# *Homological perturbation theory and Homological mirror symmetry : the $\mathbb{R}^2$ case*

Hiroshige Kajiura

Research Institute for Mathematical Sciences,
Kyoto University

Higher Structures in Geometry and Physics

∼ *In honor of Murray Gerstenhaber's 80th and Jim Stasheff's 70th birthday* ! ∼

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**Motivation**: To construct an explicit example of $A_\infty$-structures associated to geometry (Fukaya category of Lagrangian submanifolds)

(resolving the transversality problem)

Another motivation: Homological mirror symmetry (Kontsevich'94) of tori (mention at the end of this talk).

**Plan of talk**:

- Def. of $A_\infty$-algebras, $A_\infty$-categories, etc.
- A Fukaya category $Fuk(\mathbb{R}^2)$ on $\mathbb{R}^2$ and the deRham model $\mathcal{C}_{DR}(\mathbb{R})$
- Main theorem on the homotopy equivalence between $Fuk(\mathbb{R}^2)$ and $\mathcal{C}_{DR}(\mathbb{R})$
- Idea of the proof.

**Def. [$A_\infty$-algebra (Stasheff'63)]**

$(V, \mathfrak{m} := \{m_n\}_{n\geq 1})$ *is an* $A_\infty$-**algebra** $\Leftrightarrow$

$V = \oplus_{r\in\mathbb{Z}} V^r$ *: $\mathbb{Z}$-graded vector space,*

$\mathfrak{m} := \{m_n : V^{\otimes n} \to V\}_{n\geq 1}$ *: a collection of degree* $(2-n)$ *multilinear maps s.t.*

$$0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm\, m_k(v_1, \cdots, v_j,$$

$$m_l(v_{j+1}, \cdots, v_{j+l}), v_{j+l+1}, \cdots, v_n)\ ,$$

*for* $n = 1, 2, ...,$ *where* $v_i \in V^{|v_i|}$, $i = 1, ..., n$, *and* $|m_n| = (2-n)$ *implies*

$$|m_n(v_1, ..., v_n)| = (2-n) + |v_1| + \cdots + |v_n|.$$

The $A_\infty$-relations for $n = 1, 2, 3$ :

for $m_1 = d$, $m_2 = \cdot$, $\quad x, y, z \in V$ :

$$i) \quad d^2 = 0 \ ,$$

$$ii) \quad d(x \cdot y) = d(x) \cdot y + (-1)^{|x|} x \cdot d(y) \ ,$$

$$iii) \quad (x \cdot y) \cdot z - x \cdot (y \cdot z) = d(m_3)(x, y, z).$$

$i) \Leftrightarrow (V, d)$ forms a complex.

$ii) \Leftrightarrow$ Leibniz rule of $d$ w.r.t. to product $\cdot$.

$iii)$ $\cdot$ is associative **up to homotopy**.

In particular, if $m_3 = 0$, the product $\cdot$ is strictly associative. An $A_\infty$-algebra $(V, \mathfrak{m})$ with $m_3 = m_4 = \cdots = 0$ is called a differential graded (DG) algebra.

**Def. [$A_\infty$-morphism]**

*Given two $A_\infty$-algebras $(V, \mathfrak{m})$ and $(V', \mathfrak{m}')$, an $A_\infty$-morphism $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ is a collection of degree $(1-k)$ multilinear maps*

$$\mathfrak{f} := \{f_k : V^{\otimes k} \to V'\}_{k \geq 1} \text{ s.t.}$$

$$\sum_{i\geq 1} \sum_{k_1+\cdots+k_n=n} \pm m'_i(f_{k_1} \otimes \cdots \otimes f_{k_i})(v_1, ..., v_n)$$

$$= \sum_{\substack{i+1+j=k \\ i+l+j=n}} \pm f_k(\mathbf{1}^{\otimes i} \otimes m_l \otimes \mathbf{1}^{\otimes j})(v_1, ..., v_n)$$

*for* $n = 1, 2, ....$

**Note**: the above relation for $n = 1$ implies $f_1 : V \to V'$ forms a chain map

$$f_1 : (V, \mathfrak{m}) \to (V', \mathfrak{m}').$$

**Def.** *An* $A_\infty$*-morphism* $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$ *is called an* $A_\infty$-**quasi-isomorphism** *iff* $f_1 : (V, m_1) \to (V', m_1')$ *induces an isom. on the cohomologies of the two complexes.*

**Rem.** *For a given* $A_\infty$*-quasi-isomorphism* $\mathfrak{f} : (V, \mathfrak{m}) \to (V', \mathfrak{m}')$*, there always exists an inverse* $A_\infty$*-quasi-isomorphism*

$$\mathfrak{f}' : (V', \mathfrak{m}') \to (V, \mathfrak{m}).$$

*Thus,* $A_\infty$*-quasi-isomorphisms define (homotopy) equivalence between* $A_\infty$*-algebras.*

We need a categorical version of these terminologies.

**Def. [$A_\infty$-category (Fukaya'93)]**
*An* $A_\infty$-**category** $\mathcal{C}$ $\Leftrightarrow$

$\mathrm{O}b(\mathcal{C}) = \{a, b, \cdots\}$ *: a set of objects*

$V_{ab} := \mathrm{Hom}_{\mathcal{C}}(a, b)$ *: $\mathbb{Z}$-graded vector space for* $\forall a, b \in \mathrm{O}b(\mathcal{C})$

*a collection of multilinear maps*

$$\mathfrak{m} := \{m_n : V_{a_1 a_2} \otimes \cdots \otimes V_{a_n a_{n+1}} \to V_{a_1 a_{n+1}}\}_{n \geq 1}$$

*degree* $(2-n)$ *defining an* $A_\infty$*-structure.*

*In particular,* $\mathcal{C}$ *with* $m_3 = m_4 = \cdots = 0$ *is called a* **DG-category**.

