Studying Contact structures via Orderings of Mapping class groups

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ⓐ

Thurston-type ordering of MCG

algebra

↓

Right-veering ness of MCG

↓ openbook, Honda-Kazez-Matic's work

- Tightness criterion
- (Non)-vanishing of Ozsvath-Szabo invariant

(· various examples)

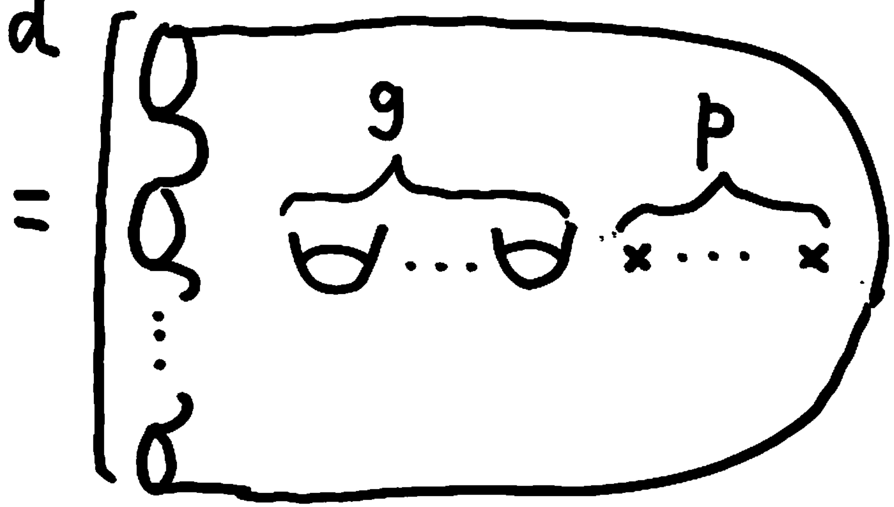
Ordering : Local information of the action of MCG

↓ "local" to "grobal"

Contact structure : Grobal information of the topology and geometry of 3-manifold

§I. Thurston-type ordering

$$\Sigma = \Sigma_{g,d,p} =$$



$$(d \geqq 1,\ \chi(\Sigma) < 0)$$

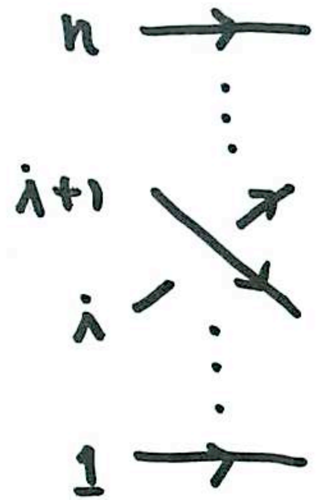
Mapping class group

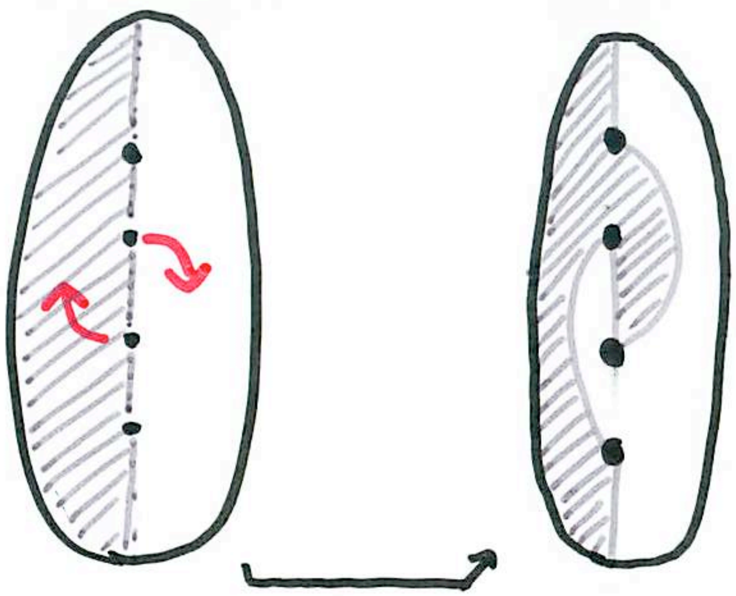
$$MCG(\Sigma, \partial\Sigma) = \left\{ f : \Sigma \xrightarrow{\cong} \Sigma \;\middle|\; f|_{\partial\Sigma} = id \right\} \Big/ \underset{\text{isotopy}}{\sim}$$

ex.) Braid group

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & (|j-i|=1) \\ \sigma_i \sigma_j = \sigma_j \sigma_i & (|j-i|>1) \end{array} \right\rangle$$

$$\cong MCG(\Sigma_{0,1,n} ; \partial\Sigma)$$





(**left**-twist)

$\sigma_i \longleftrightarrow$ Half-twist

Thurston-type ordering ③

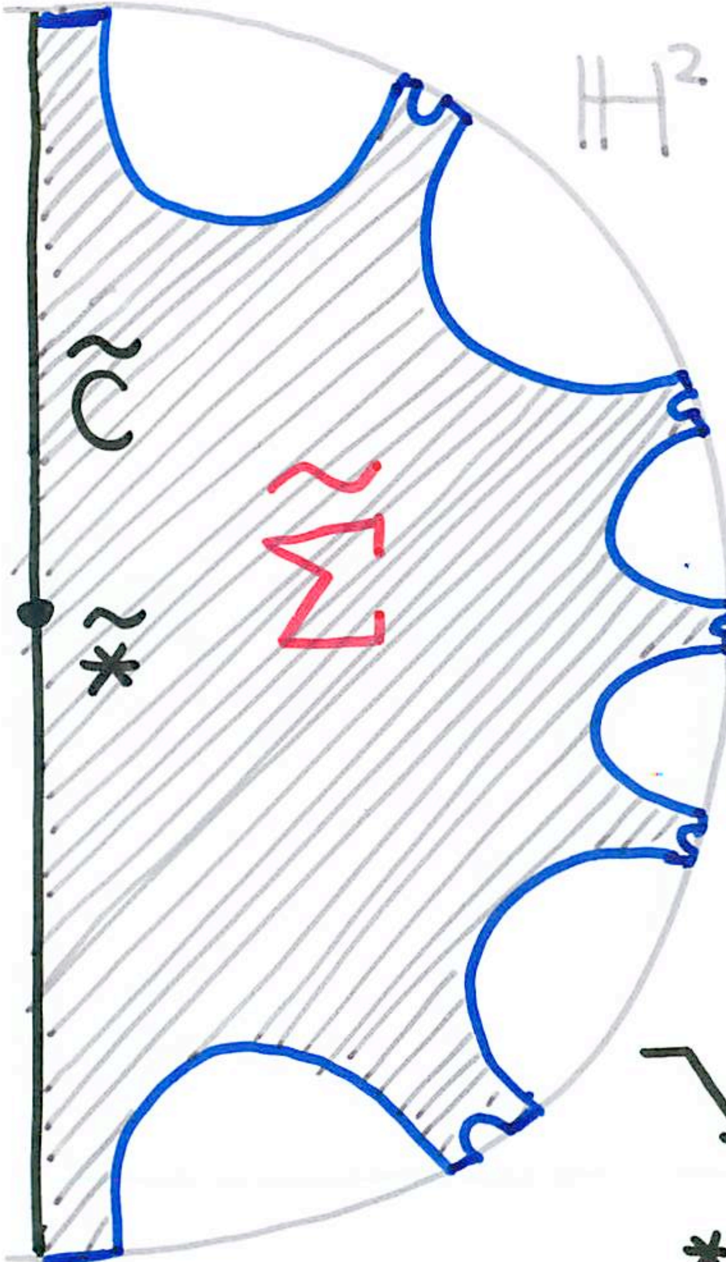
- Regard Σ as a hyperbolic surface.

