{Z}^8$.

Since φ is minimal, $H^0(X, \varphi)$ is always isomorphic to $\mathbb{Z}$.
The top-dimensional cohomology $H^d(X, \varphi)$ is isomorphic to

$$C(X, \mathbb{Z})/\langle f - f \circ \varphi^n \mid f \in C(X, \mathbb{Z}), n \in \mathbb{Z}^d\rangle,$$

which is called the coinvariants.

The following problem corresponds to the top-dimensional part of the conjecture above.

Problem 11. Let $j : C(X) \to C^*(X,\varphi)$ be the canonical inclusion and let $j_* : C(X,\mathbb{Z}) \to K_0(C^*(X,\varphi))$ be the induced homomorphism on K-groups. Is the kernel of j_* equal to

$$\langle f - f \circ \varphi^n \mid f \in C(X,\mathbb{Z}), n \in \mathbb{Z}^d \rangle?$$

As for the values of projections by traces, it is known that the Gap Labeling Conjecture holds (Bellissard-Benedetti-Gambaudo (2006), Benameur-Oyono-Oyono (2001), Kaminker-Putnam (2003)).

Theorem 12. *Let (X,φ) be a Cantor minimal $\mathbb{Z}^d$-system. Let μ be a φ-invariant probability measure on X and let τ_μ be the trace induced by μ. Then one has*

$$\tau_\mu(K_0(C(X))) = \tau_\mu(K_0(C^*(X,\varphi))).$$

F. Gähler (2004) found 5-torsion in the top-dimensional cohomology of the Tübingen Triangle Tiling.

We would like to construct torsion in coinvariants in a much easier way.

Let G be a finite abelian group and let (X_i, φ_i) be Cantor minimal $\mathbb{Z}$-systems for $i = 1, 2$.
Let $\xi_i : X_i \to G$ be a continuous map such that the transformation

$$(x, g) \mapsto (\varphi_i(x), g + \xi_i(x))$$

is a minimal homeomorphism on $X_i \times G$ for each $i = 1, 2$.
We define a Cantor minimal $\mathbb{Z}^2$-system (Y, ψ) as follows:

$$Y = X_1 \times X_2 \times G$$

$$\psi_1(x_1, x_2, g) = (\varphi_1(x_1), x_2, g + \xi_1(x_1))$$

$$\psi_2(x_1, x_2, g) = (x_1, \varphi_2(x_2), g + \xi_2(x_2)).$$

It is easy to see that ψ_1 and ψ_2 are commuting.

Theorem 13 (M). *The torsion part of the top-dimensional cohomology $H^2(Y, \psi)$ is isomorphic to the wedge product $G \wedge G$.*

For example, if $G = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$, then $G \wedge G$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

$C^*(Y, \psi)$ is isomorphic to the crossed product of $C^*(X_1, \varphi_1) \otimes C^*(X_2, \varphi_2)$ by an action of $\widehat{G}$. By the result of Putnam, $C^*(X_i, \varphi_i)$ has tracial rank zero.

Corollary 14. *There exists a Cantor minimal $\mathbb{Z}^2$-system (Y, ψ) such that $C^*(Y, \psi)$ is not an AT algebra but does have tracial rank zero.*

Proof. Let G be a finite non-cyclic abelian group. Choose (X_i, φ_i) and ξ_i so that (X_i, φ_i) and (Y, ψ) are uniquely ergodic. Then, we can verify that the action of $\widehat{G}$ on $C^*(X_1, \varphi_1) \otimes C^*(X_2, \varphi_2)$ has the tracial Rohlin property. It follows from Phillips's theorem that $C^*(Y, \psi)$ has tracial rank zero. From the theorem above, $C^*(Y, \psi)$ is not an AT algebra. $\square$

Topological orbit equivalence

Let (X, φ) be a Cantor minimal $\mathbb{Z}^d$-system and let

$$R_\varphi = \{(x, \varphi^n(x)) \mid x \in X, n \in \mathbb{Z}^d\}$$

be the associated equivalence relation on X. Let (Y, ψ) be another Cantor minimal $\mathbb{Z}^{d'}$-system. We say that (X, φ) and (Y, ψ) are orbit equivalent, if there exists a homeomorphism $h : X \to Y$ such that $h \times h(R_\varphi) = R_\psi$.

Let M_φ denote the set of all φ-invariant probability measures on X.

Conjecture 15. The following are equivalent.