**Def.** *Given two $A_\infty$-categories $\mathcal{C}$ and $\mathcal{C}'$, $\mathfrak{f} := \{f, f_1, f_2, ...\} : \mathcal{C} \to \mathcal{C}'$ is an $A_\infty$-***functor**
$\Leftrightarrow$

$f : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ *a map of objects;*

*a collection of multilinear maps*

$$f_k : \mathrm{Hom}_{\mathcal{C}}(a_1, a_2) \otimes \cdots \otimes \mathrm{Hom}_{\mathcal{C}}(a_k, a_{k+1}) \\ \to \mathrm{Hom}_{\mathcal{C}'}(f(a_1), f(a_{k+1})), \quad k = 1, 2, ...$$

*degree $(1-k)$ satisfying the defining equation of an $A_\infty$-morphism.*

*We call $\mathfrak{f}$* **homotopy equivalence** *iff $f : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C}')$ is bijection and $f_1 : \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(f(a), f(b))$ induces an isom. on the cohomologies for $\forall a, b \in \mathrm{Ob}(\mathcal{C})$.*

## Fukaya category and its deRham model

### Fukaya category $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ for $\mathbb{R}^2$

Let $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ be an $A_\infty$-category satisfying the following two conditions:

For a fixed integer $N \geq 2$, let $\{f_1, ..., f_N\}$ be a collection of functions on $\mathbb{R}$ s.t.

$$L_a : y = \frac{df_a}{dx} = t_a x + s_a \ , \qquad t_a, s_a \in \mathbb{R}$$

is a line in $\mathbb{R}^2$ with coordinates $(x, y)$ ($a = 1, ..., N$ ).

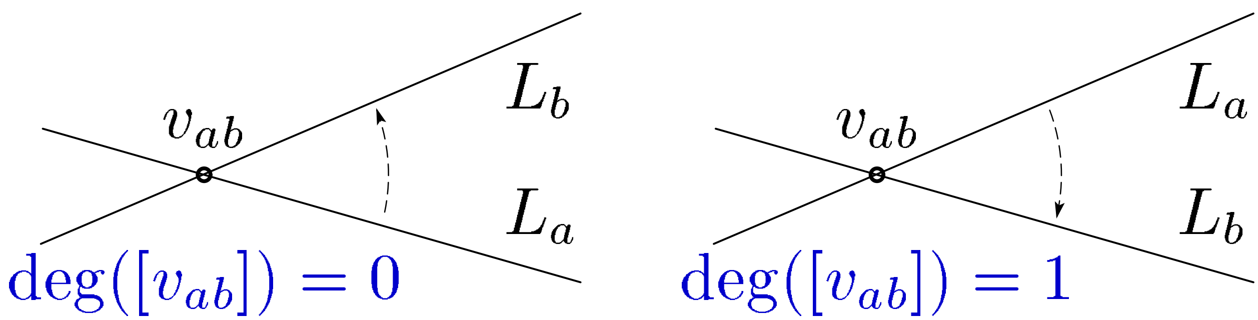
Denote by $\mathfrak{F}_N := \{f_1, ..., f_N\} = \{1, ..., N\}$ such a collection satisfying:

- $t_a \neq t_b$ for $\forall a, b \in \mathfrak{F}_N$.
- Not more than three lines intersect at the same point in $\mathbb{R}^2$.

**Condition 1** $\forall a \neq b \in \mathfrak{F}_N$,

- $V^0_{ab} = \mathbb{R} \cdot [v_{ab}], \quad V^1_{ab} = 0, \quad (t_a < t_b)$,
- $V^0_{ab} = 0, \quad V^1_{ab} = \mathbb{R} \cdot [v_{ab}], \quad (t_a > t_b)$.

Here, $[v_{ab}]$ is the base of $V_{ab}$ labeled by the intersection point $v_{ab}(= v_{ba})$ of $L_a$ and $L_b$.




**Condition 2 (Transversal $A_\infty$-products)**
For a fixed $n \geq 2$ and $a_1, ..., a_{n+1} \in \mathfrak{F}_N$ s.t.

$$a_j \neq a_k \text{ for } j \neq k = 1, ..., n+1,$$

$m_n : V_{a_1 a_2} \otimes \cdots \otimes V_{a_n a_{n+1}} \to V_{a_1 a_{n+1}}$ is

$$m_n([v_{a_1 a_2}], ..., [v_{a_n a_{n+1}}]) = c_{a_1 \cdots a_{n+1}} [v_{a_1 a_{n+1}}]$$

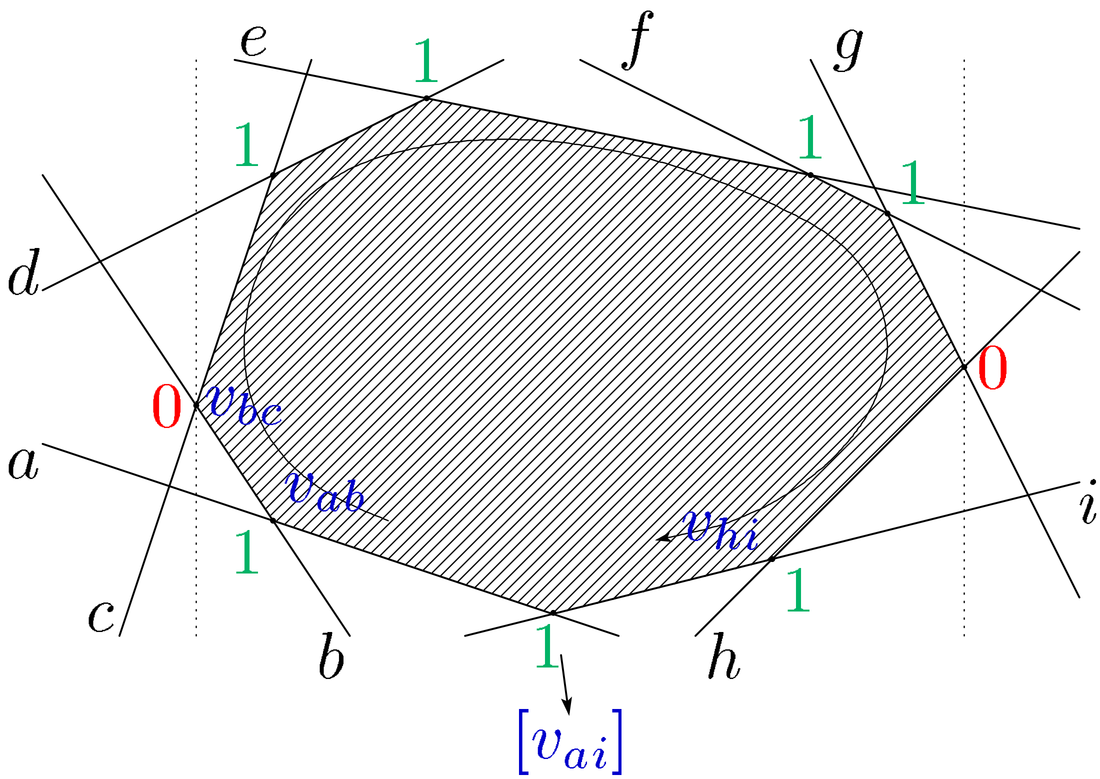
where, if $\vec{v} := (v_{a_1a_2}, \ldots, v_{a_na_{n+1}}, v_{a_{n+1}a_1})$ forms a clockwise convex $(n+1)$-gon,