$$\pi : \widetilde{\Sigma} \longrightarrow \Sigma \quad \text{universal cover}$$

$\widetilde{\Sigma} \hookrightarrow \mathbb{H}^2$ isometrically embedded

$C \subset \partial\Sigma$: connected component , $* \in C$: base point

$\widetilde{C} \subset \pi^{-1}(C) \subset \partial\widetilde{\Sigma}$, $\tilde{*} \in \widetilde{C} \subset \partial\widetilde{\Sigma}$: lift



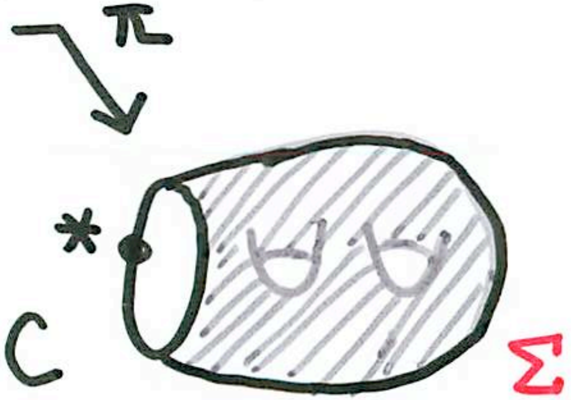
: $\mathbb{R} \cong \partial\tilde{\Sigma} - \tilde{C}$

Nielsen-Thurston map

$$\Theta : MCG(\Sigma, \partial\Sigma) \hookrightarrow \mathrm{Homeo}^+(\mathbb{R})$$

$$[f] \longmapsto \tilde{f}|_{\partial\tilde{\Sigma}-\tilde{C}}$$

(well-defined, injective)



⑤

$x \in \mathbb{R} \cong \partial\tilde{\Sigma} - \tilde{C}$ $\quad \alpha, \beta \in MCG$

$$\alpha <_x \beta$$

$$\overset{\text{def}}{\iff} \left[\Theta(\alpha)\right](x) <_{\mathbb{R}} \left[\Theta(\beta)\right](x)$$

Observation (Thurston)

$<_x$ is left-invariant partial ordering

$$\alpha <_x \beta \Rightarrow \gamma\alpha <_x \gamma\beta$$

- For generic x, $<_x$ is <u>total</u> left-ordering.

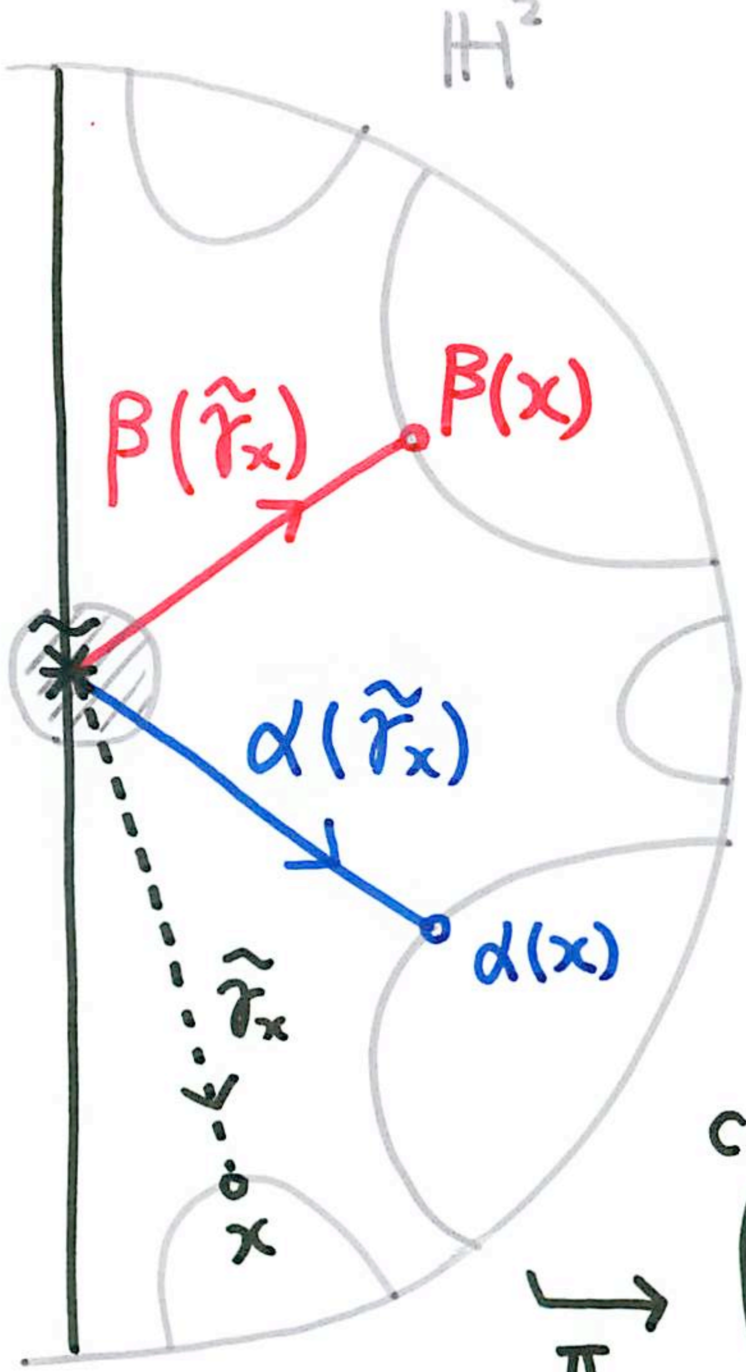
$<_x$ is called

Thurston-type ordering (having base at C)

