Ext^1 for simple modules over $U_q(L\mathfrak{sl}_2)$

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1 Introduction

Finite-dimensional simple modules over the quantum loop algebra $U_q(L\mathfrak{sl}_2)$ were classified by Chari and Pressley [CP]. In the process they determined the structure of the tensor product of every two evaluation modules and proved that any finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-module is isomorphic to the tensor product of appropriate evaluation modules.

It is known that the category of finite-dimensional $U_q(L\mathfrak{sl}_2)$-modules is not semisimple. Non-simple tensor products of evaluation modules provide many examples of non-semisimple modules. Then the natural problem to study extensions between simple modules is raised.

In this article we calculate the first extension groups Ext^1 for some finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-modules. In particular we determine the finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-modules that admit non-trivial extensions with the trivial module (Theorem 3.1.1) among other results. A conjecture concerning Ext^1 for general finite-dimensional simple modules is given at the end of the article.

2 Preliminaries on finite-dimensional modules over $U_q(L\mathfrak{sl}_2)$

2.1 The quantum loop algebra $U_q(L\mathfrak{sl}_2)$

Let q be a non-zero complex number which is not a root of unity. The quantum loop algebra $U_q(L\mathfrak{sl}_2)$ is defined as a q-deformation of the universal enveloping algebra of the loop Lie algebra $L\mathfrak{sl}_2$ (See [CP]). Let $e_i, f_i, t_i^{\pm 1} (i = 0, 1)$ be the Chevalley generators. We do not introduce the Drinfeld generators in this article although they are in fact used in the proof of Proposition 3.2.1 which we omit. The subalgebra generated by $e_1, f_1, t_1^{\pm 1}$ is isomorphic to $U_q(\mathfrak{sl}_2)$.

The coproduct, the counit and the antipode of $U_q(L\mathfrak{sl}_2)$ are defined as follows:

$$\Delta(e_i) = e_i \otimes t_i + 1 \otimes e_i,$$
$$\Delta(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i,$$
$$\Delta(t_i^{\pm 1}) = t_i^{\pm 1} \otimes t_i^{\pm 1},$$

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$$\varepsilon(e_i) = \varepsilon(f_i) = 0,$$
$$\varepsilon(t_i^{\pm 1}) = 1,$$

$$S(e_i) = -e_i t_i^{-1},$$
$$S(f_i) = -t_i f_i,$$
$$S(t_i^{\pm 1}) = t_i^{\mp 1}.$$

This makes $U_q(L\mathfrak{sl}_2)$ a Hopf algebra and hence the tensor product of modules, the trivial module, and the dual of a module are defined. Moreover, since the antipode S is invertible, two kinds of the dual for each $U_q(L\mathfrak{sl}_2)$-module M are defined: M^* by S and ${}^*M$ by S^{-1}. They are not isomorphic in general.

Remark 2.1.1. Our definition of a Hopf algebra structure above is the same as in [CP].

2.2 Evaluation modules

In the remaining of this article, $U_q(L\mathfrak{sl}_2)$-modules and $U_q(\mathfrak{sl}_2)$-modules are assumed to be those on which t_i's act by powers of q, which are called type 1 modules.

For each $a \in \mathbb{C}^\times$, we define the algebra homomorphism

$$\mathrm{ev}_a \colon U_q(L\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$$

by

$$e_0 \mapsto a f_1,$$
$$f_0 \mapsto a^{-1} e_1,$$
$$t_0^{\pm 1} \mapsto t_1^{\mp 1},$$

and the identity on $e_1, f_1, t_1^{\pm 1}$. Define the $U_q(L\mathfrak{sl}_2)$-module $V_{m,a}$ as the pull-back of the $(m+1)$-dimensional simple $U_q(\mathfrak{sl}_2)$-module by ev_a. We call it an evaluation module.