$$c_{a_1\cdots a_k} = \pm e^{-Area(v)}$$

for $Area(\vec{v})$ the area of the $(n+1)$-gon

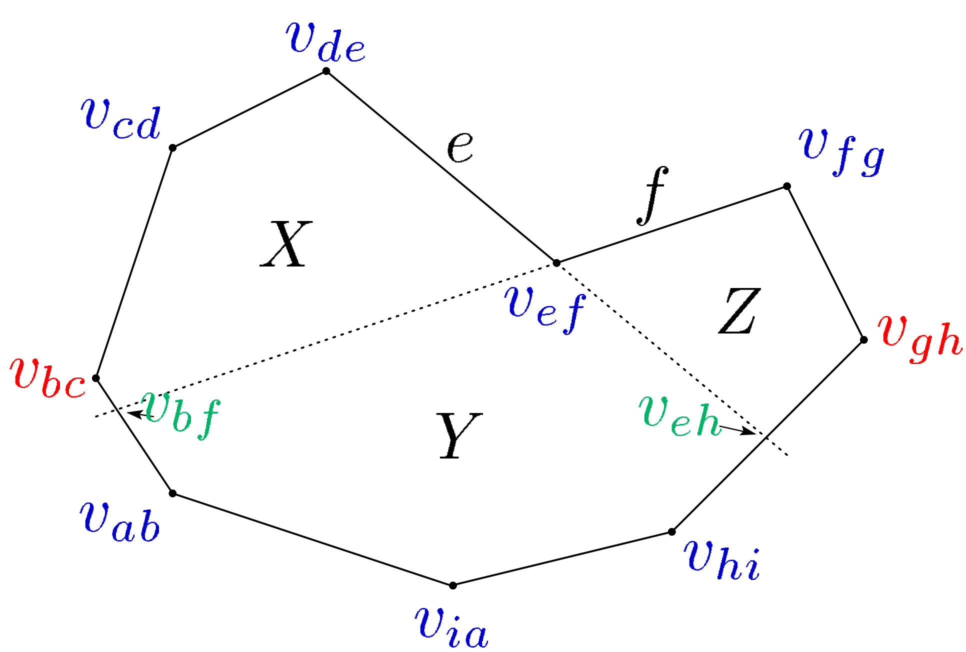
and $c_{a_1\cdots a_{n+1}} = 0$ otherwise.

$m_1 : V_{ab} \to V_{ab}$ is set to be $m_1 = 0 \; \forall a \neq b$.



For transversal $A_\infty$-products, the $A_\infty$-relation follows from a polygon which has one nonconvex vertex.

There exist two ways to divide such a polygon into two convex polygons.




In this figure, the ways of dividing the area $X + Y + Z$ into two are

(i) $X + (Y + Z)$ or (ii) $(X + Y) + Z$.

Corresponding to (i) and (ii) one has

$$(i):\ +m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi})$$
$$= e^{-X-(Y+Z)} v_{ai}\ ,$$

$$(ii):\ -m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi})$$
$$= -e^{-(X+Y)-Z} v_{ai}\ .$$

Thus, we obtain

$$0 = +m_5(v_{ab}, m_4(v_{bc}, v_{cd}, v_{de}, v_{ef}), v_{fg}, v_{gh}, v_{hi})$$
$$- m_6(v_{ab}, v_{bc}, v_{cd}, v_{de}, m_3(v_{ef}, v_{fg}, v_{gh}), v_{hi})\ ,$$

which is just one of the $A_\infty$-relations.

On the other hand, we define a DG-category $\mathcal{C}_{DR}(\mathbb{R}, \mathfrak{F}_N)$ as follows:

**Def.** **[**$\mathcal{C}_{DR}(\mathbb{R}, \mathfrak{F}_N)$ **]**

- $\mathrm{O}b(\mathcal{C}_{DR}(\mathbb{R}, \mathfrak{F}_N)) = \mathfrak{F}_N$ ;

- $\forall a, b \in \mathfrak{F}_N$, $\mathrm{Hom}(a, b) = \oplus_{r=0,1} \Omega^r_{ab}(\mathbb{R})$,
  $\Omega^0_{ab} := \mathcal{S}(\mathbb{R})$, $\Omega^1_{ab} := \mathcal{S}(\mathbb{R}) \cdot dx$,
  *where,* $\mathcal{S}(\mathbb{R})$ *is the Schwartz space,*
  *and* $dx$ *is the base of one-form on* $\mathbb{R}$ ;

- *a differential* $d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}$ *by*

$$d_{ab} := d - df_{ab} \wedge,$$

  *where* $f_{ab} := f_a - f_b$*;*

- *a product* $\Omega^{r_{ab}}_{ab} \otimes \Omega^{r_{bc}}_{bc} \to \Omega^{r_{ab}+r_{bc}}_{ac}$ *by the usual wedge product* $\wedge$*.*

