On homotopy algebras and their application to string theory

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1. A_∞-algebras

1.1 Based loop space (see MSS:book,p9∼)

X : a topological space

$Y = \Omega X$: the space of based loops in X

$x_0 \in X$: a base point

An element of Y is a map $x : [0,1] \to X$

where $x(0) = x(1) = x_0$ (Figure 1 (a)).

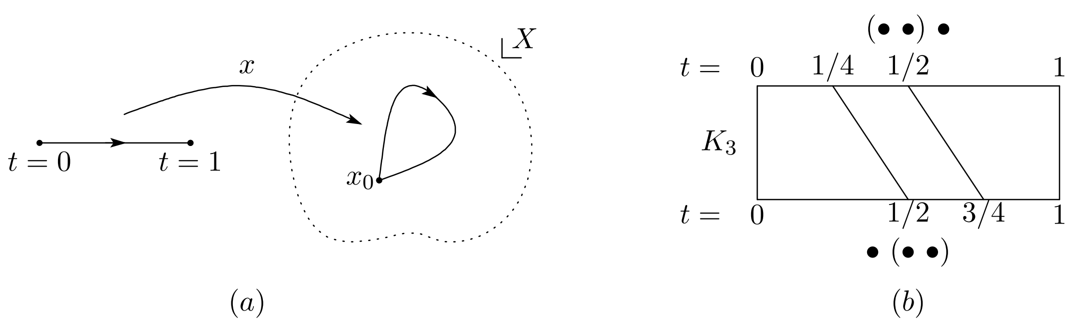


Figure 1:

We have a product as a group-like space

$$M_2 : Y \times Y \to Y \ .$$

It is given by connecting two loops as

$$M_2(x, x')(t) = x(2t) \ , \quad 0 \leq t \leq 1/2$$
$$M_2(x, x')(t) = x'(2(t - 1/2)) \ , \quad 1/2 \leq t \leq 1 \ .$$

M_2 is not associative but clearly there exists a homotopy described by an interval K_3 (Figure 1 (b))

$$M_3 : K_3 \times Y \times Y \times Y \longrightarrow Y \ .$$

When we represent the product by a trivalent planar tree, the relation above is characterized pictorially as in Figure 2(a).

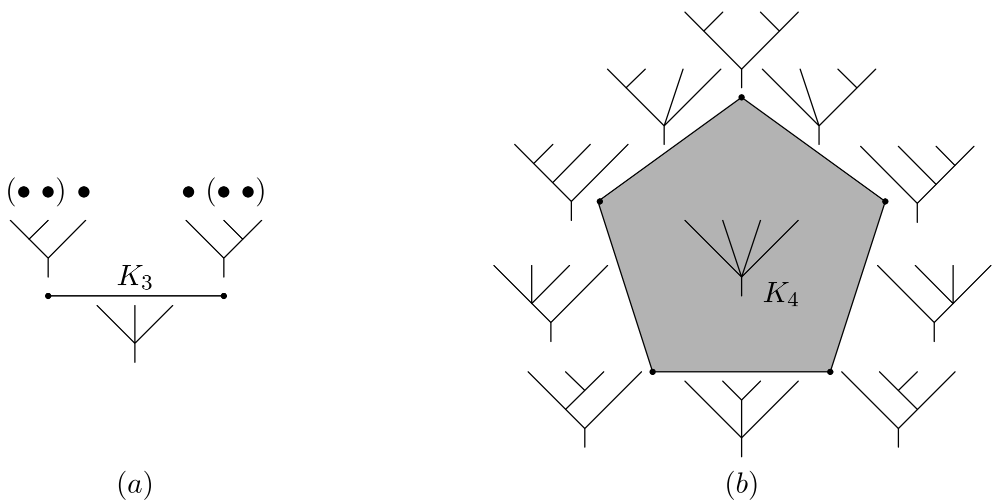


Figure 2:

Next, when considering possible operations of $(Y)^{\times 4} \to Y$ constructed from M_2, we have five tree graphs Figure 2(b).

Each edge corresponds to K_3 and K_4 bounded by these edges is a pentagon. The corresponding homotopy

$$M_4 : K_4 \times (Y)^{\times 4} \to Y$$

is then defined. Repeating this procedure produces higher homotopies

$$M_n : K_n \times (Y)^{\times n} \longrightarrow Y \ .$$

K_n, $n \geq 2$ is a polytope of dimension $(n-2)$,

where K_2 is a point.

Generally, if a topological space Y can be equipped with the structures $\{M_n, K_n\}_{n\geq 2}$ as above,

$(Y, \{M_n, K_n\}_{n\geq 2})$ is called an A_∞-**space**

(J. Stasheff'63).

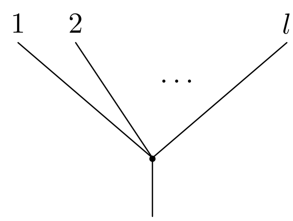
To capture the structure of the A_∞–space,

the terminology of tree graphs is convinient.

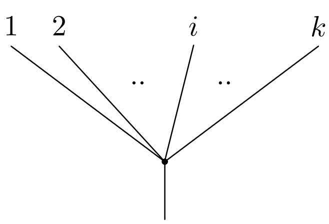
1.2 Tree graphs and A_∞-operad

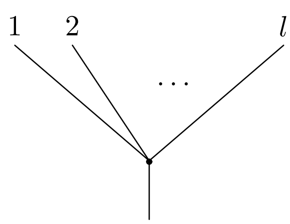
Planar tree graphs are obtained by grafting planar corollas.

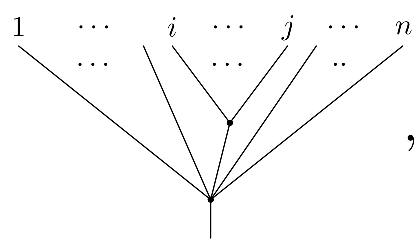
A l-**corolla** δ_l is a planar tree graph with one vertex and l-leaves all attached directly to the root.

$\delta_l :=$ 

The composite $\delta_k \bullet_i \delta_l$ is given by **grafting** the root of δ_l to the i-th leaf of δ_k, reading from left to right



$\bullet_i$


$=$


,

where $j = i + l - 1$ and $n = k + l - 1$.

Let $\mathcal{A}_\infty(n)$, $n \geq 1$ be a graded vector space spanned by planar rooted trees of n leaves with identity $e \in \mathcal{A}_\infty(1)$.

For a planar rooted tree $T \in \mathcal{A}_\infty(n)$, its grading is introduced by

$$|T| = int(T) + (2 - n) \ ,$$

where $int(T)$ is the number of the internal edges in T.

A tree $T \in \mathcal{A}_\infty(n)$, $n \geq 2$, with $int(T) = 0$ is the corolla δ_n.

Any tree T with $int(T) = 1$ is obtained by the grafting of two corollas.

Grafting of any two trees is defined in a similar way.

Any tree T with $v(T) \geq 2$ can be obtained recursively by grafting a corolla to a tree T' with $int(T') = int(T) - 1$.

One can define a **differential** d of degree one, which acts on each corolla as

$$d\,(\delta_n) = \sum_{k,l\geq 2,\ k+l=n+1}\ \sum_{i=1}^{k} \pm\, \delta_k \bullet_i \delta_l$$

and extends to one on $\mathcal{A}_\infty := \oplus_{n\geq 1}\mathcal{A}_\infty(n)$ by the following rule:

$$d(T \bullet_i T') = d(T) \bullet_i T' + (-1)^{|T|} T \bullet_i d(T')\ .$$

If we introduce the contraction of internal edges, that is, indicate by $T' \to T$ that T is obtained from T' by contracting an internal edge, the differential is equivalently given by

$$d(T) = \sum_{T' \to T} \pm T'$$

with an appropriate sign $\pm$.

Thus, $\mathcal{A}_\infty$ forms a dg operad, called the A_∞**-operad**.

• Back to the associahedra ...

$\{K_n\}_{n\geq 2}$ forms a topological operad.

Each K_n is associated with a planar tree n-corolla δ_n.

Associated to the grafting of corollas, one can consider the following inclusion map

$$K_k \circ_i K_l \hookrightarrow K_{k+l-1} \ .$$

By construction, for $\{K_n\}_{n\geq 2}$ we have

$$\partial K_n = \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{i=1}^{k} \pm \, K_k \circ_i K_l$$

for the codimension one boundary of K_n.

Namely, the differential on the operad is given by the boundary operator of the associahedra.

Thus, the cellular chain space $\{C_*(K_n)\}_{n\geq 2}$ forms an A_∞-operad.

One can see that the grading of a tree $T \in \mathcal{A}_\infty(n)$, $int(T) + 2 - n$, is equal to minus the dimension of the corresponding boundary piece of K_n.

- An algebra A over $\mathcal{A}_\infty$ is given by a map

$$\phi : \mathcal{A}_\infty(k) \to \mathrm{Hom}(A^{\otimes k}, A) \ , \quad k \geq 1 \ ,$$

for a complex (A, m_1), compatible with respect to the compositions and the differentials.

Here, the composition in $\oplus_k \mathrm{Hom}(A^{\otimes k}, A)$ is given in a similar way to that in $\mathcal{A}_\infty$, and a differential on $\oplus_k \mathrm{Hom}(A^{\otimes k}, A)$ is given by

$$d(g) = m_1 g - (-1)^{|g|} \sum_{i=1}^{k} g \circ (\mathbf{1}^{\otimes(i-1)} \otimes m_1 \otimes \mathbf{1}^{\otimes(k-i)}) \ .$$

Denote $\phi(\delta_n) =: m_n$, and the compatibility

$$\phi(d(\delta_n)) = d(\phi(\delta_n)) \ (= d(m_n))$$

turns out to be

$$m_1 m_n + \sum_{i=1}^{n} m_n(\mathbf{1}^{\otimes i-1} \otimes m_1 \otimes \mathbf{1}^{\otimes n-i})$$
$$= \sum_{\substack{k+l=n+1 \\ k\geq 2, l\geq 2}} \sum_{j=1}^{k} \pm\, m_k(\mathbf{1}^{\otimes j-1} \otimes m_l \otimes \mathbf{1}^{\otimes k-j})$$

This is the defining equation for

an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$.

Thus, an algebra over $\mathcal{A}_\infty$ is an A_∞-algebra.

Compactification of moduli spaces of disks with punctures on the boundary

It is known that K_n is obtained as the real compactification of the configuration space of $(n-2)$ distinct points in an interval.

It is equivalent to a real compactification $\bar{\mathcal{M}}_{n+1}$ of the moduli space $\mathcal{M}_{n+1}$ of a disk with $(n+1)$ points on the boundary (Figure 3 (a)).

The compactification is further described in terms of the planar tree operads (Figure 3 (b)).