$\boxed{\text{Fact}}$

Family of orderings $\{<_x\}$ only depends on C

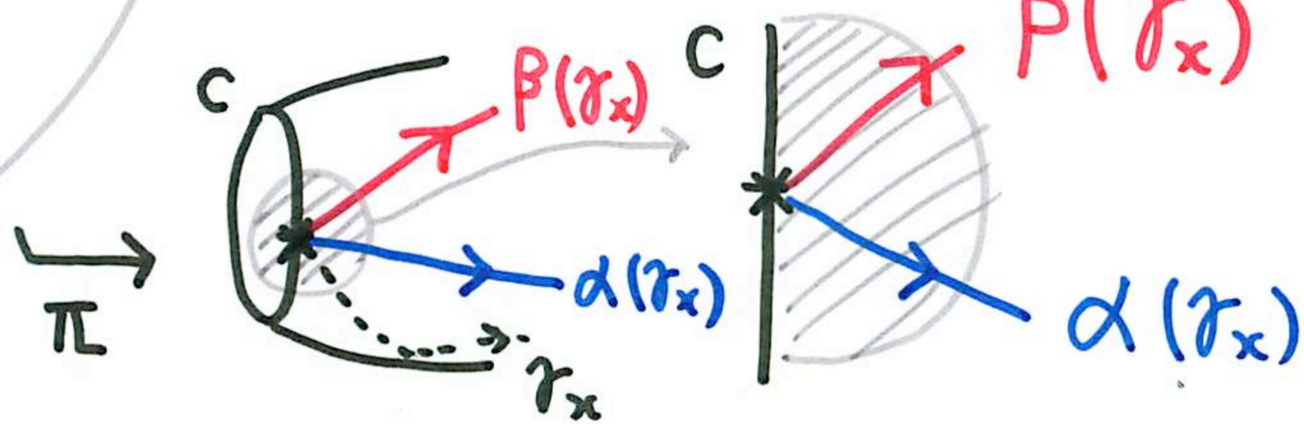
(does not depend on metric, base point etc $\cdots$)



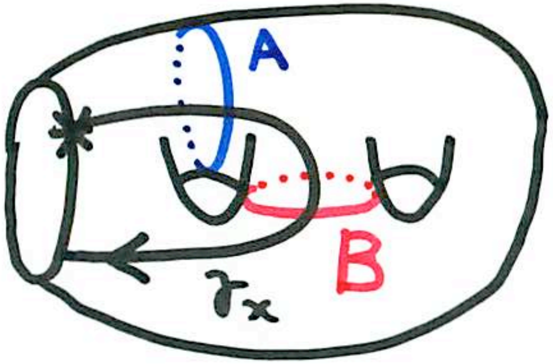
$\alpha <_x \beta$

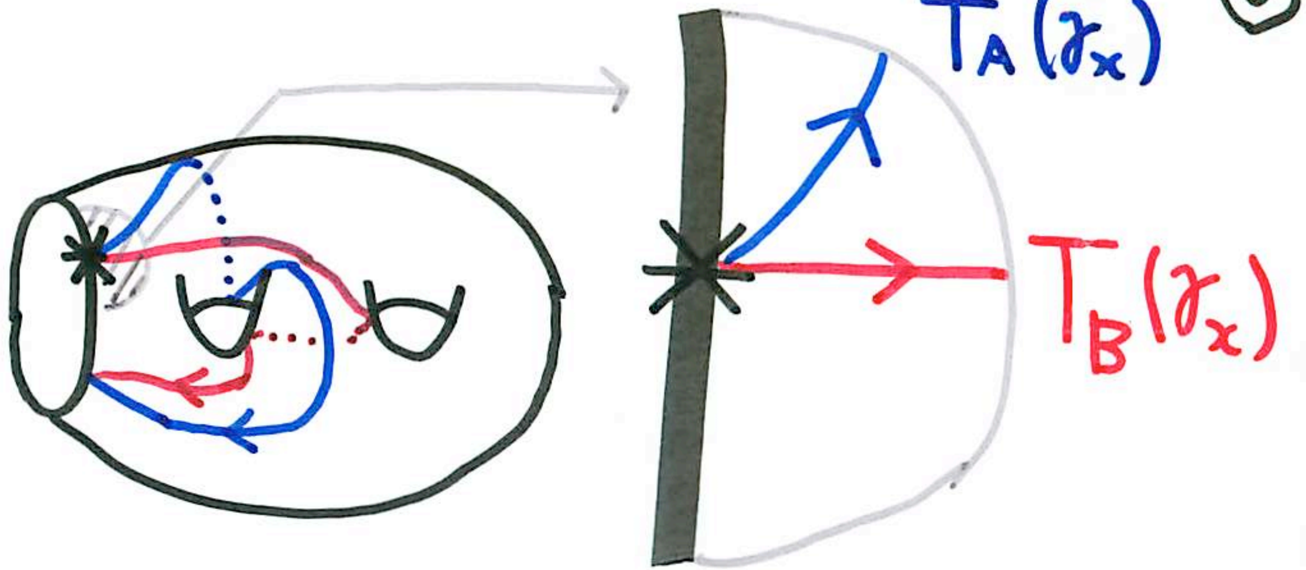
$\iff$ geodesic $\beta(\gamma_x)$ moves **left side** of $\alpha(\gamma_x)$