Theorem 2.2.1 (Chari-Pressley [CP, 4.8 Theorem])**.** (i) *The tensor product $V_{m,a} \otimes V_{n,b}$ is simple if and only if $b/a \notin \{q^{\pm(m+n-2p+2)} \mid p = 1, \dots, \min\{m,n\}\}$.*

(ii) *The tensor product $V_{m_1,a_1} \otimes \cdots \otimes V_{m_k,a_k}$ is simple if and only if $V_{m_i,a_i} \otimes V_{m_j,a_j}$ is simple for every $i \neq j$.*

It is easy to see that $(V_{m,a})^* \cong V_{m,aq^2}$ and ${}^*(V_{m,a}) \cong V_{m,aq^{-2}}$.

2.3 Classification of simple modules

We recall the classification of finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-modules in [CP] in terms of multi-segments.

For $m \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}^\times$, we put

$$\pi_{m,a} = \{aq^{-m+1}, aq^{-m+3}, \dots, aq^{m-1}\}$$

and call it a segment. The empty segment $\pi_{0,a}$ is denoted by 0. A multi-segment is defined to be a formal finite sum of segments with coefficients in $\mathbb{Z}_{\geq 0}$. We introduce the following terminology.

Definition 2.3.1. (i) Segments $\pi_{m,a}$ and $\pi_{n,b}$ are said to be in general position if $b/a \notin \{q^{\pm(m+n-2p+2)} \mid p = 1, \ldots, \min\{m,n\}\}$.

(ii) A multi-segment $\pi = \sum_{i=1}^{k} \pi_{m_i,a_i}$ is said to be in general position if π_{m_i,a_i} and π_{m_j,a_j} are in general position for every $i \neq j$.

Remark 2.3.2. The above (i) is equivalent to the usual definition, that is, segments π and π' are in general position if and only if

- $\pi \subseteq \pi'$; or
- $\pi \supseteq \pi'$; or
- $\pi \cup \pi'$ does not form a segment.

Let $\mathscr{P}^+$ be the set of all multi-segments in general position. For an element $\pi = \sum_{i=1}^{k} \pi_{m_i,a_i}$ of $\mathscr{P}^+$, put

$$V(\pi) = V_{m_1,a_1} \otimes \cdots \otimes V_{m_k,a_k},$$

where the isomorphism class of the right-hand side does not depend on the ordering of the tensor product. Hence the left-hand side is well-defined. By convention, $V(0)$ is the trivial module.

Theorem 2.3.3 (Chari-Pressley [CP, 3.4 Theorem and 4.11 Theorem])**.** *The assignment $\pi \mapsto V(\pi)$ gives a bijection between $\mathscr{P}^+$ and the set of isomorphism classes of finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-modules.*

Note that the evaluation module $V_{m,a}$ is denoted by $V(\pi_{m,a})$ in the multi-segment convention. We use the latter notation in the sequel.

We sometimes identify $\mathscr{P}^+$ with the set of all polynomials with constant term 1 by assigning

$$\pi = \sum_{i=1}^{k} \pi_{m_i,a_i} \mapsto \prod_{i=1}^{k} (1 - a_i q^{-m_i+1} u)(1 - a_i q^{-m_i+3} u) \cdots (1 - a_i q^{m_i-1} u) \in \mathbb{C}[u].$$

We denote by $P(\pi)$ the corresponding polynomial. This is nothing but the Drinfeld polynomial of the simple module $V(\pi)$.

2.4 Non-simple tensor products of evaluation modules

Recall that the tensor product $V(\pi_{m,a}) \otimes V(\pi_{n,b})$ is not simple if and only if $b/a = q^{\pm(m+n-2p+2)}$ for some $p = 1, \ldots, \min\{m,n\}$. In [CP] Chari and Pressley determined the structure of the tensor product when it is not simple. To state their result, we introduce some notation.