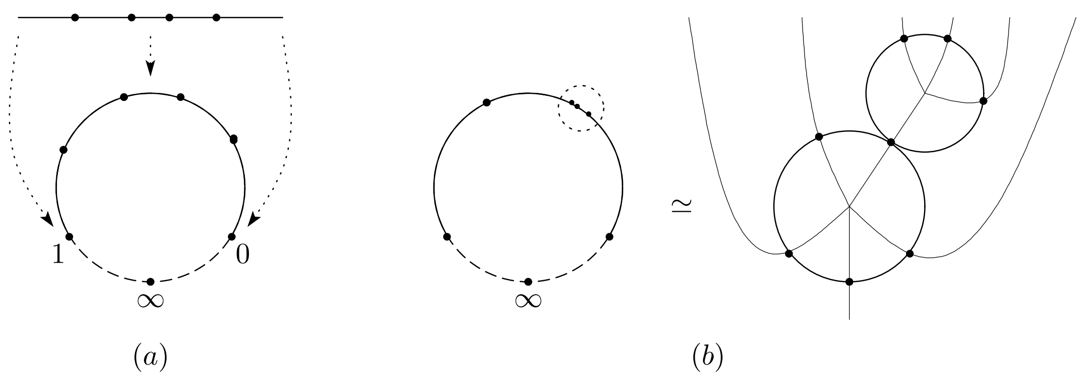


Figure 3:

$\mathcal{M}_{n+1}$ is described as the configuration space of $(n+1)$-punctures on $S^1 \sim \mathbb{R} \cup \{\infty\}$ divided by

$$x'(x) = \frac{ax+b}{cx+d} , \quad x \in \mathbb{R} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) .$$

This degree of freedom can be killed by fixing three points on the boundary. Usually we set the three points at 0, 1 and ∞.

We take the point ∞ as the 'root edge'.

(This distinction between the root and the leaves are absorbed by imposing "cyclicity" as seen later.)

Then, the interval is identified with the arc between 0 and 1 as in Figure 3 (a). Thus, we obtain:

$$\mathcal{M}_{n+1} = \{(t_2, \cdots, t_{n-1}) \mid 0 < t_2 < t_3 < \cdots < t_{n-1} < 1\} .$$

Compactification

- $\mathcal{M}_{2+1} = \{pt\} \quad \rightarrow \quad \bar{\mathcal{M}}_{2+1} \simeq K_2$,
- $\mathcal{M}_{3+1} \simeq \{t_2 \mid 0 < t_2 < 1\} \quad \rightarrow \quad \bar{\mathcal{M}}_{3+1} \simeq K_3$,
- $\mathcal{M}_{4+1} = \{(t_2, t_3) \mid 0 < t_2 < t_3 < 1\}$.

However, by $SL(2, \mathbb{R})$ transformation (cyclicity !)

$$x'(x) = \frac{1 - t_3}{1 - x} ,$$

transforming $(\infty, 0, t_2, t_3, 1) \rightarrow (0, \underline{x'(0), x'(t_2)}, 1, \infty)$,

again $\{\underline{(x'(0), x'(t_2))} \mid 0 < x'(0) < x'(t_2) < 1\}$,

but, the **dimension one** boundary $(t_2, t_3) = (t_2, 1)$

is transformed to **a point** $(x'(0), x'(t_2)) = (0, 0)$.

Similarly, for $(t_2, t_3) = (1 - \epsilon, 1 - a\epsilon)$ with a fixed $\epsilon << 1$, $0 < a < 1$, the image is

$$(x'(0), x'(t_1)) = (a\epsilon, a) .$$

Thus, $\mathcal{M}_5$ should be compactified as a pentagon instead of a triangle.

In such a way, we obtain

the associahedra with the cyclicity as $\bar{\mathcal{M}}_{n+1}$.

(Tree open) string theory should be

an algebra A over the operad $\{\bar{\mathcal{M}}_{n+1}\}_{n\geq 2}$,

where the operad map $\phi : \bar{\mathcal{M}}_{n+1} \to \mathrm{Hom}(A^{\otimes n}, A)$ is

given by the string amplitudes (correlation functions):

$$\int_{\bar{\mathcal{M}}_{n+1}} \Omega : A^{\otimes(n+1)} \to \mathbb{C} \ ,$$

for an appropiate differential form Ω on $\bar{\mathcal{M}}_{n+1}$,

with an appropriate non-degenerate inner product:

$$\eta : A \otimes A \to \mathbb{C} \ .$$

A succesful construction of $\bar{\mathcal{M}}_n$ is given by string field theory (Witten, Zwiebach, etc...).

1.4. A_∞-algebras, A_∞-categories

Def. *A weak A_∞-algebra $(V, \mathfrak{m})$ consists of a $\mathbb{Z}$ (or $\mathbb{Z}_2$-)graded vector space $V = \oplus_k V^k$ (or $= V^{even} \oplus V^{odd}$) with a collection of multilinear maps $\mathfrak{m} := \{m_n : V^{\otimes n} \to V\}_{n\geq 0}$ of degree $(2-n)$ satisfying*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^\epsilon \, m_k(v_1, \cdots, v_j, m_l(v_{j+1}, \cdots, v_{j+l}) , v_{j+l+1}, \cdots, v_n) \ ,$$

where $\epsilon = (j+1)(l+1) + l(|v_1| + \cdots + |v_j|)$.

In particular, if $m_0 : \mathbb{C} \to V^2$ is absent,

$(V, \mathfrak{m})$ is called a (strict) A_∞-algebra (J.Stasheff'63).

For $m_1 = d$, $m_2 = \cdot$, the first three relations :

$$i) \quad d^2 = 0 \ ,$$

$$ii) \quad d(x\cdot y) = d(x)\cdot y + (-1)^{|x|} x \cdot d(y) \ ,$$

$$iii) \quad (x\cdot y)\cdot z - x\cdot (y \cdot z) = d(m_3)(x, y, z)$$

for $x, y, z \in V$.

$i)$: d is nilpotent and $(\mathcal{H}, d)$ defines a complex.

$ii)$: d satisfies Leibniz rule for the product $\cdot$.

$iii)$: the product $\cdot$ is associative up to homotopy.

Rem 1 An A_∞-algebra $(V, \mathfrak{m})$ with vanishing higher products $m_3 = m_4 = \cdots = 0$ is called a **differential graded algebra (DGA)**.

Def. *We call an A_∞-algebra $(V, \mathfrak{m})$ a* **cyclic A_∞-algebra** *if a cyclic structure is given by a nondegenerate symmetric bilinear map $\eta : V \otimes V \to \mathbb{C}$ of fixed degree $|\eta| \in \mathbb{Z}$ satisfying*

$$\eta(m_n(v_1, \cdots, v_n), v_{n+1})$$
$$= (-1)^{n+(|v_2|+\cdots+|v_{n+1}|)|v_1|}\eta(m_n(v_2, \cdots, v_{n+1}), v_1) \ ,$$

for each $n \geq 1$. Here degree $|\eta|$ indicates that $\eta(v, v')$ is nonzero only if $|v| + |v'| + |\eta| = 0$.

A different definition in the degree

Def. [A_∞-algebra $(\mathcal{H}, \mathfrak{m} := \{m_k\}_{k\geq 1})$ $\Longleftrightarrow$**]**

$\mathcal{H} = \oplus_{r\in\mathbb{Z}}\mathcal{H}^r$ *:* $\mathbb{Z}$*-graded vector space*

$\{m_k : (\mathcal{H})^{\otimes k} \to \mathcal{H}\}_{k\geq 1}$, *multi-linear, degree* 1*, s.t.*

$$\sum_{k+l=n+1}\sum_{j=0}^{k-1}(-1)^{|o_1|+\cdots+|o_j|}m_k(o_1,\cdots,o_j,$$
$$m_l(o_{j+1},\cdots,o_{j+l}),o_{j+l+1},\cdots,o_n)=0 \qquad (n\geq 1)$$

Def 1 We call $(\mathcal{H}, \mathfrak{m})$ above a **cyclic** A_∞**-algebra** if it has a nondegenerate skew-symmetric inner product

$$\omega : \mathcal{H}\otimes\mathcal{H}\to\mathbb{C}\ ,$$

of fixed degree $|\omega|\in\mathbb{Z}$ satisfying the cyclicity :

$$\omega(m_n(e_1,\cdots,e_n),e_{n+1})$$
$$=(-1)^{(|e_2|+\cdots+|e_{n+1}|)|e_1|}\omega(m_n(e_2,\cdots,e_{n+1}),e_1)\ .$$

The sign can also be written as $(-1)^{(|e_2|+\cdots+|e_{n+1}|)|e_1|} = (-1)^{(-|\omega|-1-|e_1|)|e_1|} = (-1)^{|\omega||e_1|}$.

Lem 1 For a graded vector space $V = \oplus_k V^k$ with k the grading, let

$$s : V^k \to V^{k-1}[1] =: \mathcal{H}^{k-1} \ ,$$

be a degree shifting operator called **suspension**. Then two definitions of cyclic A_∞-algebras are compatible with each other through the suspension s.

proof. Let us distinguish the A_∞-structures in two notations by $\mathfrak{m}^{\mathcal{H}}$ and $\mathfrak{m}^V$. A relation between the multilinear maps is given by

$$m_n^{\mathcal{H}} = (-1)^{\sum_{i=1}^{n-1}(n-i)} s m_n^V((s^{-1})^{\otimes n})$$

or more explicitly (Getzler-Jones'90)

$$m_n^{\mathcal{H}}(e_1, \cdots, e_n) \\ = (-1)^{\sum_{i=1}^{n-1}(n-i)|e_i|}\ s m_n^V(s^{-1}(e_1), \cdots, s^{-1}(e_n))$$

A relation between the two cyclic structures is also given by $\omega = \eta(s^{-1}\ , s^{-1}\ )$, or

$$\omega(e, e') = (-1)^e \eta(s^{-1}(e), s^{-1}(e'))\ .$$

Def. *A weak* A_∞-**category** $\mathcal{C}$ *consists of a set of objects* $\mathrm{Ob}(\mathcal{C}) = \{a, b, \cdots\}$, $\mathbb{Z}$*-graded vector space* $V_{ab} := \mathrm{Hom}(a, b)$ *for each pair of objects* $a, b \in \mathrm{Ob}(\mathcal{C})$, *and a collection of multilinear maps*

$$\mathfrak{m} := \{m_n : V_{a_1 a_2} \otimes \cdots \otimes V_{a_n a_{n+1}} \to V_{a_1 a_{n+1}}\}_{n \geq 0}$$

of degree $(2 - n)$ *satisfying*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm\, m_k(v_{12}, \cdots, v_{j(j+1)}, m_l(v_{(j+1)(j+2)}, \cdots, v_{(j+l)(j+l+1)}), v_{(j+l+1)(j+l+2)}, \cdots, v_{n(n+1)}),$$

where $\pm = (j + 1)(l + 1) + l(|v_{12}| + \cdots + |v_{j(j+1)}|)$.

In particular, if $m_0 = 0$*, it is called an* A_∞-**category**.

(Fukaya'93)

Def 2 We call a weak A_∞-category $(V, \mathfrak{m})$ a **cyclic weak A_∞-category** if a cyclic structure is given by a nondegenerate symmetric bilinear map $\eta : V_{ab} \otimes V_{ba} \to \mathbb{C}$ of fixed degree $|\eta| \in \mathbb{Z}$ satisfying

$$\begin{aligned}&\eta(m_n(v_{12}, \cdots, v_{n(n+1)}), v_{(n+1)1})\\&= (-1)^{n+(|v_{23}|+\cdots+|v_{(n+1)1}|)|v_{12}|}\eta(m_n(v_{23}, \cdots, v_{(n+1)1}), v_{12}) ,\end{aligned}$$

for each $n \geq 0$.

The suspended version of a cyclic weak A_∞-category can also be defined, where the degree of the multilinear map m_n is one for all $n \geq 0$.

1.5 Examples of A_∞-algebras

- DGA

◦ DGA $(\Omega(X), d, \wedge)$ of differential forms $\Omega(X)$

on a manifold X. This is a commutative DGA.