**Thm.** $\exists$ *an* $A_\infty$*-category* $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ *s.t.*

- $\mathrm{O}b(Fuk(\mathbb{R}^2, \mathfrak{F}_N)) = \mathfrak{F}_N$*;*
- $\mathrm{Hom}_{Fuk(\mathbb{R}^2, \mathfrak{F}_N)}(a, b)$ *satisfies the condition 1* $\forall a \neq b \in \mathfrak{F}_N$*;*
- *the* $A_\infty$*-structure* $\{m_k\}_{k \geq 1}$ *of* $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ *satisfies the condition 2;*
- $Fuk(\mathbb{R}^2, \mathfrak{F}_N)$ *is homotopic to* $\mathcal{C}_{DR}(\mathbb{R}, \mathfrak{F}_N)$ *as* $A_\infty$*-categories.*

We can prove this by constructing such an $A_\infty$-category $Fuk(\mathbb{R}^2, \mathfrak{F}_N) =: \mathcal{C}(\mathfrak{F}_N)$ explicitly based on Kontsevich-Soibelman'00's proposal combining homological perturbation theory (HPT) and Harvey-Lawson'01's argument on Morse theory.

## Idea of the proof

The construction of the $A_\infty$-category $\mathcal{C}(\mathfrak{F}_N)$ is divided into 2 steps.

I. Apply HPT to $\mathcal{C}_{DR}(\mathbb{R}, \mathfrak{F}_N) = \mathcal{C}_{DR}(\mathfrak{F}_N)$ and construct a one parameter family of $A_\infty$-categories $\tilde{\mathcal{C}}_\epsilon(\mathfrak{F}_N)$ which are homotopic to $\mathcal{C}_{DR}(\mathfrak{F}_N)$.

II. Consider the limit $\tilde{\mathcal{C}}(\mathfrak{F}_N) := \lim_{\epsilon\to 0} \tilde{\mathcal{C}}_\epsilon(\mathfrak{F}_N)$ and find the minimum subcategory $\mathcal{C}(\mathfrak{F}_N) \subset \tilde{\mathcal{C}}(\mathfrak{F}_N)$ with the same objects $\mathfrak{F}_N$.

## I. HPT and the $A_\infty$-category $\tilde{\mathcal{C}}_\epsilon(\mathfrak{F}_N)$

A version of homological perturbation theory (developed by Gugenheim, Lambe, Stasheff, Huebschmann, Kadeishvili, ... ) we shall employ is as follows.

**Thm.** *Given an $A_\infty$-algebra $(V, \mathfrak{m})$, suppose we have linear maps $h : V^r \to V^{r-1}$ and $P : V^r \to V^r$ satisfying*

$$dh + hd = Id_V - P, \quad P^2 = P, \quad (d := m_1).$$

*Then,* $\exists$ **a canonical way to construct an $A_\infty$-structure $\mathfrak{m}'$ on $P(V)$ s.t. $(P(V), \mathfrak{m}')$ is homotopy equivalent to $(V, \mathfrak{m})$.**

Note that $h$ gives a Hodge decomposition of $(V, d)$ if $dP = 0$, where $P(V) = H(V)$.

$\star$ Apply this HPT to $\mathcal{C}_{DR}(\mathfrak{F}_N)$.

Construct $h_{ab}$ on $\mathrm{Hom}_{\mathcal{C}_{DR}(\mathfrak{F}_N)}(a,b) = \Omega_{ab}$.

∘ For any $a \in \mathfrak{F}_N$, we set $h_{aa} = 0$.

∘ For $a \neq b \in \mathfrak{F}_N$, fix $\epsilon \in (0,1]$ and define $d^{\dagger}_{\epsilon;ab} : \Omega^r_{ab} \to \Omega^{r-1}_{ab}$ by

$$d^{\dagger}_{\epsilon;ab} = \epsilon\, d^{\dagger} - \iota_{\mathrm{grad}(f_{ab})}.$$

Can show that $H_\epsilon := d_{ab} d^{\dagger}_{\epsilon;ab} + d^{\dagger}_{\epsilon;ab} d_{ab}$ has only non-negative real eigenvalues.

In particular,

[ for $\epsilon = 1$ ], $H_1$ is the Hamiltonian

of a harmonic oscillator,

[ for $\epsilon =$ '0' ], $H_0 = e^{f_{ab}} \mathcal{L}_{\mathrm{grad}(f_{ab})} e^{-f_{ab}}$.

(**cf.** $d_{ab} := d - df_{ab}\wedge \ = e^{f_{ab}} \cdot d \cdot e^{-f_{ab}}$. )

Let $\psi_t : \Omega^r_{ab} \to \Omega^r_{ab}$, $t \in [0, \infty)$, be a linear map satisfying $\psi_0 = Id$ and

$$\frac{d\psi_t}{dt} = -H_\epsilon \psi_t.$$

Then, we obtain

$$d_{ab} h_{\epsilon;ab} + h_{\epsilon;ab} d_{ab} = Id_{\Omega_{ab}} - P_{\epsilon;ab},$$

$$h_{\epsilon;ab} := \int_0^\infty dt \ d^\dagger_{\epsilon;ab} \psi_t, \quad P_{\epsilon;ab} := \lim_{t \to \infty} \ \psi_t.$$

Here $P_{\epsilon;ab}$ defines a projection;

$$P_{\epsilon;ab}\Omega^0_{ab} = \mathrm{Ker}(d_{ab} : \Omega^0_{ab} \to \Omega^1_{ab}),$$
$$P_{\epsilon;ab}\Omega^1_{ab} = \mathrm{Ker}(d^\dagger_{\epsilon;ab} : \Omega^1_{ab} \to \Omega^0_{ab}).$$

Choose bases $\mathbf{e}_{\epsilon;ab}$ of $P_{\epsilon;ab}\Omega^r_{ab}$, $r = 0, 1$, by

$$\mathbf{e}_{\epsilon;ab} = const \cdot e^{f_{ab}}, \qquad t_a < t_b$$

(**Gaussian** normalize so that $\mathbf{e}_{\epsilon;ab}(x_{ab}) = 1$)

$$\mathbf{e}_{\epsilon;ab} = const \cdot e^{-\frac{1}{\epsilon}(f_{ab})}dx, \quad t_a > t_b.$$

(**Gaussian** normalize so that $\int_{-\infty}^{\infty} \mathbf{e}_{\epsilon;ab} = 1$ )

In the limit $\epsilon \to 0$, the degree one base $\mathbf{e}_{\epsilon;ab}$ $(t_a > t_b)$ becomes the **delta function** localized at the point $x_{ab}$ $(= x(v_{ab}))$.