Let $\pi_{m,a}$ and $\pi_{n,b}$ be segments not in general position. By definition there exists unique $p \in \{1, \ldots, \min\{m,n\}\}$ such that $b/a = q^{\pm(m+n-2p+2)}$. Then define the two multi-segments $\pi_{m,a} * \pi_{n,b}$ and $\pi_{m,a} \triangle \pi_{n,b}$ by

$$\pi_{m,a} * \pi_{n,b} = \pi_{m+n-p+1,bq^{\mp(m-p+1)}} + \pi_{p-1,aq^{\pm(m-p+1)}}$$

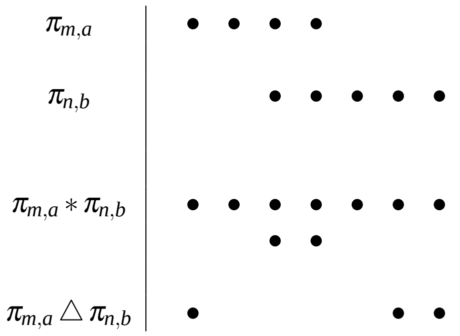
and

$$\pi_{m,a} \triangle \pi_{n,b} = \pi_{m-p,aq^{\mp p}} + \pi_{n-p,bq^{\pm p}}.$$

Each of the multi-segments defined above is in general position. The combinatorial meanings of them are:

- $*$: for two segments, taking their union (the first term) and intersection (the second term);
- $\triangle$: from the union of two segments, discarding the intersection of the two segments together with the two nearest neighbor elements.

Example 2.4.1. The figure below expresses $*$ and $\triangle$ for $\pi_{m,a}$ and $\pi_{n,b}$ satisfying $b/a = q^{m+n-2p+2}$ where $m = 4, n = 5, p = 3$.



Under the identification with the polynomials, $*$ corresponds to taking the product of them. That is, $P(\pi_{m,a} * \pi_{n,b}) = P(\pi_{m,a})P(\pi_{n,b})$.

Proposition 2.4.2 (Chari-Pressley [CP, 4.9 Proposition])**.** *If $b/a = q^{m+n-2p+2}$ for some $p = 1, \ldots, \min\{m,n\}$, then there exists a non-split exact sequence*

$$0 \to V(\pi_{m,a} \triangle \pi_{n,b}) \to V(\pi_{m,a}) \otimes V(\pi_{n,b}) \to V(\pi_{m,a} * \pi_{n,b}) \to 0.$$

If $b/a = q^{-(m+n-2p+2)}$ for some $p = 1, \ldots, \min\{m,n\}$, then there exists a non-split exact sequence

$$0 \to V(\pi_{m,a} * \pi_{n,b}) \to V(\pi_{m,a}) \otimes V(\pi_{n,b}) \to V(\pi_{m,a} \triangle \pi_{n,b}) \to 0.$$

Corollary 2.4.3. *If $b/a = q^{m+n-2p+2}$ for some $p = 1, \ldots, \min\{m,n\}$, then we have*

$$\mathrm{Ext}^1(V(\pi_{m,a} * \pi_{n,b}), V(\pi_{m,a} \triangle \pi_{n,b})) \neq 0.$$

If $b/a = q^{-(m+n-2p+2)}$ for some $p = 1, \ldots, \min\{m,n\}$, then we have

$$\mathrm{Ext}^1(V(\pi_{m,a} \triangle \pi_{n,b}), V(\pi_{m,a} * \pi_{n,b})) \neq 0.$$

Remark 2.4.4. The extension groups above turn out to be one-dimensional as a corollary of our result. See Corollary 3.2.3.

2.5 Linkage

A result of Chari and Moura [CM] implies a necessary condition that Ext^1 for simple modules does not vanish, in terms of a certain order on $\mathscr{P}^+$. We formulate it in our setting.

For elements π, π' of $\mathscr{P}^+$, we say that $\pi \succ \pi'$ if

$$P(\pi)/P(\pi') = (1 - au)(1 - aq^2u)$$

for some a. Then a partial order $>$ on $\mathscr{P}^+$ is defined to be generated by $\succ$.