◦ DGA $(V := \mathrm{End}(E) \otimes \Omega(X), d, \wedge)$

for a vector bundle $E \to X$ with a connection

$$\nabla : \Gamma(M) \otimes \Omega^k(X) \to \Gamma(M) \otimes \Omega^{k+1}(X) \ .$$

The differential $d : V^k \to V^{k+1}$ is given by

$$d(v) = \nabla \cdot v - (-1)^k v \cdot \nabla \ , \quad v \in V^k := \Gamma(M) \otimes \Omega^k(X) \ .$$

The product $\wedge : V \otimes V \to V$ is given locally

by the matrix multiplication of $\mathrm{End}(E)$

combined with the wedge product in $\Omega(X)$.

• DG category

For a fixed manifold X, consider a category $\mathcal{C}$ such that

$\mathrm{Ob}(\mathcal{C})$ is a set $\{a, b, \cdots\}$, where a is a vector bundle $E_a \to X$ with a connection ∇_a,

and for each pair (a, b), $\mathrm{Mor}(a, b) =: V_{ab} = \otimes_{k \in \mathbb{Z}} V_{ab}^k$,

$$V_{ab}^k = \mathrm{Hom}(E_a, E_b) \otimes \Omega^k(X) \ .$$

Then $\mathcal{C}$ forms a DG category, where the differential d is given by

$$d(v_{ab}) = \nabla_b \cdot v_{ab} - (-1)^k v_{ab} \cdot \nabla_a \ , \qquad v_{ab} \in V_{ab} \ .$$

The composition is defined in a natural way.

Rem. ◦ *One can replace vector bundles on a manifold with (projective) modules over a (NC) algebra $\mathcal{A}$ in this set-up.*

◦ *Instead of the DeRham complexes of differential form, one can consider a Dolbault complex. Then, we can consider a DG category on a complex manifold.*

- A geometric example

Consider a collection $\{L_a, L_b, \cdots\}$, where L_a is a line in $\mathbb{R}^2$ defined by

$$L_a : y = t_a x + u_a \ , \qquad t_a, u_a \in \mathbb{R} \ , \qquad (x, y) \in \mathbb{R}^2 \ ,$$

and we regard $a = b$ iff $(t_a, u_a) = (t_b, u_b)$.

For each pair (a, b), $V_{ab} := V^0_{ab} \oplus V^1_{ab}$ is given by :

∘ if $t_a < t_b$, $\quad V^0_{ab} = \mathbb{C} \cdot v_{ab}$, $V^1_{ab} = 0$,

∘ if $t_a > t_b$, $\quad V^0_{ab} = 0$, $V^1_{ab} = \mathbb{C} \cdot v_{ab}$,

∘ if $t_a = t_b$, $u_a \neq u_b$, $\quad V^0_{ab} = V^1_{ab} = 0$,

∘ if $a = b$, $\quad V^0_{aa} = \mathbb{C} \cdot \mathbf{1}_a$, $V^1_{aa} = \mathbb{C} \cdot \bar{\mathbf{1}}_a$,

Here, v_{ab}, $\mathbf{1}_a$, $\bar{\mathbf{1}}_a$ are the bases of the vector spaces.

v_{ab} is identified with the intersection point of (L_a, L_b).

The x-coordinate of v_{ab} is denoted by $x(v_{ab})$.

An A_∞ structure m_n of degree $(2 - n)$ is defined by $m_1 = 0$ and, for $n \geq 2$,

$$m_n(v_{a_1 a_2}, \cdots, v_{a_n a_{n+1}}) = c_{a_1 \cdots a_{n+1}} \cdot v_{a_1 a_{n+1}} \ ,$$
$$c_{a_1 \cdots a_{n+1}} = c(v) \cdot \exp\left(-A(v)\right) \ .$$

Here $A(v)$ is the area of the polygon surrounded by

$$v := (v_{a_1 a_2}, \cdots, v_{a_n a_{n+1}}, v_{a_{n+1} a_1}) \ ,$$

and $c(v) = \pm 1$ or 0 : if v forms a clockwise convex polygon, $c(v) = 1$ for even n and

$$c(v) = \frac{x(v_{a_1 a_2}) - x(v_{a_{n+1} a_1})}{|x(v_{a_1 a_2}) - x(v_{a_{n+1} a_1})|} \quad \text{for odd } n,$$

where we count $\bar{\mathbf{1}}_a$ as a convex vertex and $\mathbf{1}_a$ as a non-convex vertex $v_{a_i a_{i+1}}$ with $a_i = a_{i+1} = a$.

In the case v does not form a clockwise convex polygon,

$c(v) = 1$ if $n = 2$ and $\exists\ i$ s.t. $a_i = a_{i+1}$, and

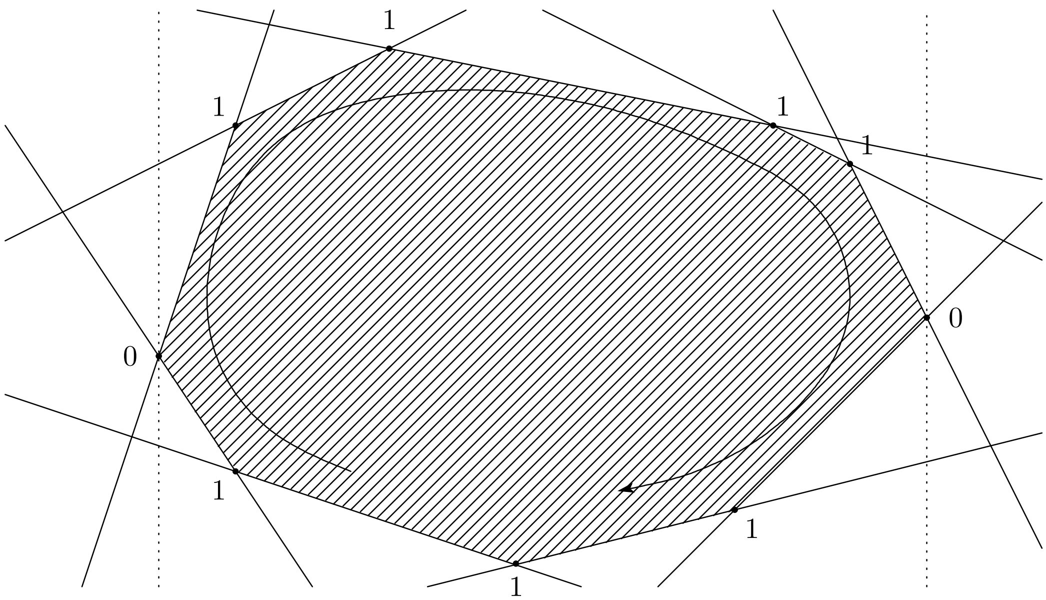
$c(v) = 0$ otherwise.

A degree minus one nondegenerate symmetric inner product $\eta : V_{ab} \otimes V_{ba} \to \mathbb{C}$ is given by

$$\eta(v_{ab}, v_{ba}) = 1 \ , \qquad t_a \neq t_b \ ,$$
$$\eta(\mathbf{1}_a, \bar{\mathbf{1}}_a) = \eta(\bar{\mathbf{1}}_a, \mathbf{1}_a) = 1 \ , \qquad a = b \ .$$

Then $(V, \eta, \mathfrak{m})$ forms a cyclic A_∞-category.

The fact that the structure constant is non-zero only when the corresponding polygon is a convex $(n+1)$-gon is equivalent to the fact that $c_{a_1 \cdots a_{n+1}}$ is nonzero only when $\sum_{i=1}^{n+1} \deg(v_{a_i a_{i+1}}) = -2 + (n+1)$.



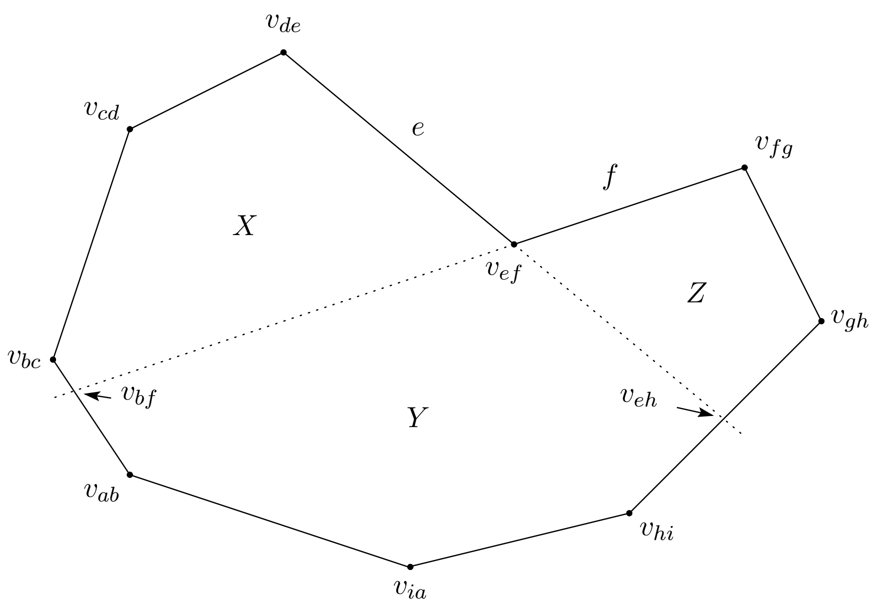
Namely, if we go around a convex $(n+1)$-gon in the clockwise direction and assign the degree k (zero or one) such that $\dim(V^k_{a_i a_{i+1}}) \neq 0$ to each vertex $v_{a_i a_{i+1}}$, we always have two degree zero vertices and $(n+1)-2$ degree one vertices as in the above Figure. Thus, we have

$$\sum_{i=1}^{n+1} \deg(v_{a_i a_{i+1}}) = (n+1) - 2$$

for any convex $(n+1)$-gon.

The A_∞-relation follows from a polygon which has one nonconvex vertex.

There exist two ways to divide the polygon into two convex polygons. The corresponding terms then appear with opposite sign and cancel with each other in the A_∞-relation.



For example, in this figure, we have the following vertices with their degree assigned:

$$\begin{array}{cccccccc} v_{ab} & v_{bc} & v_{cd} & v_{de} & v_{ef} & v_{fg} & v_{gh} & v_{hi} \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array} .$$

According to the way of dividing the area $X+Y+Z$ into (i) $X+(Y+Z)$ or (ii) $(X+Y)+Z$, we have the following composition of multilinear maps:

$$\begin{array}{l}\text{(i)} \pm v_{ab}(v_{bc}v_{cd}v_{de}v_{ef})v_{fg}v_{gh}v_{hi} \\ \text{(ii)} \pm v_{ab}v_{bc}v_{cd}v_{de}(v_{ef}v_{fg}v_{gh})v_{hi}\end{array},$$

where $(v_{bc}v_{cd}v_{de}v_{ef})$ indicates $m_4(v_{bc},v_{cd},v_{de},v_{ef})$ and so on. Then one obtains:

$$m_4(v_{bc},v_{cd},v_{de},v_{ef}) = -e^{X}v_{bf}\ ,$$

$$m_5(v_{ab},v_{bf},v_{fg},v_{gh},v_{hi}) = -e^{(Y+Z)}v_{ai}\ ,$$

$$m_3(v_{ef},v_{fg},v_{gh}) = -e^{Z}v_{eh}\ ,$$

$$m_6(v_{ab},v_{bc},v_{cd},v_{de},v_{eh},v_{hi}) = -e^{(X+Y)}v_{ai}\ .$$

Combining the first two equations leads to

$$+\, m_5(v_{ab},m_4(v_{bc},v_{cd},v_{de},v_{ef}),v_{fg},v_{gh},v_{hi}) \\ = e^{X+(Y+Z)}v_{ai}\ ,$$

and combining the last two gives

$$-\, m_6(v_{ab},v_{bc},v_{cd},v_{de},m_3(v_{ef},v_{fg},v_{gh}),v_{hi}) \\ = -e^{(X+Y)+Z}v_{ai}\ .$$

Thus, we obtain

$$
\begin{aligned}
0 = &+m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi}) \\
&- m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi}) \ ,
\end{aligned}
$$

which is just one of the A_∞-relations.