(1) There exists a homeomorphism $h : X \to Y$ such that $h_*(M_\varphi) = M_\psi$.

(2) (X, φ) and (Y, ψ) are orbit equivalent.

Theorem 16 (Giordano-Putnam-Skau 1995)**.** *When $d = d' = 1$, the conjecture above is true.*

Theorem 17 (Giordano-Matui-Putnam-Skau). *When $1 \le d, d' \le 2$, the conjecture above is true.*

Corollary 18. *Let (X, φ) and (Y, ψ) be Cantor minimal $\mathbb{Z}$ or $\mathbb{Z}^2$ systems. Suppose that $M_\varphi = \{\mu\}$ and $M_\psi = \{\nu\}$. Then the two systems are orbit equivalent if and only if*

$$\{\mu(U) \mid U \text{ is clopen in } X\}$$
$$= \{\nu(V) \mid V \text{ is clopen in } Y\}.$$

Our strategy for proving the conjecture is the following.

(1) Classify minimal AF relations up to orbit equivalence. (This step was already done by Giordano-Putnam-Skau (1995).)

(2) Find a 'large' AF subrelation R in R_φ.

(3) Apply the absorption theorem d-times and conclude that R_φ is orbit equivalent to the AF relation R.

We would like to describe the second step for the case of $d = 2$. Let (X, φ) be a Cantor minimal $\mathbb{Z}^2$-system.
For a clopen set $U \subset X$ and $x \in X$, we put

$$P = \{n \in \mathbb{Z}^2 \mid \varphi^n(x) \in U\}.$$

P is called 'hitting time'. As a subset of $\mathbb{R}^2$, P has the following properties:
P is separated, i.e. there exists $M_0 > 0$ such that $d(p, q) \geq M_0$ for any $p \neq q \in P$.
P is syndetic, i.e. there exists $M_1 > 0$ such that $\bigcup_{p \in P} B(p, M_1) = \mathbb{R}^2$.

Consider the Voronoi tessellation for P. Thus, for $p \in P$, we let

$$T(p) = \{x \in \mathbb{R}^2 \mid d(x, p) \leq d(x, P)\}.$$

Then

$$\mathcal{T}_P = \{T(p) \mid p \in P\}$$

is a tiling of $\mathbb{R}^2$ and called the Voronoi tessellation.

Generically,

$$\mathbb{Z}^2 = \bigcup_{p \in P} T(p) \cap \mathbb{Z}^2$$

gives a partition of $\mathbb{Z}^2$ into finite subsets, and so we obtain a finite subrelation of R_φ.

Let $U_1, U_2, \dots$ be clopen subsets of X getting smaller.
For each $x \in X$, we consider 'hitting time' $P_1, P_2, \dots$, which are separated and syndetic subsets of $\mathbb{R}^2$ getting thinner.
For each $x \in X$, we get Voronoi tessellations $\mathcal{T}_1, \mathcal{T}_2, \dots$, in which each tile is getting larger.

From this, we obtain an increasing sequence of finite subrelations

$$R_1 \subset R_2 \subset R_3 \subset \cdots \subset R_\varphi.$$

Put $R = \bigcup_{n \in \mathbb{N}} R_n$. Then R is an AF subrelation of R_φ.

This is the outline of Forrest's construction (which was used in the Phillips's theorem). To prove the orbit equivalence, we have to control the difference between R_φ and R.

Consider ‘Voronoi tessellations’ by ‘infinitely large’ polygons.

We may expect that there are only three possibilities:

(1) $\mathbb{R}^2$ is covered by a single ‘infinitely large’ polygon.

(2) $\mathbb{R}^2$ is covered by two ‘infinitely large’ polygons which share an edge.

(3) $\mathbb{R}^2$ is covered by three ‘infinitely large’ polygons which share a vertex.

If this is the case, we can apply the absorption theorem two times and conclude that R_φ is orbit equivalent to R.

But, in general, we may find four distinct Voronoi cells which are close to each other.

By some geometric argument in $\mathbb{R}^2$, we can modify the Voronoi tessellations so that disjoint cells are separated in some controlled manner. From these modified tessellations, we can construct a nice AF subrelation R in R_φ and complete the proof.

Problem 19. Is it possible to extend this result to Cantor minimal $\mathbb{Z}^d$-systems for any $d \geq 3$?

Problem 20. Can we find some dynamical notion which implies the isomorphism of cohomology or K-groups ?