In the limit $\epsilon \to 0$, $h_{ab} := \lim_{\epsilon\to 0} h_{\epsilon;ab}$ and $P_{ab} := \lim_{\epsilon\to 0} P_{\epsilon;ab}$ turn out to be

$$h_{ab} = \int_0^\infty dt e^{f_{ab}} \varphi_t^* (e^{-f_{ab}} \ \iota_{\mathrm{grad}(f_{ab})}),$$

$$P_{ab} = \lim_{t\to\infty} e^{f_{ab}} \varphi_t^* e^{-f_{ab}} \ ,$$

where $\varphi_t : \mathbb{R} \to \mathbb{R}$ is the flow defined by

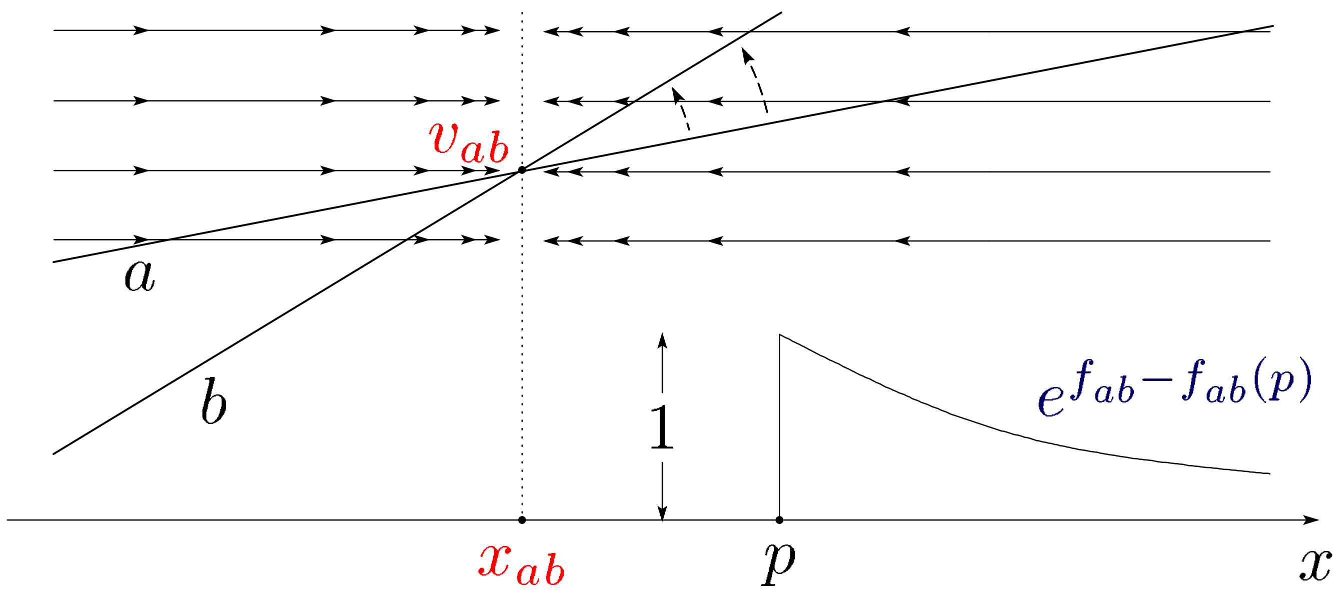
$$\frac{d\varphi_t}{dt} = \mathrm{grad}(f_{ab}), \qquad \varphi_0 = Id.$$

Let us consider the following case:

$$h_{ab}(\delta(x-p)dx)$$
$$= \int_0^\infty dt e^{f_{ab}} \varphi_t^* e^{-f_{ab}} \delta(x-p) \frac{df_{ab}}{dx}(x)$$
$$= e^{f_{ab}} (\varphi_t^* e^{-f_{ab}})|_{\varphi_t(x)=p}(x).$$

$h_{ab}(\delta(x-p)dx)$ for $t_a < t_b$ and $x_{ab} < p$

Flow of $\mathrm{grad}(f_{ab})$




(**step function** twisted by $e^{f_{ab}}$)