(ordering is very "local")



example





T_A, T_B : Dehn twist along A, B

↳ $T_B <_x T_A$

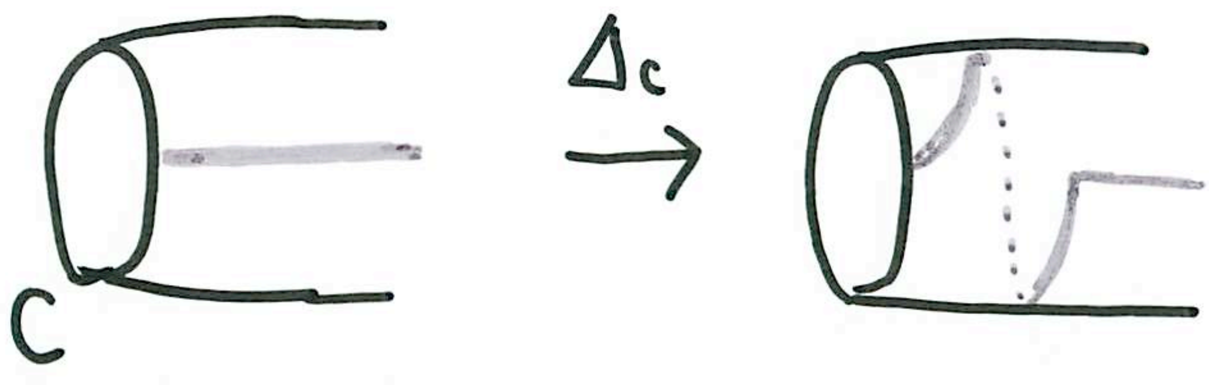
Facts

① Thurston-type orderings are "easy" to compute (Computed in polynomial time)

② $\exists$ Infinitely many (uncountably many !) "essentially different" Thurston-type orderings.

§II. Ordering floor

- $C \subset \partial\Sigma$: boundary component of Σ
- $<_x$: Thurston-type ordering of MCG having base at C
- Δ_C : (left) Dehn twist along C



Def

Ordering floor of $\alpha \in MCG(\Sigma, \partial\Sigma)$

$[\alpha]_x = N \in \mathbb{Z}$ if $\Delta_c^N \leq_x \alpha <_x \Delta_c^{N+1}$

Lemma

- $\left| [\alpha\beta]_x - [\alpha]_x - [\beta]_x \right| \leqq 1$
- $\left| [\alpha\beta\alpha^{-1}]_x - [\beta]_x \right| \leqq 1$

Def

Stable ordering floor

$$S_c(\alpha) \overset{\text{def}}{=} \lim_{N\to\infty} \frac{[\alpha^N]_x}{N} \in \mathbb{R}$$

Lemma.

- S_c does not depend on a choice of ordering $<_x$.
- $|S_c(\alpha\beta) - S_c(\alpha) - S_c(\beta)| \leqq 1$
- $S_c(\alpha^{-1}\beta\alpha) = S_c(\beta)$, $S_c(\alpha^N) = N S_c(\alpha)$

Theorem A (I.)

Let $N = N_\Sigma = \begin{cases} 4g+2 & (p=0,\ d=1) \\ 4g+2d+2p-4 & (\text{otherwise}) \end{cases}$

Then,

(i) $S_c(\alpha) \in \{ p/q \in \mathbb{Q} \mid |q| \leq N \} \ (\overset{\text{def}}{=} \mathbb{Q}_N)$

for all $\alpha \in MCG(\Sigma, \partial\Sigma)$

(ii) $S_c(\alpha) = \left\{ r \in \mathbb{Q}_N \,\middle|\, \left| \dfrac{[\alpha^{N(N+1)}]_x}{N(N+1)} - r \right| \text{ is minimal} \right\}$

↳ $S_c(\alpha)$ is "easy" to compute.

$\boxed{\text{example}}$

$$B_3 = MCG(\Sigma_{0,1,3}, \partial\Sigma), \quad \Delta_c = (\sigma_1\sigma_2\sigma_1)^2$$

- $S_c(\sigma_1\sigma_2) = \frac{1}{3}$, $S_c(\sigma_1\sigma_2\sigma_1) = \frac{1}{2}$
- $S_c(\sigma_2^k) = 0 \quad {}^\forall k \in \mathbb{Z}$
- $S_c(\Delta_c \sigma_2^k) = 1$

$(\because)$
$$\left| [(\Delta_c\sigma_2^k)^N]_x - \underbrace{[\Delta_c^N]_x}_{= N} - \underbrace{[\sigma_2^{Nk}]_x}_{= 0} \right| \leq 1$$

$$N \to \infty \quad \Rightarrow \quad S_c(\Delta_c\sigma_2^k) = 1.$$

§III Openbook and contact structure of 3-manifolds

M : closed, oriented 3-manifold

$\xi \subset TM$: oriented plane field

ξ is contact structure on M

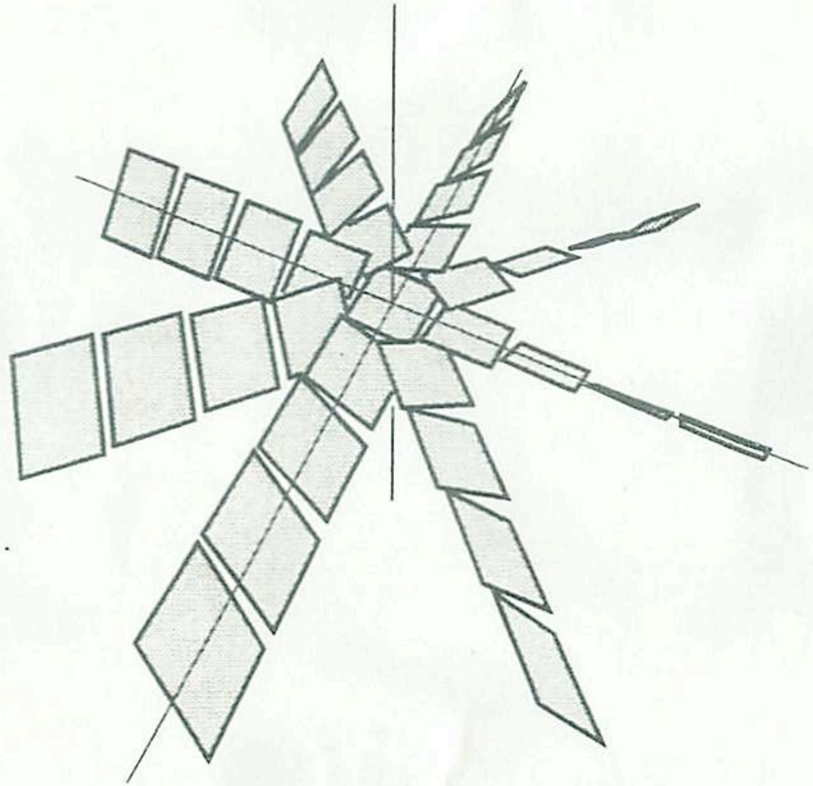
$\overset{\text{def}}{\Longleftrightarrow}$ $\xi = \mathrm{Ker}\,\omega$ $\quad \omega$: 1-form on M

s.t $\omega \wedge d\omega > 0$

Ex.