Rem. *If we consider lines in a* $\mathbb{T}^2$ *instead of* $\mathbb{R}^2$*, one obtains a Fukaya's* A_∞*-category on a (non)commutative two-torus*

(see Polishchuk'00, H.K'04).

Rem. *Lines are in general replaced by Lagrangian submanifolds in a symplectic manifold.*

1.6 The coalgebra description

Def 3 (Coalgebra, coassociativity) Let C be a (generally infinite dimensional) graded vector space. $(C, \triangle)$ is called a **coalgebra**, if the **coproduct** $\triangle : C \longrightarrow C \otimes C$ is **coassociative**, *i.e.*

$$(\triangle \otimes \mathbf{1})\triangle = (\mathbf{1} \otimes \triangle)\triangle \ .$$

The coassociativity is expressed as

$$\begin{tikzcd} C \arrow[r, "\triangle"] \arrow[d, "\triangle"] & C \otimes C \arrow[d, "\triangle\otimes\mathbf{1}"] \\ C \otimes C \arrow[r, "\mathbf{1}\otimes\triangle"] & C \otimes C \otimes C \ . \end{tikzcd}$$

Def 4 (Coderivation) A linear operator $\mathfrak{m} : C \to C$ is called a **coderivation** when it satisfies

$$\triangle\mathfrak{m} = (\mathfrak{m} \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{m})\triangle$$

Here, for $x, y \in C$, the sign is defined as

$$(\mathbf{1} \otimes \mathfrak{m})(x \otimes y) = (-1)^{|x||\mathfrak{m}|}(x \otimes \mathfrak{m}(y)) \ .$$

The condition of a coderivation can be written as

$$\begin{array}{ccc} C & \xrightarrow{\mathfrak{m}} & C \\ \downarrow \triangle & & \downarrow \triangle \\ C \otimes C & \xrightarrow{\mathbf{1}\otimes\mathfrak{m}+\mathfrak{m}\otimes\mathbf{1}} & C \otimes C \end{array}$$

Def 5 (Coalgebra homomorphism) Given two coalgebras C and C', a **coalgebra homomorphism** $\mathcal{F} : C \to C'$ is a map of degree zero which satisfies the condition

$$\triangle\mathcal{F} = (\mathcal{F} \otimes \mathcal{F})\triangle \ .$$

The condition of a cohomomorphism is summarized as the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\mathcal{F}} & C' \\ \downarrow \triangle & & \downarrow \triangle \\ C \otimes C & \xrightarrow{\mathcal{F}\otimes\mathcal{F}} & C' \otimes C' \end{array}$$

For a $\mathbb{Z}$-graded vector space A, consider

$T^cA := \oplus_{k\geq 0}(A)^{\otimes k}$ (the **bar construction**) .

∘ **Coproduct** $\triangle : T^cA \to T^cA \otimes T^cA$ is given by

$$\Delta(o_1\otimes\cdots\otimes o_n) = \sum_{p=0}^{n}(o_1\otimes\cdots\otimes o_p)\otimes(o_{p+1}\otimes\cdots\otimes o_n) .$$

∘ $\mathrm{Hom}(T^cA, A) \simeq \mathrm{Coder}(T^cA)$ holds :

for $h \in \mathrm{Hom}(A^{\otimes k}, A)$, $\mathfrak{h} \in \mathrm{Coder}(T^cA)$ is given by

$$\mathfrak{h}(o_1 \otimes \cdots \otimes o_n) = \sum_{j=0}^{n-k}(-1)^{(|o_1|+\cdots+|o_j|)|\mathfrak{h}|}$$

$$o_1 \otimes \cdots \otimes o_j \otimes h(o_{j+1}, \cdots, o_{j+k}) \otimes o_{j+k+1} \otimes \cdots \otimes o_n$$

for $k \leq n$ and zero otherwise.

$(\mathrm{Coder}(T^cA), [\ ,\ ])$ forms a Graded Lie Algebra,

where $[\ ,\ ]$ is the commutator in $\mathrm{Coder}(T^cA)$.

Then $\mathrm{Hom}(T^cA, A) \simeq \mathrm{Coder}(T^cA)$

as Graded Lie algebras.

For a weak A_∞-algebra $(A, \{m_k\}_{k\geq 0})$, $|m_k| = 1$,

each $m_k \in \mathrm{Hom}(A^{\otimes k}, A)$ is lift to $\mathfrak{m}_k \in \mathrm{Coder}(T^c A)$.

• $(A, \mathfrak{m})$ is a weak A_∞-algebra $\Longleftrightarrow$

$$\mathfrak{m} := \sum_k \mathfrak{m}_k \text{ is a } \textbf{codifferential} : \quad (\mathfrak{m})^2 = 0 \ .$$

• For two weak A_∞-algebras $(A, \mathfrak{m})$ and $(A', \mathfrak{m}')$,

∘ $\{f_k : (A)^{\otimes k} \to A'\}_{k\geq 0}$: multi-linear, degree 0.

∘ $\{f_k\}$ is in one-to-one correspondence with
a **coalgebra homomorphism** $\mathfrak{f} : T^c A \to T^c A'$:

$$\mathfrak{f}(o_1 \otimes \cdots \otimes o_n) = \sum_{i\geq 1} \sum_{1\leq k_1 \leq k_2 \cdots \leq k_i = n} f_{k_1}(o_1, \cdots, o_{k_1}) \otimes f_{k_2 - k_1}(o_{k_1+1}, \cdots, o_{k_2}) \otimes \cdots \cdots \otimes f_{n-k_{i-1}}(o_{k_{i-1}+1}, \cdots, o_n) \ .$$

Def. $\mathfrak{f} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ *is a* **weak A_∞-morphism**
$\Longleftrightarrow \quad \mathfrak{m}' \, \mathfrak{f} = \mathfrak{f} \, \mathfrak{m}$.

In particular, for two A_∞-algebras $(A, \mathfrak{m})$ and $(A', \mathfrak{m}')$, a weak A_∞-morphism $\mathfrak{f} = \{f_0, f_1, \cdots\}$ with $f_0 = 0$ is called **an A_∞-morphism**.

1.7 On some homotopy algebraic properties

Def. *An A_∞-morphism $\mathcal{F} : (A, \mathfrak{m}) \to (A', \mathfrak{m}')$ is*

- **A_∞-isom.** $\iff$ $f_1 : A \to A'$ *is isom.*
- **A_∞-quasi-isom.** $\iff$ $f_1 : (A, m_1) \to (A', m'_1)$ *induces isom. on their cohomologies.*

Def. *An A_∞-algebra $(A, \mathfrak{m})$ is*

- **min***imal* $\Leftrightarrow$ $m_1 = 0$,
- *linear* **cont***ractible* $\Leftrightarrow$ $m_2 = m_3 = \cdots = 0,$ *cohomology of (A, m_1) is trivial.*

Thm. [Minimal model theorem (Kadeishvili '82)]

$(A, \mathfrak{m})$ *: given* $\Longrightarrow$

$$^{\exists} A_\infty\text{-quasi-isom. } \mathcal{F} : {}^{\exists}(A, \mathfrak{m})_{min} \to (A, \mathfrak{m}) .$$

Thm. [Decomposition theorem (cf. H.K'03)]

$$^{\forall}(A, \mathfrak{m}) \overset{A_\infty-isom}{\simeq} (A, \mathfrak{m})_{min} \oplus (A, \mathfrak{m})_{cont} .$$

2. L_∞-algebra

2.1. L_∞-algebras

Def. [weak L_∞-algebra $(L, \{l_k\}_{k\geq 1})$ $\Longleftrightarrow$]

L : *$\mathbb{Z}$-graded vector space*

$\{l_k : (L)^{\otimes k} \to L\}_{l\geq 0}$: *multi-linear, degree* 1,

graded symmetric on $(L)^{\otimes k}$*, s.t.*

$$\sum_{k+l=n+1} \sum_{\sigma\in\mathfrak{S}_n} \frac{\epsilon(\sigma)}{l!(n-l)!}$$

$$l_k(l_l(c_{\sigma(1)}, \cdots, c_{\sigma(l)}), c_{\sigma(l+1)}, \cdots, c_{\sigma(n)}) = 0 \quad (n \geq 1) ,$$

where $\epsilon(\sigma)$ is the sign associated to the permutation

$$(c_1, \cdots, c_n) \to (c_{\sigma(1)}, \cdots, c_{\sigma(n)}) .$$

In particular, if $l_0 = 0$, it is called an L_∞-**algebra**.

(Lada-Stasheff'92)

After the desuspension $s^{-1} : L^k \to (L[-1])^{k+1}$,

and for $l_1 = d$, $l_2 = [\ ,\ ]$, the first three relations :

$$
\begin{array}{ll}
i) & d^2 = 0 \ , \\
ii) & d[x,y] = [d(x), y] + (-1)^{|x|}[x, d(y)] \ , \\
iii) & [[x,y],z] \pm [[y,z],x] \pm [[z,x],y] = d(l_3)(x,y,z)
\end{array}
$$

for $x, y, z \in L[-1]$.

$i)$: d is nilpotent and (L, d) defines a complex.

$ii)$: d satisfies Leibniz rule for the bracket $[\ ,\ ]$.

$iii)$: the bracket $[\ ,\ ]$ satisfies the Jacobi identity up to homotopy.

Rem 2 An L_∞-algebra $(L, \mathfrak{l})$ with vanishing higher products $l_3 = l_4 = \cdots = 0$ is called a **differential graded Lie algebra (DGLA)**.

2.2 Tree graph description

The tree operad description of L_∞-algebras uses non-planar rooted trees, which can be expressed as a planar rooted tree but with arbitrary ordered labels for the leaves. In particular, corollas obtained by permuting the labels are identified (Figure 4).

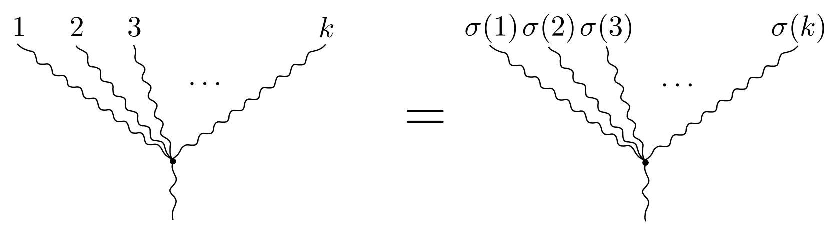


Figure 4:

Let $\mathcal{L}_\infty(n)$, $n \geq 1$ be a graded vector space generated by those non-planar rooted trees of n leaves.

For a tree $T \in \mathcal{L}_\infty(n)$, a permutation $\sigma \in \mathfrak{S}_n$ of the labels for leaves generates a different tree in general, but sometimes the same one because of the symmetry of the corollas above.

The grafting, $\circ_i$, to the i-th leaf is defined as in the planar case.

Any non-planar rooted tree is obtained by grafting corollas $\{l_k\}_{k\geq 2}$ recursively, as in the planar case, together with the permutations of the labels for the leaves.

