Affine Yangians and rectangular W-algebras

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Abstract

We construct algebra homomorphisms from affine Yangians to rectangular W-algebras both in type A. The construction is given via the coproduct and the evaluation map for the affine Yangians. This is a joint work with Mamoru Ueda [KU].

1 Introduction

Ragoucy-Sorba [RS] and Brundan-Kleshchev [BK] have constructed surjective algebra homomorphisms from the Yangian of $\mathfrak{gl}_n$ to the finite W-algebras associated with nilpotent elements of rectangular type in $\mathfrak{gl}_N$. Our main result (Theorem 3.1) is an affine analog of their construction.

Our work is motivated by the Alday-Gaiotto-Tachikawa (AGT) correspondence for parabolic sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. Using Brundan-Kleshchev's result, Braverman-Feigin-Finkelberg-Rybnikov [BFFR] have constructed an action of the finite W-algebra on the localized equivariant cohomology of the moduli space of flags of sheaves on $\mathbb{P}^1$ (They call this result "a finite analog of the AGT relation"). We expect that our result can be used to construct an action of the W-algebra on the cohomology of the moduli space of parabolic sheaves.

Please also see expository papers [Kod1], [Kod2] written in Japanese.

2 Yangians and rectangular finite W-algebras

We introduce finite W-algebras in the case $\mathfrak{g} = \mathfrak{gl}_N$ with a nilpotent element whose Jordan form is of rectangular type.

Let $\mathfrak{g} = \mathfrak{gl}_N$ be the complex general linear Lie algebra consisting of $N \times N$ matrices. We denote by e_{ij} the matrix unit whose (i,j)-entry is 1. We assume $N = ln$ with nonzero positive integers l and n. As vector spaces, we have an isomorphism

$$\mathfrak{gl}_l \otimes \mathfrak{gl}_n \cong \mathfrak{g} = \mathfrak{gl}_N, \qquad e_{st} \otimes e_{ij} \mapsto e_{(s-1)n+i,(t-1)n+j}. \tag{2.1}$$

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We define a $\mathbb{Z}$-grading of $\mathfrak{gl}_l$ by $\deg e_{st} = t - s$ and denote by $(\mathfrak{gl}_l)_j$ its degree j component. It induces a $\mathbb{Z}$-grading of $\mathfrak{g}$ by

$$\mathfrak{g}_j = (\mathfrak{gl}_l)_j \otimes \mathfrak{gl}_n$$

using the identification (2.1). In particular, we have $\mathfrak{g}_0 \cong \mathfrak{gl}_n^{\oplus l}$. We set

$$e_{ij}^{[s]} = e_{(s-1)n+i,(s-1)n+j}$$

for $s = 1, \ldots, l$ and $i, j = 1, \ldots, n$. Then $\{e_{ij}^{[s]} \mid i, j = 1, \ldots, n\}$ forms a $\mathbb{C}$-basis of the s-th component of $\mathfrak{g}_0 \cong \mathfrak{gl}_n^{\oplus l}$.

We take a nilpotent element f in $\mathfrak{g}$ as

$$f = \sum_{s=1}^{l-1} \sum_{i=1}^{n} e_{sn+i,(s-1)n+i} = \begin{bmatrix} O & & & & O \\ I_n & O & & & \\ & I_n & \ldots & & \\ & & & O & \\ O & & & I_n & O \end{bmatrix}.$$

This is called rectangular nilpotent since its Jordan type is (l^n) and the corresponding Young diagram has an $n \times l$ rectangular shape. Set

$$\mathfrak{p} = \bigoplus_{j \leq 0} \mathfrak{g}_j = \bigoplus_{1 \leq t \leq s \leq l} \bigoplus_{i,j=1}^{n} \mathbb{C} e_{sn+i,tn+j},$$

$$\mathfrak{m} = \bigoplus_{j > 0} \mathfrak{g}_j = \bigoplus_{1 \leq s < t \leq l} \bigoplus_{i,j=1}^{n} \mathbb{C} e_{sn+i,tn+j}$$

and let $\chi = (f, -) \colon \mathfrak{m} \to \mathbb{C}$ be the one-dimensional representation of $\mathfrak{m}$ which corresponds to f via the trace form $(,)$. Let I_χ be the left ideal of $U(\mathfrak{g})$ generated by $\{x - \chi(x) \mid x \in \mathfrak{m}\}$. We have a decomposition

$$U(\mathfrak{g}) = U(\mathfrak{p}) \oplus I_\chi \tag{2.2}$$

of a $\mathbb{C}$-vector space.