Proposition 2.5.1 (Chari-Moura [CM, Proof of Lemma 8.5 (i)])**.** *Let π, π' be elements of $\mathscr{P}^+$. If* $\mathrm{Ext}^1(V(\pi), V(\pi')) \neq 0$ *then* $\pi \geq \pi'$ *or* $\pi' \geq \pi$*.*

Remark 2.5.2. A similar result can be obtained for general quantum loop algebras with an appropriate order on the set of Drinfeld polynomials, since the result of Chari and Moura [CM] was verified in the general situation. This order is a loop analog of the usual one on the weight lattice of semisimple Lie algebras.

2.6 Adjointness

We recall an important fact which will be used repeatedly in the next section. Let M be a finite-dimensional $U_q(L\mathfrak{sl}_2)$-module. Then exact functors $-\otimes M$ and $M \otimes -$ are defined. By the general theory of Hopf algebras, the following are adjoint pairs:

$$(-\otimes M, -\otimes M^*),\ (-\otimes {}^*M, -\otimes M),\ (M\otimes -, {}^*M \otimes -),\ (M^*\otimes -, M\otimes -).$$

This fact immediately implies the following.

Proposition 2.6.1. *We have natural isomorphisms*

$$\mathrm{Ext}^1(V\otimes M, V') \cong \mathrm{Ext}^1(V, V'\otimes M^*),\ \mathrm{Ext}^1(V, V'\otimes M) \cong \mathrm{Ext}^1(V\otimes {}^*M, V'),$$
$$\mathrm{Ext}^1(M\otimes V, V') \cong \mathrm{Ext}^1(V, {}^*M\otimes V'),\ \mathrm{Ext}^1(V, M\otimes V') \cong \mathrm{Ext}^1(M^*\otimes V, V')$$

for any finite-dimensional $U_q(L\mathfrak{sl}_2)$-modules V, V', M.

3 Calculation of extension groups

3.1 The main theorem

Now we state the main theorem.

Theorem 3.1.1. *Let π be an element of $\mathscr{P}^+$. Then we have*

$$\begin{aligned}\dim \mathrm{Ext}^1(V(\pi), V(0)) &= \dim \mathrm{Ext}^1(V(0), V(\pi)) \\ &= \begin{cases} 1 & \text{if } \pi = \pi_{m,a} * \pi_{m,aq^2} \text{ for some } m \in \mathbb{Z}_{\geq 1} \text{ and } a \in \mathbb{C}^\times, \\ 0 & \text{otherwise}. \end{cases}\end{aligned}$$

A sketch of the proof will be given in the next subsection.

Remark 3.1.2. There exist natural exact sequences

$$0 \to V(0) \to V(\pi_{m,a}) \otimes V(\pi_{m,a})^* \to \mathrm{Coker} \to 0$$

and

$$0 \to \mathrm{Ker} \to V(\pi_{m,a})^* \otimes V(\pi_{m,a}) \to V(0) \to 0$$

by the general theory of Hopf algebras. Since $V(\pi_{m,a})^* \cong V(\pi_{m,aq^2})$, we see by Proposition 2.4.2 that both the cokernel and the kernel are isomorphic to $V(\pi_{m,a} * \pi_{m,aq^2})$ (Note that $\pi_{m,a} \triangle \pi_{m,aq^2} = 0$). We can check directly that the extensions between $V(\pi_{m,a} * \pi_{m,aq^2})$ and $V(0)$ coming from Proposition 2.4.2 are equivalent to the natural ones up to scalar. Theorem 3.1.1 asserts that these extensions are unique classes in the extension groups up to scalar.

3.2 Proof of the main theorem

We can prove the following by an easy calculation, where we investigate the actions of the Drinfeld generators. The details are omitted.

Proposition 3.2.1. *We have*

$$\begin{aligned}\dim \operatorname{Ext}^1(V(\pi_{m,a}),V(0)) &= \dim \operatorname{Ext}^1(V(0),V(\pi_{m,a}))\\ &= \begin{cases} 1 & \text{if } m=2,\\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

The following proposition is crucial to the proof of the main theorem.