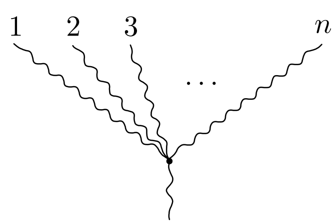
A degree one differential

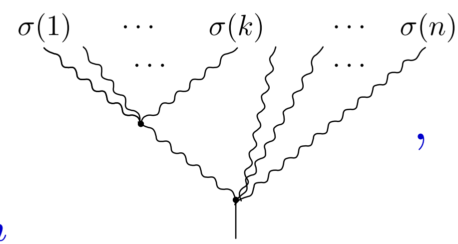
$$d : \mathcal{L}_\infty(n) \to \mathcal{L}_\infty(n)$$

is given in a similar way; for $T' \to T$ indicating that T is obtained from T' by the contraction of an internal edge,

$$d(T) = \sum_{T' \to T} \pm T' \; .$$

In particular, for each corolla one gets

$$d\left(\vphantom{\sum} \right.$$ 

$$\left. \vphantom{\sum} \right) = - \sum_{\substack{\sigma \in \mathfrak{S} \\ k+l=n}}$$ 

,

and

$$d(T \circ_i T') = d(T) \circ_i T' + (-1)^{|T|} T \circ_i d(T')$$

again holds,

where $|T| := int(T) + (3 - 2k)$ for $T \in \mathcal{L}_\infty(k)$.

Thus, $\mathcal{L}_\infty := \oplus_{n \geq 1} \mathcal{L}_\infty(n)$ forms a dg operad, called the L_∞-*operad*.

An algebra L over $\mathcal{L}_\infty$ obtained by a map

$$\phi : \mathcal{L}_\infty(k) \to \mathrm{Hom}(L^{\otimes k}, L)$$

then forms an L_∞-algebra $(L, \mathfrak{l})$.

In the double desuspended notation $L[-2]$, the degree of a multi-linear map l_k turns into $1-2(k-1) = 3-2k$.

Thus, the grading of a tree $T \in \mathcal{L}_\infty(n)$,

$$int(T) + 1 \ + (3 - 2n) \ ,$$

is equal to minus the dimension of the corresponding boundary piece of the compactified moduli space of a sphere with k marked points.

2.3 Compactification of moduli spaces of spheres with punctures

The compactification corresponding to an L_∞-structure is the **real** compactification $\bar{\mathcal{M}}_{0,n}$ of the moduli spaces $\mathcal{M}_{0,n}$ of spheres with n punctures

(Kimura-Stasheff-Voronov'93, see also Zwiebach'92).

$\mathcal{M}_{0,n}$ is the configuration space of n points on

a sphere $\simeq \mathbb{C} \cup \{\infty\}$ modulo $SL(2,\mathbb{C})$ action

$$w'(w) = \frac{aw+b}{cw+d} , \quad w \in \mathbb{C} \cup \{\infty\} , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) .$$

This $SL(2,\mathbb{C})$ action can be killed by fixing

three points; usually $0, 1$ and ∞.

- $\mathcal{M}_{0,2+1} \simeq \{pt\} \simeq \bar{\mathcal{M}}_{0,2+1} \leftrightarrow l_2 = [\ ,\ ]$

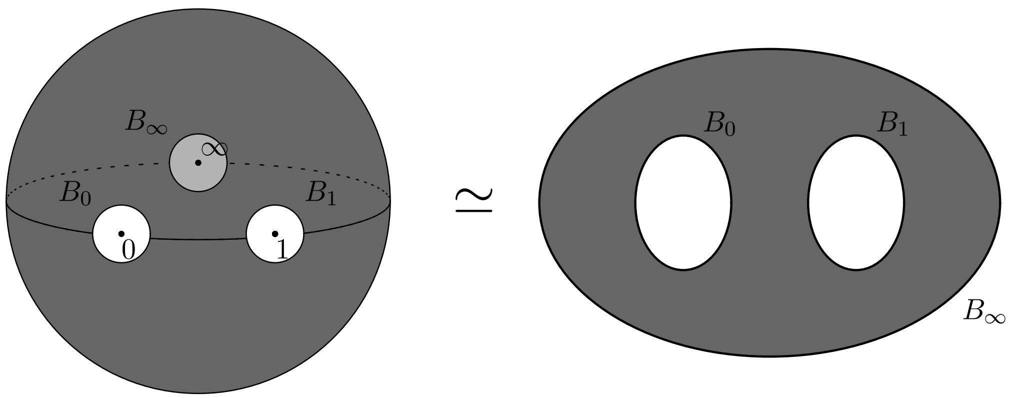
Corresponding to the relation $\partial \bar{\mathcal{M}}_{0,2+1} = 0$, we have

$$d(l_2) = 0 , \qquad \text{the Leibniz rule} .$$

• $\mathcal{M}_{0,4} \simeq (\mathbb{C} \cup \infty) - \{0, 1, \infty\}$ ($\leftrightarrow l_3$),

which is the configuration space of four points $0, 1, w, \infty$ with the subtraction of the 'diagonal'.

the real compactification of $\mathcal{M}_{0,4}$ is ...



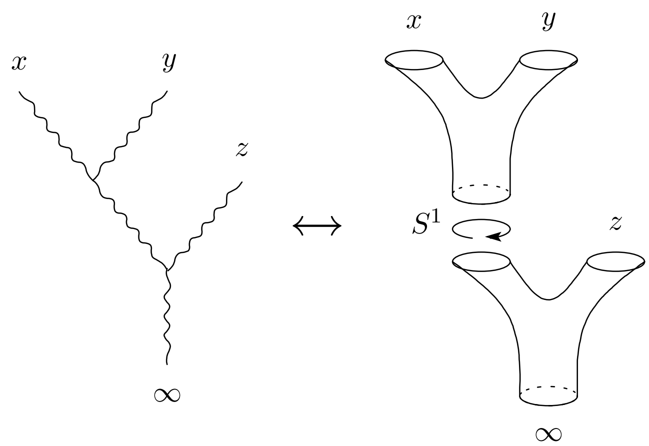
$\bar{\mathcal{M}}_{0,4}$ has $\underline{\mathrm{codim}_{\mathbb{R}} = 1}$ boundaries B_0, B_1, B_∞.

If we associate points $0, 1, w$ to $x, y, z \in L[-1]$ and ∞ to the root edge, we get the correspondence:

$$\begin{array}{ccc} B_0 & \leftrightarrow & \pm[[x, z], y] \\ B_1 & \leftrightarrow & \pm[[y, z], x] \\ B_\infty & \leftrightarrow & \pm[[x, y], z] \; . \end{array}$$

Namely, a grafting of closed string edges

produces moduli S^1:



Thus, $\bar{\mathcal{M}}_{0,n}$ is not contractible.

Corresponding to the relation

$$\partial(\bar{\mathcal{M}}_{0,4}) = B_0 + B_1 + B_\infty, \text{ we obtain :}$$

$$d(l_3)(x,y,z) = [[x,y],z] \pm [[y,z],x] \pm [[z,x],y] \ .$$

• In general, $\bar{\mathcal{M}}_{0,n}$ is a manifold with corners of real dimension $2n-6$.

2.4 The coalgebra description

Consider the graded symmetric coalgebra $C(L)$ on a graded vector space L .

One can define a coproduct $\Delta : C(L) \to C(L) \otimes C(L)$.

Then, for a weak L_∞-algebra $(L, \mathfrak{l})$, $\{l_k\}_{k\geq 0}$ provides a coderivation differential $\mathfrak{l} : C(L) \to C(L)$, $[\mathfrak{l}, \mathfrak{l}] = 0$.

Also, a collection of degree preserving graded symmetric multilinear maps

$$\{f_k : L^{\otimes k} \to L'\}_{k\geq 0}$$

is again in one-to-one correspondence with a coalgebra homomorphism

$$\mathfrak{f} : C(L) \to C(L') \ .$$

Then, a weak L_∞-morphism $\mathfrak{f} : (L, \mathfrak{l}) \to (L', \mathfrak{l}')$ is defined by

$$\mathfrak{f} \circ \mathfrak{l} = \mathfrak{l}' \circ \mathfrak{f} \ .$$

In particular, for $(L, \mathfrak{l})$ and $(L', \mathfrak{l}')$ are two L_∞-algebras, a weak L_∞-morphism $\mathfrak{f} : (L, \mathfrak{l}) \to (L', \mathfrak{l}')$ with $f_0 = 0$ is called an L_∞-morphism.

3. Open-closed homotopy algebra

3.1 The definition (H.K-Stasheff'04)

Def. [Open-closed homotopy algebra (OCHA)]

∘ $\mathcal{H} = \mathcal{H}_o \oplus \mathcal{H}_c$: *a* $\mathbb{Z}$*-graded vector space*

∘ *degree one multi-linear maps*

$$\{l_k : (\mathcal{H}_c)^{\otimes k} \to \mathcal{H}_c\}_{k\geq 0} \ ,$$
$$\{n_{p,q} : (\mathcal{H}_c)^{\otimes p} \otimes (\mathcal{H}_o)^{\otimes q} \to \mathcal{H}_o\}_{p,q\geq 0}$$

which are graded symmetric on $(\mathcal{H}_c)^{\otimes p}$*.*

∘ $T(\mathcal{H})$: *the bar construction of* $\mathcal{H} = \mathcal{H}_c \oplus \mathcal{H}_o$*,*

whose elements are expressed as

$$(c_1 \otimes \cdots \otimes c_m) \otimes (o_1 \otimes \cdots \otimes o_n) \ .$$

∘ *One can define a coproduct on it.*

∘ l_k *and* $n_{p,q}$ *are extended to coderivations on* $T(\mathcal{H})$*,*

which we denote by $\mathfrak{l}_k$ *and* $\mathfrak{n}_{p,q}$*.*

○ *Let us define the total coderivation by*

$$\mathfrak{l}+\mathfrak{n}=\sum_{k\geq 0}\mathfrak{l}_k+\sum_{p\geq 0,q\geq 0}\mathfrak{n}_{p,q}\ .$$

• $(\mathcal{H},\mathfrak{l},\mathfrak{n})$ *is a* **weak open-closed homotopy algebra**

$\Longleftrightarrow$ *the total coderivation is a codifferential;*

$$(\mathfrak{l}+\mathfrak{n})^2=0\ . \qquad (**)$$

In particular if $l_0=n_{0,0}=0$, *we call* $(\mathcal{H},\mathfrak{l},\mathfrak{n})$

an **open-closed homotopy algebra (OCHA)**.