The finite W-algebra $U(\mathfrak{g}, f)$ is defined by

$$U(\mathfrak{g}, f) = \mathrm{End}_{\mathfrak{g}}(U(\mathfrak{g})/I_\chi)^{\mathrm{op}}.$$

By identifying $U(\mathfrak{g})/I_\chi$ with $U(\mathfrak{p})$ via (2.2), it is regarded as a subalgebra of $U(\mathfrak{p})$:

$$U(\mathfrak{g}, f) = \{x \in U(\mathfrak{p}) \mid [y, x] \in I_\chi \text{ for all } y \in \mathfrak{m}\}.$$

Example 2.1 ($n = 1$ and $l = 2$)**.** We consider the following case:

$$\mathfrak{g} = \mathfrak{gl}_2, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have

$$\mathfrak{p} = \left\{ \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} \right\}, \qquad \mathfrak{m} = \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right\}.$$

Then $U(\mathfrak{g}, f)$ is generated by

$$e_{11} + e_{22}, \qquad e_{11}e_{22} - e_{21} + e_{22}.$$

These elements are the images of elements

$$e_{11} + e_{22}, \qquad e_{11}e_{22} - e_{21}e_{12} + e_{22}$$

of the center of $U(\mathfrak{gl}_2)$ by the projection to $U(\mathfrak{p})$ via (2.2). This correspondence induces an algebra isomorphism between the center of $U(\mathfrak{gl}_2)$ and $U(\mathfrak{g}, f)$ in this example. More generally, Kostant [Kos] proved that $U(\mathfrak{g}, f)$ is isomorphic to the center of $U(\mathfrak{g})$ in the case $n = 1$ and an arbitrary l (principal nilpotent case).

Ragoucy-Sorba [RS] and Brundan-Kleshchev [BK] related the finite W-algebra $U(\mathfrak{g}, f)$ to the Yangian.

Theorem 2.2 (Ragoucy-Sorba [RS], Brundan-Kleshchev [BK])**.** *There exists a surjective algebra homomorphism from the Yangian of* $\mathfrak{gl}_n$ *to* $U(\mathfrak{g}, f)$*.*

Let $\xi \colon \mathfrak{p} \to \mathfrak{g}_0$ be the natural projection and denote by the same letter the induced algebra homomorphism $U(\mathfrak{p}) \to U(\mathfrak{g}_0) \cong U(\mathfrak{gl}_n)^{\otimes l}$. The restriction of ξ to $U(\mathfrak{g}, f)$ is known to be injective. It is called the Miura map. In Example 2.1, $\xi(U(\mathfrak{g}, f))$ is generated by

$$e_{11} + e_{22}, \qquad e_{11}e_{22} + e_{22}.$$

We define an algebra automorphism η on $U(\mathfrak{g}_0)$ by

$$\eta(e_{ij}^{[s]}) = e_{ij}^{[s]} + \delta_{ij}\, n(l - s)$$

for $s = 1, \ldots, l$ and $i, j = 1, \ldots, n$. We set

$$\mathcal{W}_l^{\mathrm{fin}} = \eta^{-1}\xi(U(\mathfrak{g}, f)).$$

In Example 2.1, since we have $\eta(e_{11}e_{22}) = (e_{11} + 1)e_{22} = e_{11}e_{22} + e_{22}$, $\mathcal{W}_l^{\mathrm{fin}}$ is generated by

$$e_{11} + e_{22}, \qquad e_{11}e_{22}.$$

Brundan-Kleshchev gave an observation that the homomorphism in Theorem 2.2 is related to the coproduct and the evaluation map for the Yangian $Y(\mathfrak{gl}_n)$. Here

- the coproduct is an algebra homomorphism $\Delta \colon Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)^{\otimes 2}$;

- the evaluation map is an algebra homomorphism $\mathrm{ev}\colon Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$.