$$\xi_{std} = \ker(dz + r^2 d\theta)$$

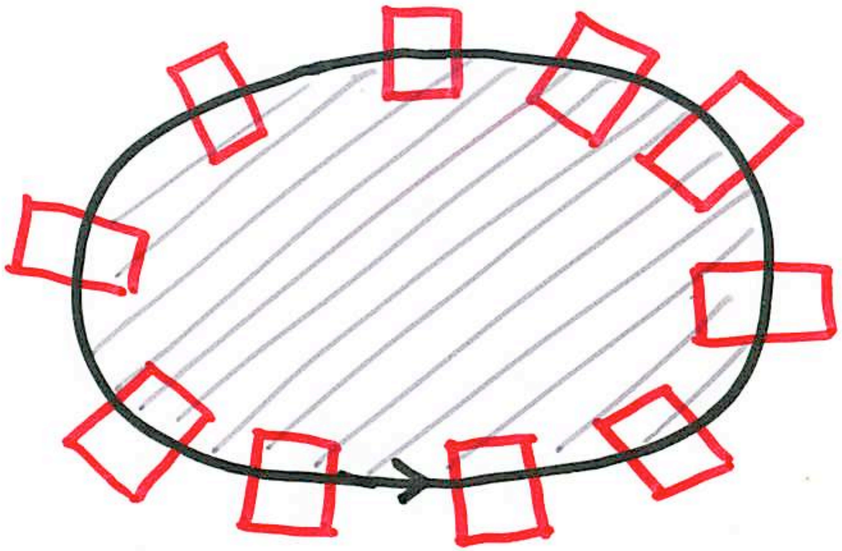
standard contact structure



Tight vs. Overtwisted

$D^2 \hookrightarrow (M, \xi)$ is overtwisted Disc

$\Leftrightarrow T_p D^2 \subset \xi_p \quad {}^\forall p \in \partial D^2$



ex.) (S^3, ξ_{std}) contains

no overtwisted Disc.

Def

Contact structure ξ is

Overtwisted $\overset{\text{def}}{\iff}$ $\exists$ overtwisted Disc

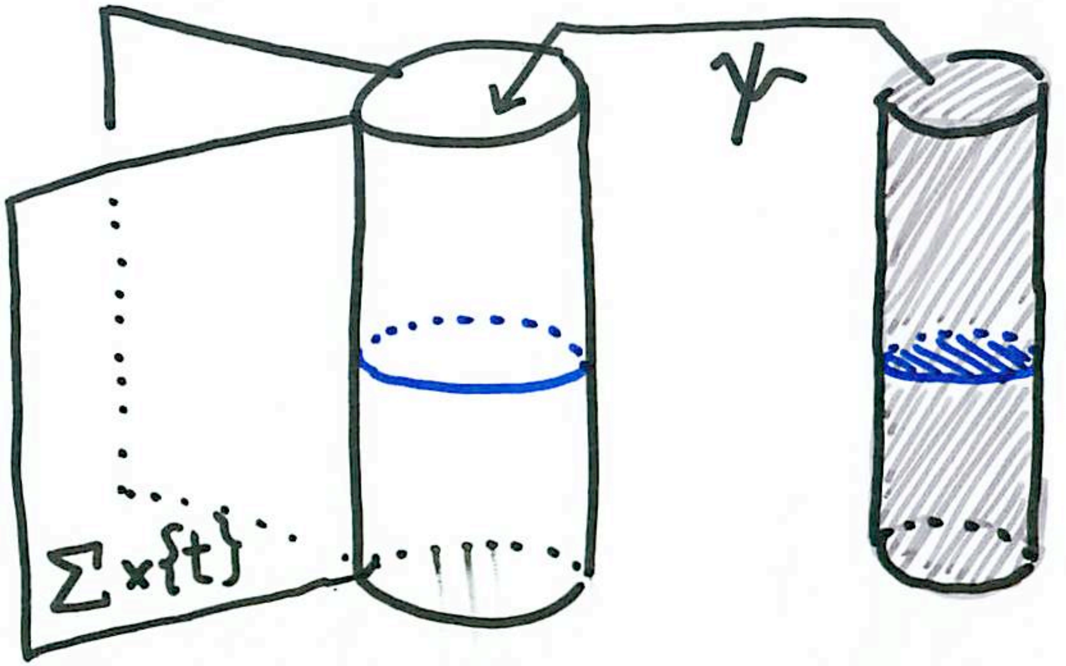
Tight $\overset{\text{def}}{\iff}$ $\nexists$ overtwisted Disc

- Classification of overtwisted contact structure is known (Eliashberg)