Rem. [$(\mathcal{H},\mathfrak{l},\mathfrak{n})$ includes L_∞ & A_∞]

• *Restrict (the image of) eq.$(**)$ to* $T(\mathcal{H}_c)\subset T(\mathcal{H})$

$\Rightarrow$ *one gets* $\mathfrak{l}^2=0$ *i.e.* $(\mathcal{H}_c,\mathfrak{l})$ *is an* L_∞*-algebra.*

• *Evaluate eq.$(**)$ on* $T(\mathcal{H}_o)\subset T(\mathcal{H})$

$\Rightarrow$ *one gets* $(\mathfrak{m})^2=0$ *with* $\mathfrak{m}:=\sum_q\mathfrak{n}_{0,q}$

i.e. $(\mathcal{H}_o,\mathfrak{m})$ *is an* A_∞*-algebra.*

The defining equation is written down explicitly as the L_∞-condition for $(\mathcal{H}_c, \mathfrak{l})$ and

$$0 = \sum_{\sigma \in \mathfrak{S}} (-1)^{\epsilon(\sigma)} \sum_{p+r=n}$$
$$n_{1+r,m}(l_p(c_{\sigma(1)}, \cdots, c_{\sigma(p)}), c_{\sigma(p+1)} \cdots, c_{\sigma(n)}; o_1, \cdots, o_m)$$
$$+ \sum_{\sigma \in \mathfrak{S}} \sum_{p+r=n} \sum_{i+s+j=m} (-1)^{\mu_{p,i}(\sigma)}$$
$$n_{p,i+1+j}(c_{\sigma(1)}, \cdot\cdot, c_{\sigma(p)}; o_1, \cdot\cdot, o_i,$$
$$n_{r,s}(c_{\sigma(p+1)}, \cdot\cdot, c_{\sigma(n)}; o_{i+1}, \cdot\cdot, o_{i+s}), o_{i+s+1}, \cdot\cdot, o_m)\ ,$$

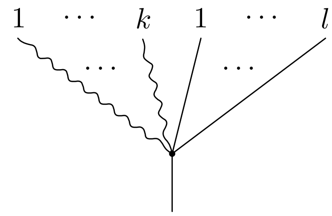
where $c_i \in \mathcal{H}_c, o_i \in \mathcal{H}_o$, and

$$\mu_{p,i}(\sigma) = \epsilon(\sigma) + (c_{\sigma(1)} + \cdots + c_{\sigma(p)}) + (o_1 + \cdots + o_i)$$
$$+ (o_1 + \cdots + o_i)(c_{\sigma(p+1)} + \cdots + c_{\sigma(n)})\ ,$$

corresponding to the signs effected by the interchanges.

3.2 The tree description of an OCHA

For a tree operad description of an OCHA $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$, we introduce **mixed corolla** $n_{k,l}$:

$n_{k,l} \quad \longleftrightarrow$  ,

which is partially symmetric (non-planar), that is,

only symmetric with respect to the k leaves.

Let us consider such corollas for $2k+l+1 \geq 3$ together with non-planar corollas $\{l_k\}_{k\geq 2}$.

Since we have two kinds of edges,

we have two kinds of grafting;

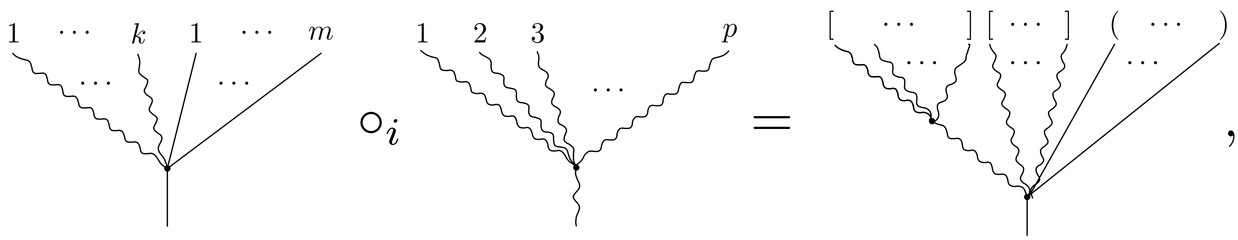
grafting $\circ_i$ for $\mathcal{H}_c$ (closed string edges),

and grafting $\bullet_i$ for $\mathcal{H}_o$ (open string edges).

We have three types of the composite;

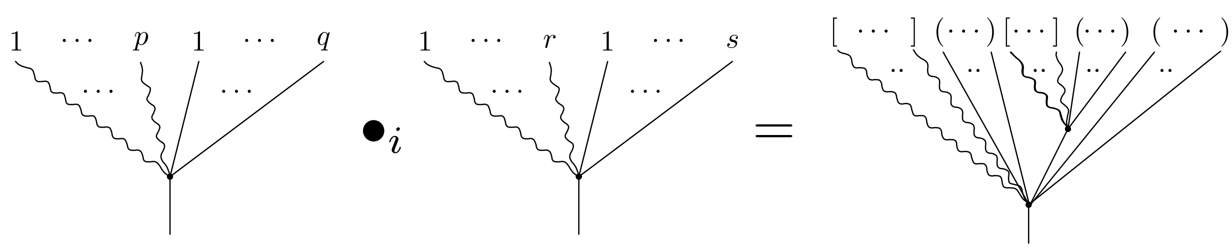
in addition to $l_{1+k} \circ_i l_l$ in $\mathcal{L}_\infty$,

a composite $n_{k,m} \circ_i l_p$ described by



where in the right hand side the labels are given by $[i, \cdots, i+p-1][1, \cdots, i-1, i+p, \cdots, p+k-1](1, \cdots, m)$,

and the composite $n_{p,q} \bullet_i n_{r,s}$



with labels $[1, \cdots, p](1, \cdots, i-1)[p+1, \cdots, p+r](i, \cdots, i+s-1)(i+s, \cdots, q+s-1)$.

Let us consider tree graphs obtained by repeating these grafting,

together with the action of permutations of the labels for closed string leaves.

Each of them has a closed string root edge or an open string root edge.

As explained previously, the tree graphs with closed string root edge, generate $\mathcal{L}_\infty$.

On the other hand, the tree graphs with open string root edge are new; the graded vector space generated by them with k closed string leaves and l open string leaves we denote by $\mathcal{N}_\infty(k;l)$.

In particular, we formally add the identity e_o generating $\mathcal{N}_\infty(0;1)$, and $\mathcal{N}_\infty(1;0)$ is generated by a corolla $n_{1,0}$.

For $\mathcal{N}_\infty := \oplus_{k,l}\mathcal{N}_\infty(k;l)$, the tree operad relevant here is

$$\mathcal{OC}_\infty := \mathcal{L}_\infty \oplus \mathcal{N}_\infty \ .$$

We introduce the grading of $T \in \mathcal{N}_{k,l}$ by

$$|T| = int(T) + (2 - 2k - l) \ .$$

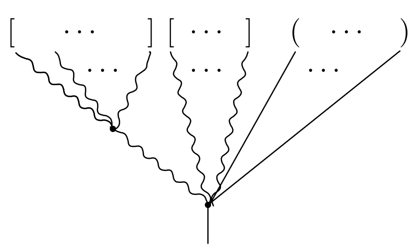
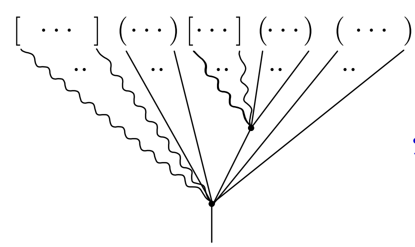
For trees in $\mathcal{OC}_\infty$, let $T' \to T$ indicate that T is obtained from T' by contracting a closed or an open internal edge. A degree one differential $d : \mathcal{OC}_\infty \to \mathcal{OC}_\infty$ is given by

$$d(T) = \sum_{T' \to T} \pm T' \ ,$$

so that the following compatibility holds:

$$d(T \circ_i T') = d(T) \circ_i T' + (-1)^{|T|} T \circ_i d(T') \ ,$$
$$d(T \bullet_i T'') = d(T) \bullet_i T'' + (-1)^{|T|} T \bullet_i d(T'') \ .$$

Thus, $\mathcal{OC}_\infty$ forms a dg operad. In particular, $d(l_k)$ is that given previously, and $d(n_{n,m})$ is:

$$- \sum_{\substack{\sigma \in \mathfrak{S} \\ p+r=n}}$$  $$- \sum_{\substack{\sigma \in \mathfrak{S} \\ p+r=n \\ i+s+j=m}}$$  ,

where the labels for the first and the second terms are $[\sigma(1), \cdots, \sigma(p)][\sigma(p+1), \cdots, \sigma(n)](1, \cdots, m)$ and $[\sigma(1), \cdots, \sigma(p)](1, \cdots, i)[\sigma(p+1), \cdots, \sigma(n)](i+1, \cdots, i+s)(i+s+1, \cdots, m)$, respectively.

An algebra $\mathcal{H} := \mathcal{H}_c \oplus \mathcal{H}_o$ over $\mathcal{OC}_\infty$ is obtained by a representation

$$\phi : \mathcal{L}_\infty(k) \to \mathrm{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c) \ ,$$
$$\phi : \mathcal{N}_\infty(k;l) \to \mathrm{Hom}((\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$$

which is compatible with respect to the grafting $\circ_i$, $\bullet_i$ and the differential d.

Here, regarding elements in both $\mathrm{Hom}(\mathcal{H}_c^{\otimes k}, \mathcal{H}_c)$ and $\mathrm{Hom}((\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l}, \mathcal{H}_o)$ as those in $\mathrm{Coder}(C(\mathcal{H}_c) \otimes T^c(\mathcal{H}_o))$, the differential in the algebra side is given by

$$d := [l_1 + n_{0,1}, \ \ ] \ .$$

Then, the comptibility of ϕ with the differentials turns to the condition of an OCHA.

In the notation $\mathcal{H}_c[-2]$ and $\mathcal{H}_o[-1]$,

the degree of l_k is $(3-2k)$ as stated previously,

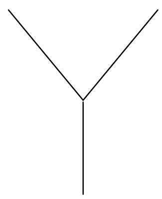
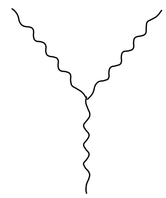
and the degree of $n_{k,l}$ turns to be

$$1+(1-l)-2k = 2-(2k+l)\ .$$

The grading of $T \in \mathcal{N}_\infty(k;l)$, $int(T)+(2-2k-l)$, is equal to minus the dimension of the corresponding boundary piece of the compactified moduli space of a disk with k points interior and l points on the boundary.

3.3 Examples of the moduli spaces

The moduli spaces associated to an OCHA are generated by grafting

$n_{0,2} = m_2 \leftrightarrow$  , $l_2 \leftrightarrow$  , $n_{1,0} \leftrightarrow$  .

- $\{m_n = n_{0,n}\}$: Associahedra $\{K_n\}_{n\geq 2}$ is generated by grafting planar trivalent trees m_2.
- L_∞ $\{l_n\}$: by non-planar trivalent trees $l_2 = [\ ,\ ]$
  - $d(l_2) = 0$
  - $d(l_3)(x,y,z) = [[x,y],z] \pm [[y,z],x] \pm [[z,x],y]$
  - $\cdots\ \cdots$

- $\{n_{1,q}\}$: generated by $n_{1,0}, m_2 = n_{0,2}$.

The resulting moduli space is known as the **Cyclohedra** $\{W_{q+1}\}$, which is obtained by the moduli space of configuration space of points on S^1 modulo rotation (Bott - Taubes'94, see MSS:book,p241)).

- ∘ $d(n_{1,0}) = 0 \qquad \leftrightarrow W_1 = \{pt\}$,

- ∘ $d(n_{1,1})(c; o) = m_2(n_{0,1}(c), o) \pm m_2(o, n_{0,1}(c))$

  $\leftrightarrow W_2 =$ an interval,

- ∘ $d(n_{1,2})(c; o, o') = m_2(n_{1,1}(c; o), o')$

  $\pm m_2(o, n_{1,1}(c; o')) \pm n_{1,1}(c; m_2(0, 0'))$

  $\sum_{i=1}^{3} \pm(m_3 \bullet_i n_{1,0})(c; o, o')$

  $\leftrightarrow W_3 =$ a hexagon,

- ∘ $\cdots \ \cdots$

In general $\{W_n\}$ are contractible polytopes respecting that l_2 is not used and closed string edges are not grafted.