Proposition 3.2.2. *Let π be an element of $\mathscr{P}^+$ satisfying $\pi_{m,a} > \pi$. Then we have*

$$\begin{aligned}\dim \operatorname{Ext}^1(V(\pi_{m,a}),V(\pi)) &= \dim \operatorname{Ext}^1(V(\pi),V(\pi_{m,a}))\\ &= \begin{cases} 1 & \text{if } \pi_{m,a} \succ \pi,\\ 0 & \text{otherwise}.\end{cases}\end{aligned}$$

Proof. We introduce some notation. For a segment $\pi = \pi_{m,a} = \{aq^{-m+1}, aq^{-m+3}, \dots, aq^{m-1}\}$, we put $i(\pi) = aq^{-m+1}$ and $e(\pi) = aq^{m-1}$, the elements at the initial and the end of the segment respectively. For $\pi = \pi_{m,a}$ and an integer l, we put $q^l\pi = \pi_{m,aq^l}$. We extend this notion to multi-segments by applying to each term. Then for a multi-segment π, we have $V(\pi)^* \cong V(q^2\pi)$ and ${}^*V(\pi) \cong V(q^{-2}\pi)$ in this notation.

We only consider $\operatorname{Ext}^1(V(\pi),V(\pi_{m,a}))$ since the proof is similar for another one. By the assumption $\pi_{m,a} > \pi$, we can express π as $\pi = \sum_{i=1}^k \pi^i$ where $\pi^i = \pi_{m_i,a_i}$'s satisfy $i(\pi^{i+1})/e(\pi^i) = q^{2l_i+2}$ for some even integers l_i's greater than or equal to 2. Recall that $V(\pi) \cong V(\pi^1) \otimes \cdots \otimes V(\pi^k)$.

We prove the assertion by the induction on $\sum_{i=1}^k m_i$. The case $\sum_{i=1}^k m_i = 0$, equivalently $\pi = 0$, is proved in Proposition 3.2.1. Assume that $\sum_{i=1}^k m_i > 0$ and $\pi^k \neq 0$. Put $\pi' = \sum_{i=1}^{k-1} \pi^i$, then $V(\pi) \cong V(\pi') \otimes V(\pi^k)$.

We may assume that $e(\pi^k) = e(\pi_{m,a})$. Indeed, if not, then we have $e(\pi_{m,a})/e(\pi^k) = q^{2l}$ for some even integer l greater than or equal to 2, and the following isomorphisms hold:

$$\begin{aligned}\operatorname{Ext}^1(V(\pi),V(\pi_{m,a})) &\cong \operatorname{Ext}^1(V(\pi'),V(\pi_{m,a}) \otimes V(q^2\pi^k))\\ &\cong \operatorname{Ext}^1(V(\pi'),V(q^2\pi^k) \otimes V(\pi_{m,a}))\\ &\cong \operatorname{Ext}^1(V(q^4\pi^k) \otimes V(\pi'),V(\pi_{m,a}))\\ &\cong \operatorname{Ext}^1(V(\pi') \otimes V(q^4\pi^k),V(\pi_{m,a})),\end{aligned}$$

where the first and third isomorphisms are due to the adjointness in Proposition 2.6.1, while the second and fourth follow from the facts $V(\pi_{m,a}) \otimes V(q^2\pi^k) \cong V(q^2\pi^k) \otimes V(\pi_{m,a})$ and $V(q^4\pi^k) \otimes V(\pi') \cong V(\pi') \otimes V(q^4\pi^k)$ since each tensor product is simple by Theorem 2.2.1. By repeating this, the calculation reduces to the case $e(\pi^k) = e(\pi_{m,a})$.