Q. How to know ξ is tight ?

$$\Sigma = \Sigma_{g,d,0} \qquad \alpha \in MCG(\Sigma, \partial\Sigma)$$

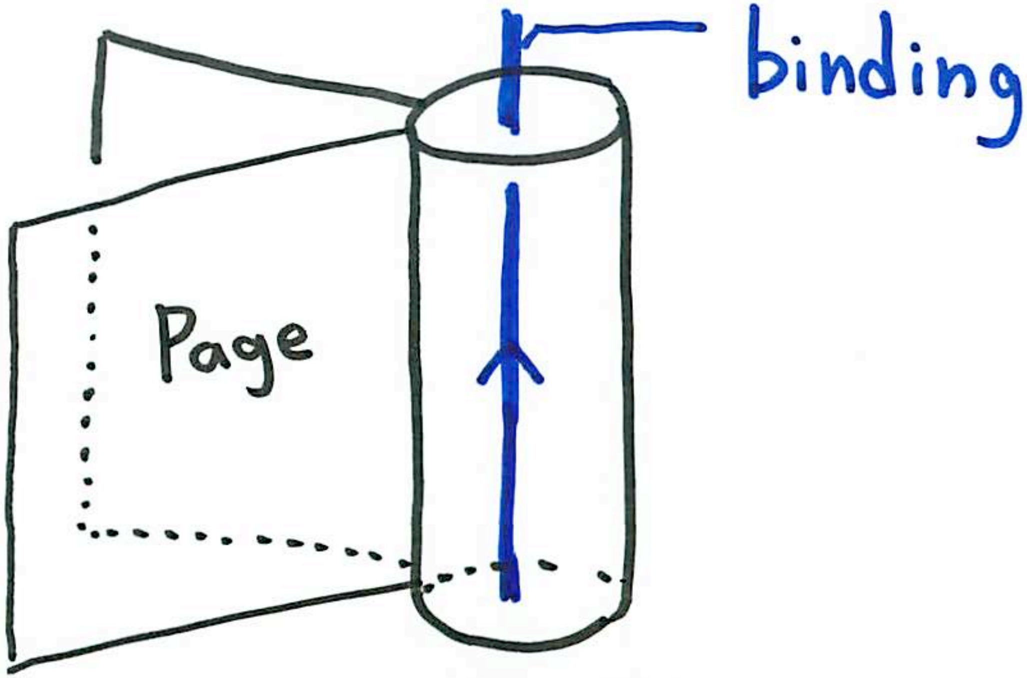
$$M_\alpha = \underbrace{\left[\Sigma \times [0,1] \Big/ (x,0) \sim (\alpha(x),1) \right]}_{\text{Mapping Torus}} \cup_\psi \underbrace{\left[\coprod_d S^1 \times D^2 \right]}_{\text{Solid Tori}}$$



$$\begin{array}{ccc} \{p\} \times S^1 & \overset{\psi}{\cong} & \{q\} \times \partial D^2 \\ \cap & & \cap \\ \partial\Sigma & & S^1 \end{array}$$

(Σ, α) is called **openbook decomposition** of 3-Manifold $M = M_\alpha$

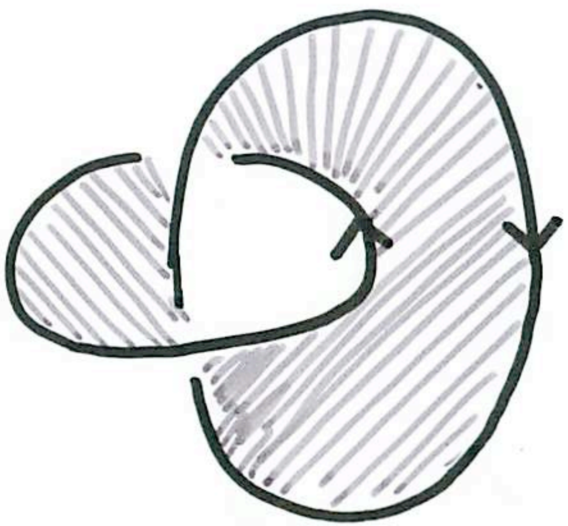
- Binding $=$ core of solid tori $\subset M$
- Page $= \Sigma \times \{t\} \subset M$



(openbook $\doteqdot$ fibered link)

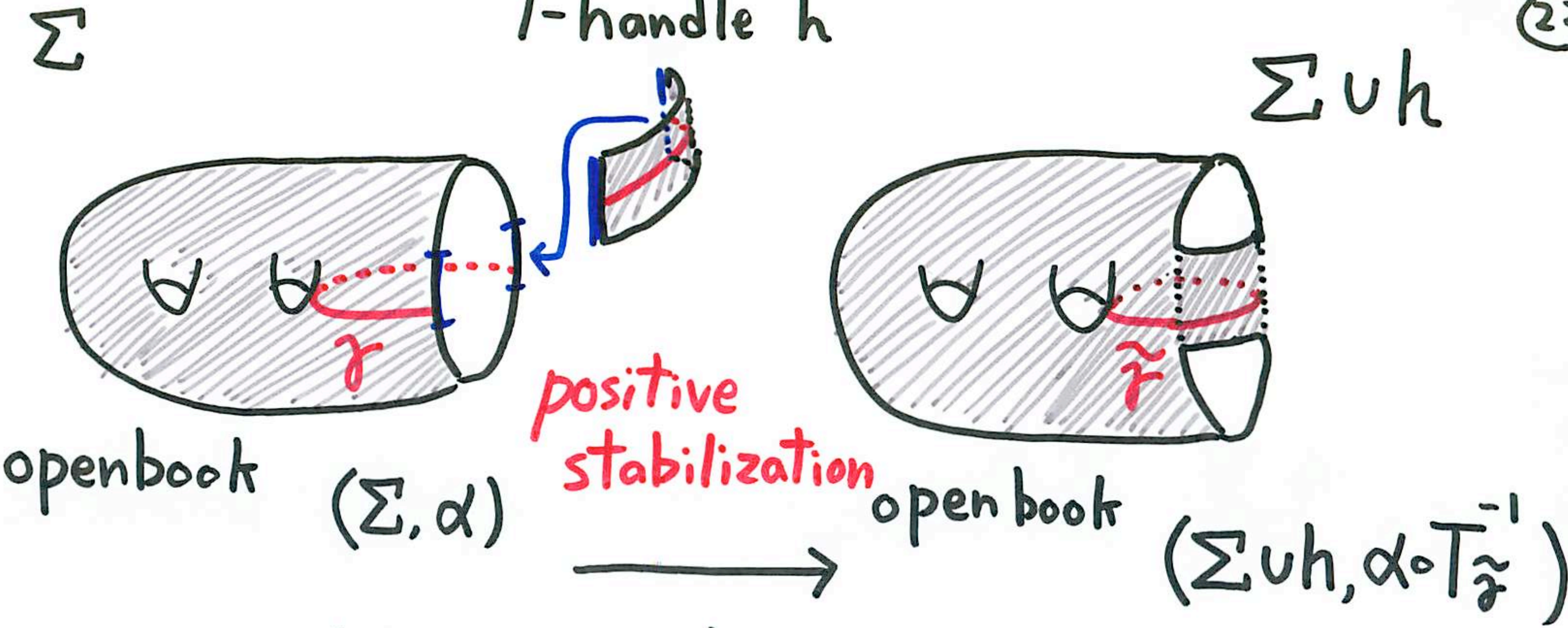
ex.)

α : Dehn-twist along the core of an annulus A



- $(A, \alpha^{\pm 1})$ is openbook of S^3

($\alpha^{\pm 1}$ is monodromy of pos/neg Hopf link.)