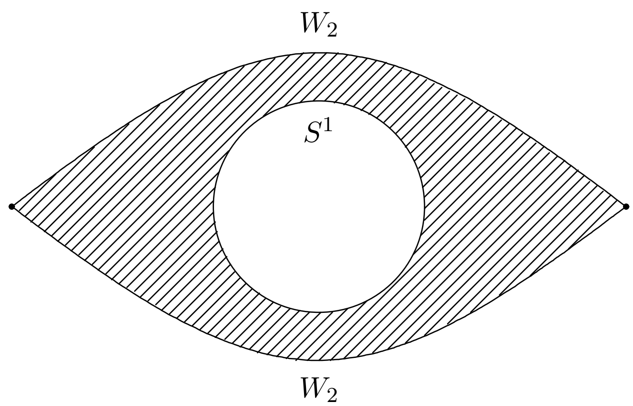
- $n_{2,q}$: disk with two closed strings

: generated by $m_2, n_{1,0}, l_2$

  - $d(n_{2,0})(c,c';\,) = n_{1,0}l_2(c,c')$

$$\pm n_{1,1}(c; n_{1,0}(c')) + n_{1,1}(c'; n_{1,0}(c))$$

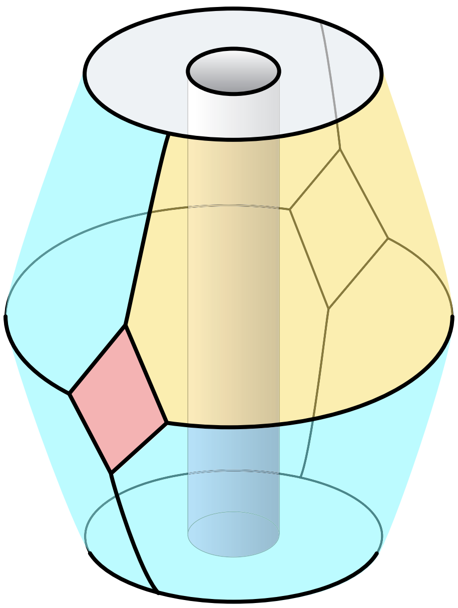
The Eye in Kontsevich'97



$\circ\ d(n_{2,1})'' ='' \text{(the eye } \times K_2\text{)}\ (\times 2) + W_3 \times W_1\ (\times 4)$

$+ \text{ rectangle } W_2 \times W_2\ (\times 2) + \text{cylinder } S^1 \times W_2$

: topologically a solid torus



(picture made by S.Devadoss)

$\circ \cdots \quad \cdots$

The moduli space associated with $n_{p,q}$ is not simply connected for $p \geq 2$ because of the inclusion of l_2.

3.4 Some homotopy algebraic structures

Def. *An OCHA* $(\mathcal{H}, \mathfrak{l}, \mathfrak{n})$ *is*

- **minimal** $\Leftrightarrow$ $l_1 = 0$ *on* $\mathcal{H}_c$ *and* $n_{0,1} = 0$ *on* $\mathcal{H}_o$

- **linear contractible** $\Leftrightarrow$ $l_2 = l_3 = \cdots = 0$,

  $n_{p,q} = 0$ *except for* $(p, q) = (0, 1)$, *and*

  cohomologies of $(\mathcal{H}_c, l_1)$ *and* $(\mathcal{H}_o, n_{0,1})$ *are trivial.*

- An OCHA morphism is constructed by

  degree zero multi-linear maps

$$f_k : (\mathcal{H}_c)^{\otimes k} \to \mathcal{H}_c \,, \qquad f_{k,l} : (\mathcal{H}_c)^{\otimes k} \otimes (\mathcal{H}_o)^{\otimes l} \to \mathcal{H}_o \,.$$

OCHA-isom. and **OCHA quasi-isom.** are defined in a similar way.

Thm. *The decomposition theorem holds for OCHAs.*

$$(\mathcal{H}, \mathfrak{l}, \mathfrak{n}) \overset{isom}{\simeq} (\mathcal{H}, \mathfrak{l}, \mathfrak{n})_{min} \oplus (\mathcal{H}, \mathfrak{l}, \mathfrak{n})_{cont} \,.$$

Thus, one can say OCHA is a homotopy algebra !!

Def. [Cyclic open-closed homotopy algebra]

∘ $(\mathcal{H}, \mathfrak{l}, \mathfrak{m})$: *an open-closed homotopy algebra,*

∘ $\omega_o : \mathcal{H}_o \otimes \mathcal{H}_o \to \mathbb{C}$, $\qquad \omega_c : \mathcal{H}_c \otimes \mathcal{H}_c \to \mathbb{C}$;

non-deg. shew sym. inner products of $\deg = d \in \mathbb{Z}$

∘ *Define two degree* $(d+1)$ *multi-linear maps by*

$$\mathcal{V}_{k+1} = \omega_c(l_k \otimes \mathbf{1}_c) : (\mathcal{H}_c)^{\otimes(k+1)} \to \mathbb{C} \ ,$$
$$\mathcal{V}_{p,q+1} = \omega_o(n_{p,q} \otimes \mathbf{1}_o) : (\mathcal{H}_c)^{\otimes p} \otimes (\mathcal{H}_o)^{\otimes(q+1)} \to \mathbb{C} \ .$$

- $(\mathcal{H}, \omega = \omega_c \oplus \omega_o, \mathfrak{l}, \mathfrak{n})$ *is a* **cyclic OCHA.**

$\Longleftrightarrow \quad \mathcal{V}_{p,q+1}$ *is cyclic symmetric ;*

$$\begin{aligned}&\mathcal{V}_{p,q+1}(c_1, \cdots, c_p; o_1, \cdots, o_{q+1}) \\ &\qquad = \pm\mathcal{V}_{p,q+1}(c_1, \cdots, c_p; o_2, \cdots, o_{q+1}, o_1) \ ,\end{aligned}$$

and $\mathcal{V}_{k+1}$ *satisfies the graded symmetry*

$$\mathcal{V}_{k+1}(c_1, \cdots, c_{k+1}) = \pm\mathcal{V}_{k+1}(c_{\sigma(1)}, \cdots, c_{\sigma(k+1)}) \ .$$

3.5 Dual supermanifold description

For $\mathcal{H}_c$ $\{\mathbf{e}_{c,i}\}$: basis $\Leftrightarrow$ $\{\psi^i\}$: dual coord.

For $\mathcal{H}_o$ $\{\mathbf{e}_{o,i}\}$: basis $\Leftrightarrow$ $\{\phi^i\}$: dual coord.

Degree is set to be $\deg(\psi^i) = -\deg(\mathbf{e}_{c,i})$, etc.

• For the structure constants ;

$$l_k(\mathbf{e}_{c,i_1}, \cdots, \mathbf{e}_{c,i_k}) = \mathbf{e}_{c,j} c^j_{i_1 \cdots i_k} \ ,$$

$$n_{p,q}(\mathbf{e}_{c,i_1}, \cdots, \mathbf{e}_{c,i_p}; \mathbf{e}_{o,j_1}, \cdots, \mathbf{e}_{o,j_q}) = \mathbf{e}_{o,j} c^j_{i_1 \cdots i_p; j_1 \cdots j_q} \ ,$$

Define a degree one **formal vect. field** $\delta := \delta_S + \delta_D$,

$$\delta_S = \sum_{k \geq 1} \frac{1}{k!} \frac{\overleftarrow{\partial}}{\partial \psi^j} c^j_{i_1 \cdots i_k} \psi^{i_k} \cdots \psi^{i_1} \ ,$$

$$\delta_D = \sum_{p+q \geq 1} \frac{1}{l!} \frac{\overleftarrow{\partial}}{\partial \phi^j} c^j_{i_1 \cdots i_p; j_1 \cdots j_q} (\phi^{j_q} \cdots \phi^{j_1})(\psi^{i_p} \cdots \psi^{i_1}) \ ,$$

acting on polynomials of ψ and ϕ. Then

$$\boxed{(\mathfrak{l} + \mathfrak{n})^2 = 0 \quad \text{is equivalent to} \quad (\delta)^2 = 0}$$

Dual descriptoin of a cyclic OCHA

∘ For the component description of the symp. str. ;

$$\omega_{o,ij} := \omega_o(\mathbf{e}_{o,i}, \mathbf{e}_{o,j}) \ , \qquad \omega_{c,ij} := \omega_c(\mathbf{e}_{c,i}, \mathbf{e}_{c,j}) \ ,$$

define the corresponding Poisson brackets

on the spaces of polynomials of ψ and ϕ by

$$(\ \ , \ \ )_o = \frac{\overleftarrow{\partial}}{\partial\phi^i}\omega_o^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} \ , \qquad (\ \ , \ \ )_c = \frac{\overleftarrow{\partial}}{\partial\phi^i}\omega_c^{ij}\frac{\overrightarrow{\partial}}{\partial\phi^j} \ .$$

(ω_o^{ij}, ω_c^{ij} : the inverse matrices of $\omega_{o,ij}$, $\omega_{c,ij}$)

∘ For the component description of $\mathcal{V}_k$ and $\mathcal{V}_{p,q}$;

$$\mathcal{V}_k(\mathbf{e}_{c,i_1}, \cdots, \mathbf{e}_{c,i_k}) := \mathcal{V}_{i_1\cdots i_k} \in \mathbb{C} \ ,$$
$$\mathcal{V}_{p,q}(\mathbf{e}_{c,i_1}, \cdots, \mathbf{e}_{c,i_p}; \mathbf{e}_{o,j_1}, \cdots, \mathbf{e}_{o,j_q}) := \mathcal{V}_{i_1\cdots i_p;j_1\cdots j_q} \in \mathbb{C}$$

consider "action" $S(\phi,\psi) = S_S(\psi) + S_D(\phi,\psi)$, where

$$S_S = \sum_l \frac{1}{l!}\mathcal{V}_{i_1\cdots i_l}\psi^{i_l}\cdots\psi^{i_1} \ ,$$

$$S_D = \sum_{p,q} \frac{1}{p! \ q}\mathcal{V}_{i_1\cdots i_p;j_1\cdots j_q}(\phi^{j_q}\cdots\phi^{j_1})(\psi^{i_p}\cdots\psi^{i_1}) \ .$$

- The differential $\delta = \delta_S + \delta_D$ is then given by

$$\delta_S = (\ \ , S_S)_c \ , \qquad \delta_D = (\ \ , S_D)_o \ .$$

- $(\delta)^2 = 0$ can also be expressed as

$$0 = \frac{1}{2}(S_S, S_S)_c \ , \qquad 0 = \frac{1}{2}(S_D, S_D)_o + (S_D, S_S)_c \ ,$$

which are just sub-recursion relations of

Batalin-**V**ilkovisky **master equation**

for Zwiebach's QUANTUM open-closed SFT !!

(Zwiebach'97)

(when $d = -1 \Leftrightarrow$ degree of action S is zero.)

4. Homotopy algebraic structures in B-twisted topological Landau-Ginzburg model

4.1 Weighted homogeneous polynomials and matrix factorization

Def. *For $\mathcal{A} = \mathbb{C}[x_1, \cdots, x_d]$, $f \in \mathcal{A}$ is a called a* **weighted homogeneous polynomial** *iff $\exists$ relatively prime positive integer (***weight***) $(a_1, \cdots, a_d; h)$ s.t.*

$$\sum_{i=1}^{d} a_i x_i \frac{\partial}{\partial x_i} f = h \cdot f$$

holds.