Thus assume that $e(\pi^k) = e(\pi_{m,a})$. Again we have

$$\operatorname{Ext}^1(V(\pi),V(\pi_{m,a})) \cong \operatorname{Ext}^1(V(\pi'),V(\pi_{m,a}) \otimes V(q^2\pi^k))$$

and now $V(\pi_{m,a}) \otimes V(q^2\pi^k)$ is not simple. According to Proposition 2.4.2, there exists the exact sequence

$$0 \to V(\pi_{m,a} \triangle q^2\pi^k) \to V(\pi_{m,a}) \otimes V(q^2\pi^k) \to V(\pi_{m,a} * q^2\pi^k) \to 0.$$

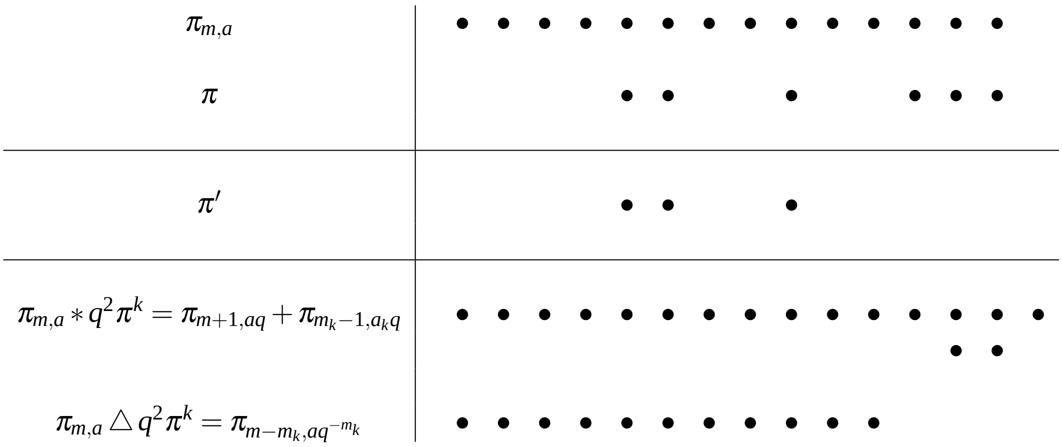
We calculate the multi-segments as

$$\pi_{m,a} * q^2\pi^k = \pi_{m+1,aq} + \pi_{m_k-1,a_kq},$$

$$\pi_{m,a} \triangle q^2\pi^k = \pi_{m-m_k,aq^{-m_k}}.$$

(We present an example in Figure 1 below.)

Figure 1



Applying $\mathrm{Hom}(V(\pi'), -)$ to the exact sequence, we obtain

$$\begin{aligned}
&\mathrm{Hom}(V(\pi'), V(\pi_{m,a} * q^2\pi^k)) \\
&\to \mathrm{Ext}^1(V(\pi'), V(\pi_{m,a} \triangle q^2\pi^k)) \\
&\to \mathrm{Ext}^1(V(\pi'), V(\pi_{m,a}) \otimes V(q^2\pi^k)) \\
&\to \mathrm{Ext}^1(V(\pi'), V(\pi_{m,a} * q^2\pi^k)).
\end{aligned}$$

Since $V(\pi')$ and $V(\pi_{m,a} * q^2\pi^k)$ are non-isomorphic simple modules,

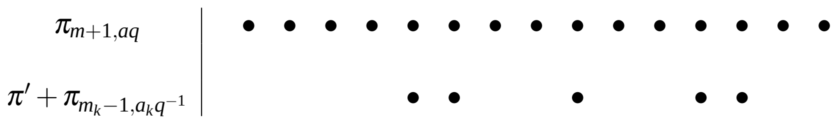
$$\mathrm{Hom}(V(\pi'), V(\pi_{m,a} * q^2\pi^k)) = 0.$$

Since $V(\pi_{m,a} * q^2\pi^k) \cong V(\pi_{m+1,aq}) \otimes V(\pi_{m_k-1,a_kq})$,

$$\mathrm{Ext}^1(V(\pi'), V(\pi_{m,a} * q^2\pi^k)) \cong \mathrm{Ext}^1(V(\pi') \otimes V(\pi_{m_k-1,a_kq^{-1}}), V(\pi_{m+1,aq}))$$

again by the adjointness. Then by the induction hypothesis, the right-hand side is equal to zero (Figure 2).