(Σ, α), $(\Sigma \cup h, \alpha \circ T_{\tilde{\gamma}}^{-1})$ define open book decomposition of the same 3-manifold

($T_{\tilde{\gamma}}^{-1}$: Right-handed Dehn twist)

Giroux Correspondence

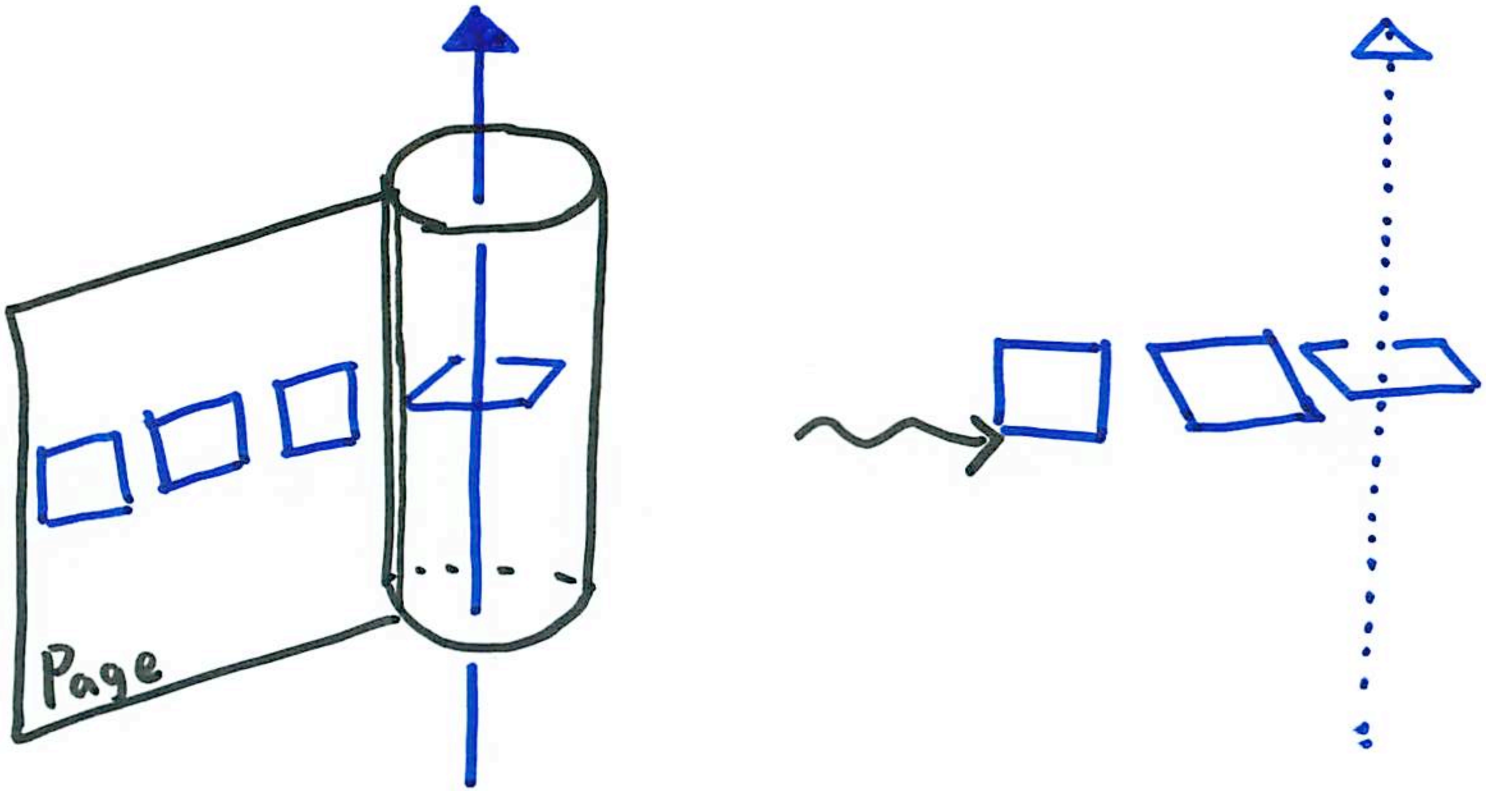
Theorem (Giroux)

$$\{\text{contact structures on } M\} \Big/ \underset{\text{isotopy}}{\sim}$$

$$\Big\updownarrow \ 1:1$$

$$\{\text{Open book decompositions of } M\} \Big/ \underset{\text{positive stabilization}}{\sim}$$

(Openbook ⇒ contact structure)



plane field

- tangent to pages
- transverse to binding

→ contact structure

§IV Orderings ⇒ contact structure

Theorem A^* (I)

(Σ, α) : open book

ξ_α : contact structure on M_α given by (Σ, α)

$S_c(\alpha) > 0$ for some $C \subset \partial\Sigma$

$\Rightarrow \xi_\alpha$ is over-twisted

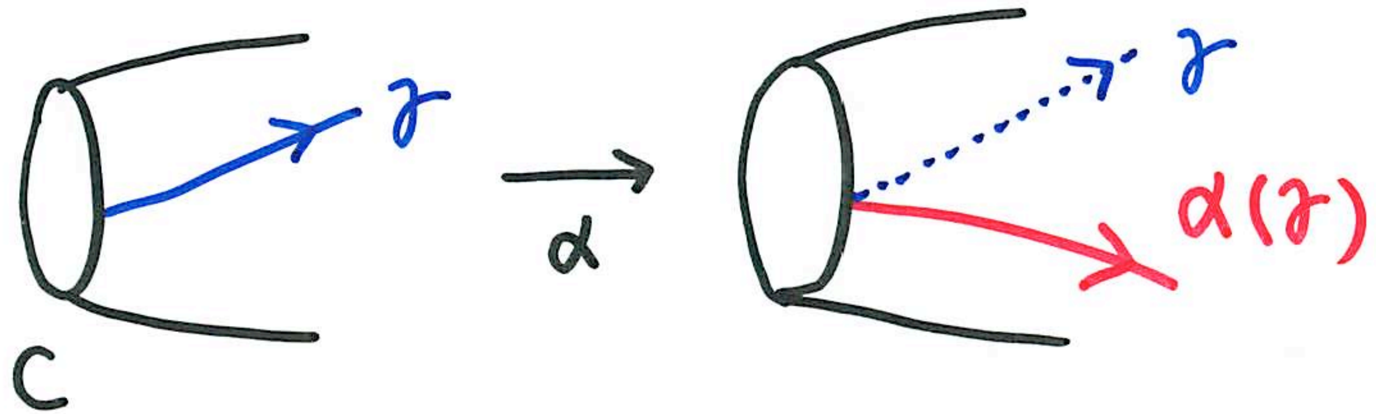
[Proof]

Def

$\alpha \in MCG(\Sigma, \partial\Sigma)$ is right-veering on C

$\Longleftrightarrow$ $\forall$ geodesic γ emanating from C,

$\alpha(\gamma)$ moves right side of γ



$\Leftrightarrow \forall <_x$: Thurston-type ordering having base at C

$$\alpha <_x 1$$

Theorem (Honda-Kazez-Matic)

(Σ, α) defines tight contact structure

$\Rightarrow \alpha$ is right-veering.