For a weighted homogeneous polynomial $f \in \mathcal{A}$ with weight $(a_1, \cdots, a_d; h)$, we define the *charge (degree)* for $x_1, \cdots, x_d$ by $q_i := \deg(x_i) = 2a_i/h$.

By definition

$$Ef = 2 \cdot f \ , \qquad E := \sum_i q_i x_i \frac{\partial}{\partial x_i}$$

holds. E is called the *Euler vector field*.

Def 6 For a weighted homogeneous function $f \in \mathcal{A}$, a **matrix factorization** of f is a pair

$$M^0 \underset{\varphi}{\overset{\psi}{\rightleftarrows}} M^1$$

of morphisms between free $\mathcal{A}$-modules s.t.

$$\varphi \circ \psi = f \cdot \mathbf{1}_{M^0} \ , \qquad \psi \circ \varphi = f \cdot \mathbf{1}_{M^1} \ .$$

Namely, to find

$$Q = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix} \ , \quad \varphi, \psi \in Mat_{r\times r}(\mathcal{A}) \ , \qquad \text{s.t.} \ ,$$

$f \cdot \mathbf{1}_{2r\times 2r} = (Q)^2$ for each $r \in \mathbb{N}$.

We have

$$(Q)^2 = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix} = \begin{pmatrix} \varphi\cdot\psi & \\ & \psi\cdot\varphi \end{pmatrix} \ ,$$

so, $\varphi \cdot \psi \ (= \psi \cdot \varphi) = f \cdot \mathbf{1}_{r\times r}$.

We regard $M := M^0 \oplus M^1$, $M^0 = M^1 = (\mathcal{A})^{\oplus r}$, as a $\mathbb{Z}_2$-graded $\mathcal{A}$-free module where M^0 and M^1 are even and odd, respectively. (brane anti-brane system)

Ex 1 (Trivial MF) For $f \in \mathcal{A}$,

$$Q = \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix} , \quad \text{or} \quad Q = \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}$$

are trivial MFs. These trivial MFs will be ignored, or play no role even if included.

Ex 2 (MF for A_n) For $f = x^{n+1} \in \mathbb{C}[x]$,

$$Q = \begin{pmatrix} 0 & x^k \\ x^{n+1-k} & 0 \end{pmatrix} , \qquad k = 1, \cdots , n$$

is MFs for $f = x^{n+1}$.

Ex 3 (MF) For $f = g_1 g_1' + g_2 g_2'$, one can construct a MF of the form

$$Q = \begin{pmatrix} 0 & 0 & g_1 & g_2 \\ 0 & 0 & g_2' & -g_1' \\ g_1' & g_2 & 0 & 0 \\ g_2' & -g_1 & 0 & 0 \end{pmatrix} .$$

In fact,

$$\begin{pmatrix} g_1 & g_2 \\ g_2' & -g_1' \end{pmatrix} \begin{pmatrix} g_1' & g_2 \\ g_2' & -g_1 \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} .$$

Ex 4 (MFs for $f = x^2y + y^{n-1}$ (essentially D_n))
Factorize f to $f = x \cdot y^2 + y^k \cdot y^{n-1-k}$. The corresponding MF:

$$\begin{pmatrix} 0 & 0 & x & y^k \\ 0 & 0 & y^{n-1-k} & -y^2 \\ y^2 & y^k & 0 & 0 \\ y^{n-1-k} & -x & 0 & 0 \end{pmatrix}$$

Def 7 (Charge matrix S) Suppose that a matrix factorization (M, Q) satisfies the following identity

$$SQ - QS + EQ = 1 \cdot Q \ , \qquad E := \sum_{i=1}^{d} q_i x_i \frac{\partial}{\partial x_i}$$

for a diagonal $2r \times 2r$ matrix

$$S := \mathrm{diag}(s_1^+, \cdots, s_r^+; s_1^-, \cdots, s_r^-)$$

with entries in $\mathbb{Q}$. We call S a *charge matrix* of Q.

Hereafter we consider only matrix factorizations equipped with charge matrices.

For two matrix factorizations $a = (M_a, Q_a)$ and $b = (M_b, Q_b)$, denote by $V(a, b)$ the space of homomorphism from M_a to M_b.

(open string stretching between D-brane M and M')
Its elements are described as :

$$V(a, b) \ni \Phi = \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} ,$$

where ϕ^i for each $i = 1, \cdots, 4$ is a $r_b \times r_a$ matrix with entries $\mathcal{A}$.

The diagonal part, ϕ^1 and ϕ^4, is regarded as an even element in $V(a, b)$ in the sense of $\mathbb{Z}_2$-grading. We denote by $V^0(a, b)$ the corresponding subspace. Similarly, the off-diagonal matrices ϕ^2 and ϕ^3 defines an odd element; the corresponding space is denoted by $V^-(a, b)$.

One can define the charge for elements in $V(a,b)$.

Def 8 (Charge for Hom**)** Given two matrix factorizations $a = (M_a, Q_a, S_a)$, $b = (M_b, Q_b, S_b)$ of f, $V(a,b)$ can be decomposed into the direct sum

$$V(a,b) = \oplus_{q\in\mathbb{Q}} V_q(a,b)$$

such that each element $\Phi \in V(a,b)$ satisfies the following identity

$$S_b\Phi - \Phi S_a + E\Phi = q\Phi \ .$$

Here $q = \deg(\Phi)$ is called the *charge* of Φ.

4.2 Charged weak A_∞-algebras, categories

Def. [Charged weak A_∞-algebra] *Let* $(V = V^0 \oplus V^1, \mathfrak{m})$ *be a* $\mathbb{Z}_2$*-graded weak* A_∞*-algebra.* $(V, \mathfrak{m})$ *is called* **charged** *if* V^0, V^1 *are decomposed into the direct sum*

$$V^\sigma = \oplus_{q\in\mathbb{Q}} V^\sigma_q \ , \qquad \sigma = 0 \text{ or } 1 \ ,$$

where any element $v \in V^\sigma_q$ *has its* **charge** $||v|| = q$*, and is compatible with the CDG algebra structure:*

$$||m_n(v_1, \cdots, v_n)|| = ||m_n|| + ||v_1|| + \cdots + ||v_n|| \ ,$$

$||m_n|| := 2 - n$*, for homogeneous elements* $v_1, \cdots, v_n \in V$*.*

(see Takahashi'05, preprint)

Prop 1 Given a $\mathbb{Z}$-graded weak A_∞-algebra $(V, \mathfrak{m})$, one obtains a charged weak A_∞-algebra $(V', \mathfrak{m})$ by

$$\iota : V^{2k} \to (V')^0_{2k} \ , \qquad \iota : V^{2k+1} \to (V')^1_{2k+1}.$$

On the other hand, given a charged weak A_∞ algebra $(V', \mathfrak{m}')$, let us consider a natural projection $p : V' \to V$ given by

$$V^{2k} := (V')^0_{2k} \ , \qquad V^{2k+1} := (V')^1_{2k+1} \ ,$$

and zero otherwise.

Then, $(V, \mathfrak{m})$ forms a $\mathbb{Z}$-graded weak A_∞-algebra, where $m_n = p \circ m'_n \circ (\iota)^{\otimes n}$.

We can define

a **cyclic charged weak A_∞-algebra**,

a **(cyclic) charged weak A_∞-category**

in a similar way,

and similar Proposition holds.

4.3 Category of matrix factorizations

Lem. *Let* $f \in \mathcal{A}$ *be a weighted homogeneous polynomial. Then,* $(\mathcal{A}, f)$ *forms a charged weak* A_∞ *algebra with* $m_0 = f$, $m_1 = 0$, *and* m_2 *is the multiplication in* $\mathcal{A}$.

Def. [A differential on $V(a,b)$] *For two matrix factorizations* (M_a, Q_a, S_a) *and* (M_b, Q_b, S_b), *a differential* $D : V^\sigma(a,b) \to V^{\sigma+1}(a,b)$ *is defined by*

$$D \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} = Q_b \begin{pmatrix} \phi^1 & \phi^2 \\ \phi^3 & \phi^4 \end{pmatrix} - \begin{pmatrix} \phi^1 & -\phi^2 \\ -\phi^3 & \phi^4 \end{pmatrix} Q_a \ .$$

Lem. $D^2 = 0$.

The corresponding cohomology is denoted by $H^\sigma(a,b) = \mathrm{Ker}\, D / \mathrm{Im} D$, *or more precisely,*

$$H^\sigma(a,b) :=$$
$$\{\Phi \in V^\sigma(a,b) | D\Phi = 0\} / \{D\Phi | \Phi \in V^{\sigma+1}(a,b)\} \ .$$

Prop. [DG category of matrix factorizations $\mathcal{C}(MF)$]
For a given $f \in \mathcal{A}$,

$\circ$ Ob : *the set of all matrix factorizations* $\{a, b, \cdots\}$ *with charge matrices* $S = \mathrm{diag}(s_1^+, \cdots, s_d^+; s_1^-, \cdots, s_d^-)$,

$$s^+ \in 2\mathbb{Z}/h \ , \quad s^- \in 2\mathbb{Z}/h + 1 \ .$$

$\circ$ $V(a, b) = Mat_{2r_b \times 2r_a}(\mathcal{A})$,

$\circ$ $m_1 := D : V_{ab} \to V_{ab}$, $\quad D(\Phi) := Q_b \Phi \pm \Phi Q_a$,

$\circ$ m_2 : *the product of the matrices*

forms a charged DG-category.

As we have seen previously, by projecting this charged DG category onto a $\mathbb{Z}_2$-graded vector spaces, we get a ($\mathbb{Z}$-graded) DG-category $\mathcal{C}_{\mathbb{Z}}(MF)$.

Def 9 We denote the cohomology of a DG category $\mathcal{C}$ by $H(\mathcal{C})$, which forms a graded category (additive category).

Thm 1 For a polynomial $f \in \mathcal{A} = \mathbb{C}[x,y,z]$ of A-D-E type, $H(\mathcal{C}(MF))$ and in particular $H(\mathcal{C}_{\mathbb{Z}}(MF))$ form triangulated categories, and the following equivalence of triangulated categories holds :

$$H(\mathcal{C}_{\mathbb{Z}}(MF)) \simeq D^b(B - mod) \ ,$$

where $D^b(B - mod)$ is the derived category of modules over the path algebra of the corresponding (A-D-E) Dynkin quiver.

(H.K-Saito-Takahashi, in preparation)

Roughly speaking, this means that the objects and morphisms in $H(\mathcal{C}_{\mathbb{Z}}(MF))$ corresponds to the vertices and the arrows of the coresponding Dynkin quivers.

4.4 Charged OCHA in LG-model

We can define a charged weak OCHA in a similar way.

Prop. *For* $\mathcal{A} = \mathbb{C}[x_1, \cdots, x_d]$*,*

$f \in \mathcal{A}$ *a fixed weighted homogeneous polynomial,*

$\mathcal{R} := \mathcal{A}/(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d})$ *the Jacobi ring (fix a basis),*

let us set $V_c = \mathcal{R}$*,* $V_o = \mathcal{A}$*,*

$n_{0,0} = f$*,*

$n_{1,0} : V_c \to V_o$ *the inclusion,*

$n_{0,2} : V_o \otimes V_o \to V_o$ *the (commutative) pointwise multiplication in* V_o*,*

and $n_{p,q} = 0$ *for others*

and also $l_k = 0$ *for* $k = 0, 1, \cdots$*.*

Then, $(V := V_c \oplus V_o, \mathfrak{l}, \mathfrak{n})$ *forms a charged weak OCHA.*

In a similar way, we can define a charged OC homotopy category of matrix factorizations.

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