Figure 2



Therefore we have

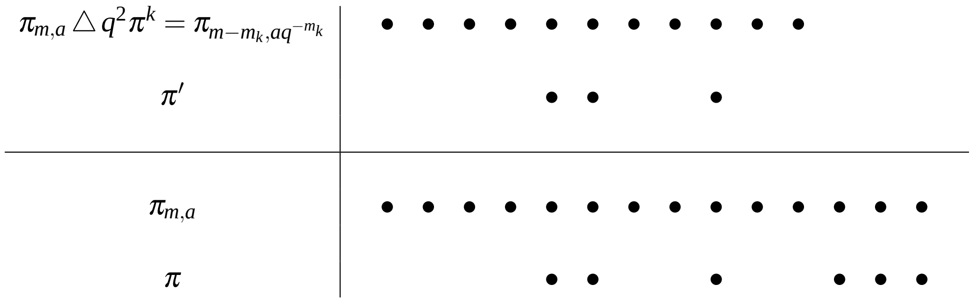
$$\mathrm{Ext}^1(V(\pi'), V(\pi_{m,a} \triangle q^2\pi^k)) \cong \mathrm{Ext}^1(V(\pi'), V(\pi_{m,a}) \otimes V(q^2\pi^k)).$$

Since $\pi_{m,a} \triangle q^2\pi^k = \pi_{m-m_k,aq^{-m_k}}$, we can apply the induction to the left-hand side (Figure 3). The result is

$$\dim \mathrm{Ext}^1(V(\pi'), V(\pi_{m-m_k,aq^{-m_k}})) = \begin{cases} 1 & \text{if } \pi_{m-m_k,aq^{-m_k}} \succ \pi', \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\pi_{m-m_k,aq^{-m_k}} \succ \pi'$ is equivalent to $\pi_{m,a} \succ \pi$ (Figure 3). This completes the proof.

Figure 3



□

Proof of Theorem 3.1.1. Using the adjointness and the linkage condition in Proposition 2.5.1, we see that if $\mathrm{Ext}(V(\pi), V(0)) \neq 0$ then π must be of the form $\pi = \pi_{m,a} + \sum_{i=1}^{k} \pi^i$ where π^i's are pairwise disjoint segments satisfying $\pi^i \subseteq \pi_{m,a}$. Then by Proposition 3.2.2 together with the adjointness and the linkage condition, the assertion is proved. □

We obtain the following corollary as mentioned in Remark 2.4.4.

Corollary 3.2.3. *If $b/a = q^{m+n-2p+2}$ for some $p = 1, \dots, \min\{m, n\}$, then we have*

$$\dim \mathrm{Ext}^1(V(\pi_{m,a} * \pi_{n,b}), V(\pi_{m,a} \triangle \pi_{n,b})) = 1.$$

If $b/a = q^{-(m+n-2p+2)}$ for some $p = 1, \dots, \min\{m, n\}$, then we have

$$\dim \mathrm{Ext}^1(V(\pi_{m,a} \triangle \pi_{n,b}), V(\pi_{m,a} * \pi_{n,b})) = 1.$$

Proof. We only consider the first case. Put

$$\pi^1 = \pi_{m+n-p+1,bq^{-(m-p+1)}},\ \pi^2 = \pi_{p-1,aq^{m-p+1}},$$

$$\pi^3 = \pi_{m-p,aq^{-p}},\ \pi^4 = \pi_{n-p,bq^p}$$

so that $V(\pi_{m,a} * \pi_{n,b}) \cong V(\pi^1) \otimes V(\pi^2)$ and $V(\pi_{m,a} \triangle \pi_{n,b}) \cong V(\pi^3) \otimes V(\pi^4)$. We have

$$\mathrm{Ext}^1(V(\pi_{m,a} * \pi_{n,b}), V(\pi_{m,a} \triangle \pi_{n,b})) \cong \mathrm{Ext}^1(V(\pi^1), V(\pi^3) \otimes V(\pi^4) \otimes V(q^2\pi^2)).$$