Hence we have the composite

$$\mathrm{ev}^{\otimes l} \circ \Delta^{l-1} \colon Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)^{\otimes l}$$

of the iterated coproduct $\Delta^{l-1} = (\Delta \otimes \mathrm{id}^{\otimes(l-2)}) \circ \cdots \circ (\Delta \otimes \mathrm{id}) \circ \Delta$ and $\mathrm{ev}^{\otimes l}$. Theorem 2.2 can be restated as the following.

Theorem 2.3 (Brundan-Kleshchev [BK] Section 12)**.** *The image of* $\mathrm{ev}^{\otimes l} \circ \Delta^{l-1}$ *coincides with* $\mathcal{W}_l^{\mathrm{fin}}$.

3 Affine Yangians and rectangular W-algebras

Given a Lie algebra $\mathfrak{a}$, let $\hat{\mathfrak{a}}$ denote its affine Lie algebra, that is, the one-dimensional central extension of $\mathfrak{a}[t, t^{-1}]$. We write the element xt^m ($x \in \mathfrak{a}$ and $m \in \mathbb{Z}$) by $x(m)$.

The W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is defined from the data $\mathfrak{g}, f$ as Section 2 and $k \in \mathbb{C}$ by the BRST cohomology (see [AM] for the definition and construction of generators). It is a subalgebra of a certain completion $U(\hat{\mathfrak{p}})_{\mathrm{comp}}$ of the universal enveloping algebra of $\hat{\mathfrak{p}}$. It also admits the Miura map

$$\xi \colon \mathcal{W}^k(\mathfrak{g}, f) \to U(\hat{\mathfrak{g}}_0)_{\mathrm{comp}} \cong U(\hat{\mathfrak{gl}}_n)^{\otimes_{\mathrm{comp}} l},$$

which is injective. We define an algebra automorphism η on $U(\hat{\mathfrak{g}}_0)_{\mathrm{comp}}$ by

$$\eta(e_{ij}^{[s]}(m)) = e_{ij}^{[s]}(m) + \delta_{m,0}\delta_{ij}\,(k + N - n)(l - s)$$

for $s = 1, \ldots, l,\ i, j = 1, \ldots, n$, and $m \in \mathbb{Z}$. We set

$$\mathcal{W}_l = \eta^{-1}\xi(\mathcal{W}^k(\mathfrak{g}, f)).$$

The affine Yangian $Y(\hat{\mathfrak{sl}}_n)$ is defined by generators and relations (see [Kod2]). Let us summarize its properties:

- it is a two-parameter deformation of the universal enveloping algebra of the universal central extension of the Lie algebra $\hat{\mathfrak{sl}}_n[s]$;
- it admits the coproduct $\Delta\colon Y(\hat{\mathfrak{sl}}_n) \to Y(\hat{\mathfrak{sl}}_n)^{\otimes_{\mathrm{comp}} 2}$;
- it admits the evaluation map $\mathrm{ev}\colon Y(\hat{\mathfrak{sl}}_n) \to U(\hat{\mathfrak{gl}}_n)_{\mathrm{comp}}$.

The composite

$$\mathrm{ev}^{\otimes l} \circ \Delta^{l-1} \colon Y(\hat{\mathfrak{sl}}_n) \to U(\hat{\mathfrak{gl}}_n)^{\otimes_{\mathrm{comp}} l}$$

is defined in a way similar to Section 2. The following is our main result. It gives an affine analog of Theorem 2.3.

Theorem 3.1 (Kodera-Ueda [KU])**.** *Let* $n \geq 3$. *The image of* $\mathrm{ev}^{\otimes l} \circ \Delta^{l-1}$ *is contained in* $\mathcal{W}_l$. *It coincides with* $\mathcal{W}_l$ *if* $k + N - n \neq 0$.

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