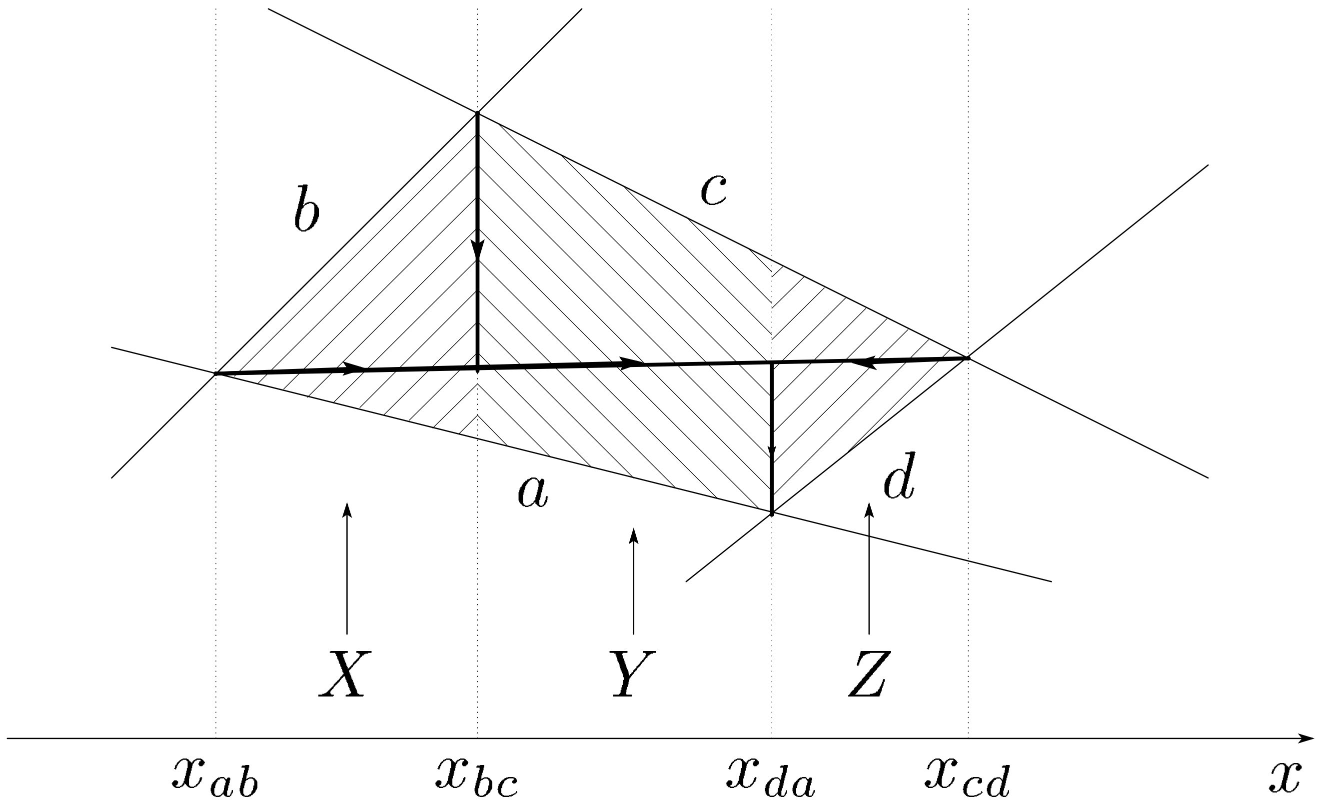
⋆ Now, let us derive the $A_\infty$-products $\{m'_n\}$ of $\tilde{\mathcal{C}}(\mathfrak{F}_N)$ with the identifications

$$\lim_{\epsilon\to 0} P_{\epsilon;ab}\Omega_{ab} =: \mathrm{Hom}_{\tilde{\mathcal{C}}(\mathfrak{F}_N)}(a,b) \simeq V_{ab},$$

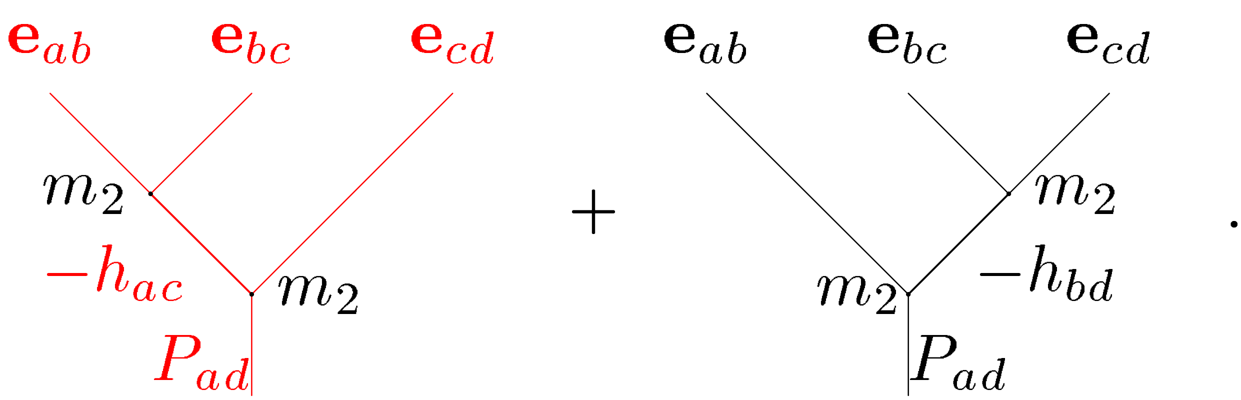
$$\lim_{\epsilon\to 0} \mathbf{e}_{\epsilon;ab} = \mathbf{e}_{ab} \;\longleftrightarrow\; [v_{ab}]$$

for $a \neq b$.