By Proposition 2.4.2, we have

$$0 \to V(\pi^4 * q^2\pi^2) \to V(\pi^4) \otimes V(q^2\pi^2) \to V(\pi^4 \triangle q^2\pi^2) \to 0,$$

where

$$\pi^4 * q^2\pi^2 = \pi_{n-1,bq},$$

$$\pi^4 \triangle q^2\pi^2 = \pi_{n-p-1,bq^{p+1}} + \pi_{p-2,aq^{m-p+2}}.$$

Applying $\mathrm{Hom}(V(\pi^1), V(\pi^3) \otimes -)$, we obtain

$$\begin{aligned} 0 &\to \mathrm{Ext}^1(V(\pi^1), V(\pi^3) \otimes V(\pi^4 * q^2\pi^2)) \\ &\to \mathrm{Ext}^1(V(\pi^1), V(\pi^3) \otimes V(\pi^4) \otimes V(q^2\pi^2)) \\ &\to \mathrm{Ext}^1(V(\pi^1), V(\pi^3) \otimes V(\pi^4 \triangle q^2\pi^2)). \end{aligned}$$

We see that the first term is one-dimensional and the third is equal to zero by Proposition 3.2.2, which proves the assertion. □

We obtain another corollary.

Corollary 3.2.4. *We have*

$$\dim \mathrm{Ext}^1(V(\pi_{m,a}), V(\pi_{m,a})) \leq 1.$$

Proof. We have

$$\mathrm{Ext}^1(V(\pi_{m,a}), V(\pi_{m,a})) \cong \mathrm{Ext}^1(V(0), V(\pi_{m,a}) \otimes V(\pi_{m,aq^2})).$$

Applying $\mathrm{Hom}(V(0), -)$ to the exact sequence

$$0 \to V(0) \to V(\pi_{m,a}) \otimes V(\pi_{m,aq^2}) \to V(\pi_{m,a} * \pi_{m,aq^2}) \to 0,$$

we obtain

$$\begin{aligned} 0 &= \mathrm{Ext}^1(V(0), V(0)) \\ &\to \mathrm{Ext}^1(V(0), V(\pi_{m,a}) \otimes V(\pi_{m,aq^2})) \\ &\to \mathrm{Ext}^1(V(0), V(\pi_{m,a} * \pi_{m,aq^2})). \end{aligned}$$

The last term is one-dimensional by Theorem 3.1.1, which proves the assertion. □

3.3 Conjectures

We state a conjecture for extensions between general non-isomorphic finite-dimensional simple modules.

Conjecture 3.3.1. *Let V, V' be non-isomorphic finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-modules. If* $\mathrm{Ext}^1(V,V') \neq 0$ *then we have*

$$V \cong V(\pi) \otimes M,$$

$$V' \cong V(\pi') \otimes M$$

for $\pi, \pi' \in \mathscr{P}^+$ such that the pair (π, π') is equal to either $(\pi_{m,a} * \pi_{n,b}, \pi_{m,a} \triangle \pi_{n,b})$ satisfying $b/a = q^{m+n-2p+2}$ for some $p = 1, \ldots, \min\{m,n\}$, or $(\pi_{m,a} \triangle \pi_{n,b}, \pi_{m,a} * \pi_{n,b})$ satisfying $b/a = q^{-(m+n-2p+2)}$ for some $p = 1, \ldots, \min\{m,n\}$, and a finite-dimensional simple $U_q(L\mathfrak{sl}_2)$-module M. Moreover, we have $\dim \mathrm{Ext}^1(V,V') = 1$ *in this case.*

The following conjecture concerns self-extensions of evaluation modules.

Conjecture 3.3.2. *We have*

$$\dim \mathrm{Ext}^1(V(\pi_{m,a}), V(\pi_{m,a})) = 1$$

for $m \geq 1$.

The author thinks that the dimension of $\mathrm{Ext}^1(V,V)$ for V a general finite-dimensional simple module can be greater than 1.

References

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