- Example for $m'_3(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd})$



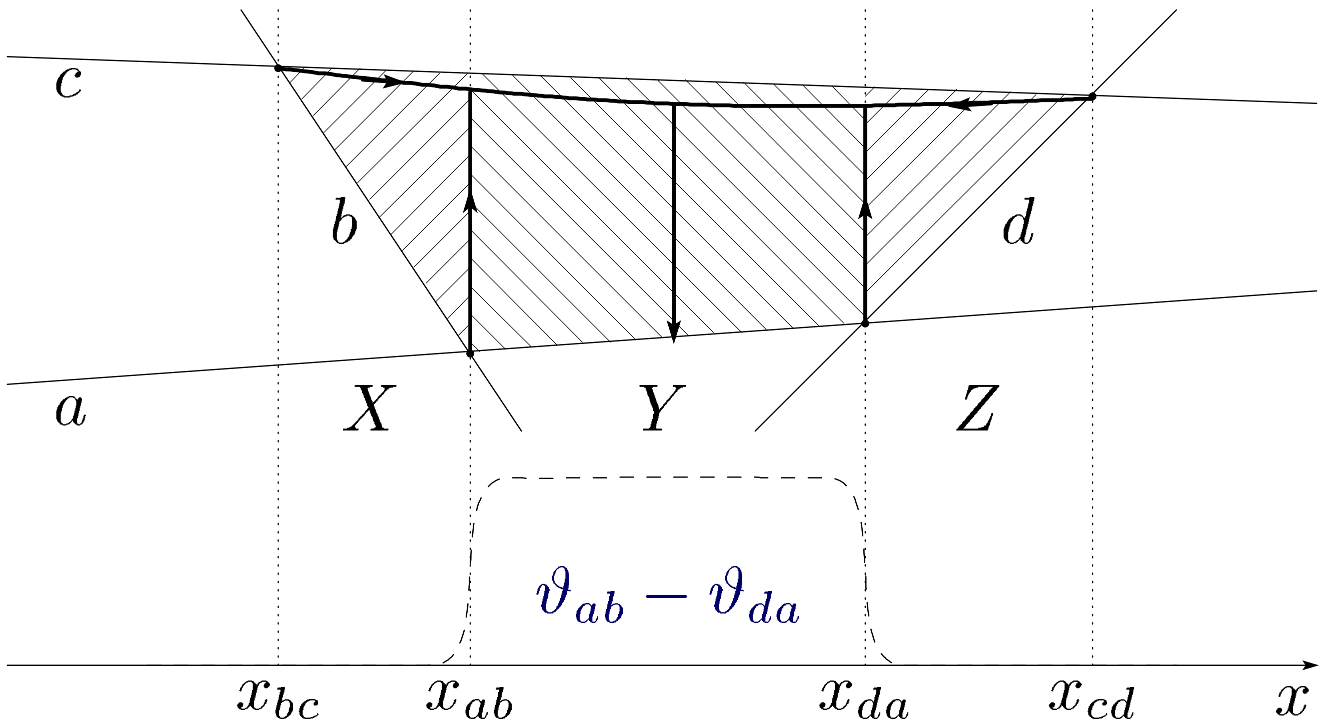

HPT implies $m'_3(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}) =$




$$= -e^{-(X+Y+Z)} \cdot \mathbf{e}_{ad}.$$

• An example of non-transversal product:

$m_3'(\mathbf{e}_{ab}, \mathbf{e}_{bc}, \mathbf{e}_{cd}, \mathbf{e}_{da})$




$$= e^{-(X+Y+Z)} \cdot (\vartheta_{ab} - \vartheta_{da}).$$

By observations as above, we will define

$$V_{aa} = \mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a,a) \subset \mathrm{Hom}_{\tilde{\mathcal{C}}(\mathfrak{F}_N)}(a,a)$$

by introducing $\vartheta_{ab} = \vartheta_{v_{ab}}$ (**step function**),

etc., as its generators.

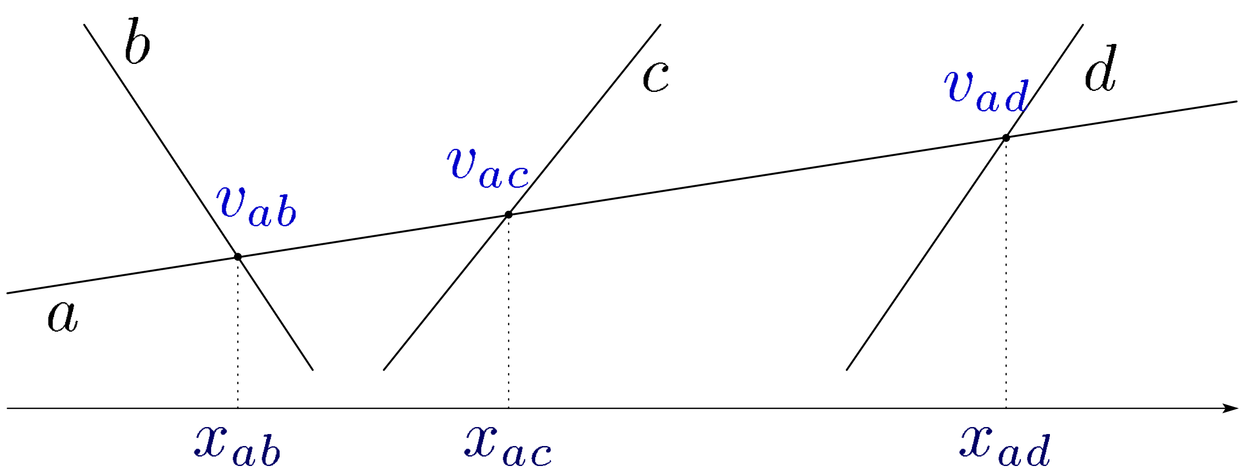
## II. Subcategory of $\tilde{\mathcal{C}}(\mathfrak{F}_N) := \lim_{\epsilon\to 0}\tilde{\mathcal{C}}_\epsilon(\mathfrak{F}_N)$

Consider the minimum subcategory $\mathcal{C}(\mathfrak{F}_N) \subset \tilde{\mathcal{C}}(\mathfrak{F}_N)$ with the same set of objects $\mathfrak{F}_N$ and

$$\mathrm{Hom}_{\mathcal{C}(\mathfrak{F}_N)}(a,b) = \mathrm{Hom}_{\tilde{\mathcal{C}}(\mathfrak{F}_N)}(a,b) = V_{ab}$$

for $a \neq b$.

Then, for any $a \in \mathfrak{F}_N$, $V_{aa}$ (with comm. DGA structure) is defined purely algebraically by the following idea.



For any $v \in \mathfrak{F}_N - \{a\}$,

∘ introduce degree zero generator $\vartheta_v$ and degree one generator $\delta_v$ which are supposed to be

$$\delta_{v_{ab}} = \lim_{\epsilon \to 0} (\mathbf{e}_{\epsilon;ab} \wedge \mathbf{e}_{\epsilon;ba}),$$
$$\vartheta_{v_{ab}}(x) = \int_{-\infty}^{x} dx' \delta_{v_{ab}}(x').$$

∘ appropriate relations

$$\vartheta_v \cdot \vartheta_{v'} = \vartheta_{v'} \text{ for } x(v) < x(v'), \text{ etc.,}$$

∘ $V^0_{aa}$ and $V^1_{aa}$ are the degree zero and one vector space of elements generated by $\vartheta_v$, $\delta_v$ s.t. they are zero at $x = \pm\infty$.

∘ differential $d : V^0_{aa} \to V^1_{aa}$ by extending $d(\vartheta_v) = \delta_v$ by the Leibniz rule.

**Note.** $(\vartheta_v)^2 \neq \vartheta_v$, etc.,

- More examples of non-transversal $A_\infty$-products of $\mathcal{C}(\mathfrak{F}_N)$

For $t_a < t_b$,

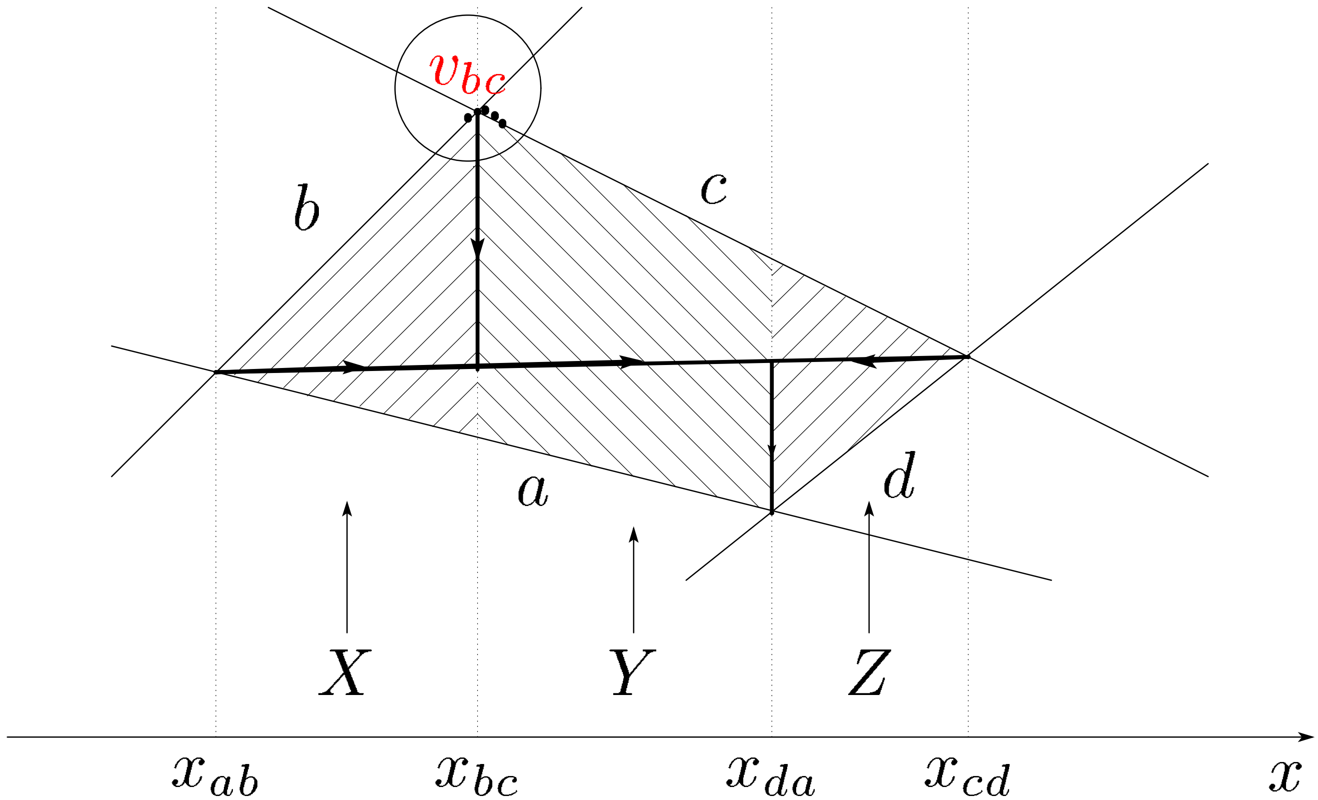
$$m_2'((\vartheta_{v_{ab}})^n, [v_{ab}]) = m_2'([v_{ab}], (\vartheta_{v_{ab}})^n) = \frac{1}{2^n}[v_{ab}],$$

$$m_3'([v_{ba}], (\vartheta_{v_{ab}})^n, [v_{ab}]) = \frac{1}{n+1}\vartheta_{v_{ab}}(1 - (\vartheta_{v_{ab}})^n),$$

for $n \geq 1$,

...

$k$ elements at $v_{bc}$




$$m'_{2+k}([v_{ab}], \delta_{v_{bc}}, \ldots, \delta_{v_{bc}}, [v_{bc}], \delta_{v_{bc}}, \ldots, \delta_{v_{bc}}, [v_{cd}])$$

$$= \frac{(-1)^k}{k!} e^{-(X+Y+Z)} \cdot [v_{ad}].$$

$\star$ The precise proof of the main theorem is given by **defining** $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ and $\tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N)$ s.t.

$$\begin{array}{ccccc} & & & \tilde{\mathcal{C}}_{DR}(\mathfrak{F}_N) & \\ & & \overset{\iota}{\nearrow} & & \overset{\iota}{\nwarrow} \\ \mathcal{C}(\mathfrak{F}_N) & \xrightarrow{\text{HPT}} & \mathcal{C}'_{DR}(\mathfrak{F}_N) & & \mathcal{C}_{DR}(\mathfrak{F}_N) \end{array}$$

Applying HPT for $\mathcal{C}'_{DR}(\mathfrak{F}_N)$ gives $\mathcal{C}(\mathfrak{F}_N)$.

## Future directions

• Generalization to higher dimensional case

(though not so straightforward)

• The $\mathbb{R}^{2n}$ case can be applied to the $T^{2n}$ case.

(**Note.** In this case, each object has identity morphism. )

⇒ ∘ application to homological mirror symmetry for tori;

⇒ ∘ can produce geometric examples of finite dim. $A_\infty$-algebra

∘ (Noncommutative, etc.,) deformation of these $A_\infty$-categories ??

∘ building block to more general mfds ?