Theorem A^{**} (I)

$S_c(\alpha) < 0 \Rightarrow \alpha$ is right-veering

Theorem A^{**} is useful to construct

right-veering map $(\leftarrow\cdots\rightarrow$ tight $)$

which is not a

product of (right) Dehn-twists $(\leftarrow\cdots\rightarrow$ Stein fillable$)$

◦ Toward generalization

Ozsvath–Szabo

- Heegaard Floer homology $\widehat{HF}(M)$
- contact class $c(\xi) \in \widehat{HF}(-M)$

$$\begin{cases} c(\xi) = 0 & \text{if } \xi \text{ is over-twisted} \\ c(\xi) \neq 0 & \text{if } \xi \text{ is Stein fillable} \end{cases}$$

Honda-Kazez-Matic

- give a description of $c(\xi)$ via open-book decomposition
- Showed α is not right-veering $\Rightarrow c(\xi) = 0$

Q. Can we use orderings (or $Sc(\alpha)$) to study $c(\xi)$?

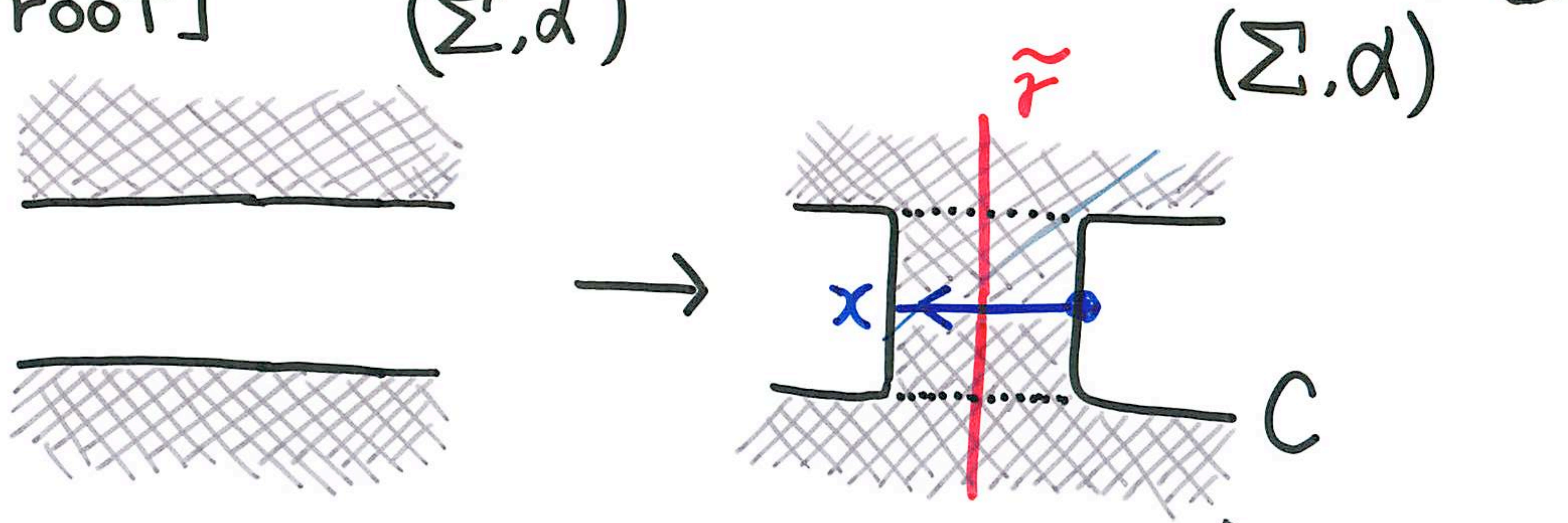
Theorem B

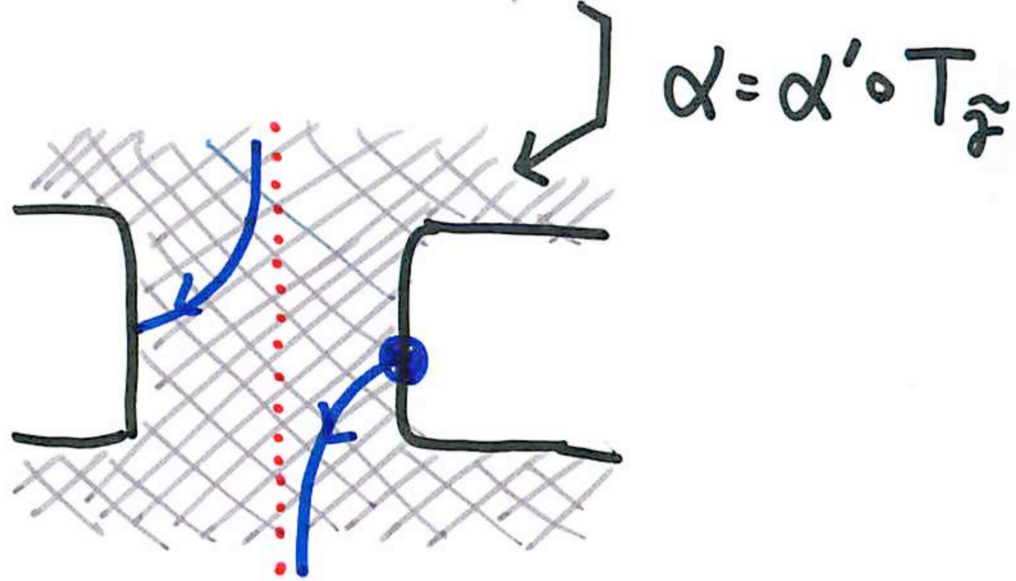
(Σ, α) is positive stabilization of another open book (Σ', α')

$\Rightarrow |S_C(\alpha)| \leqq 1$ for some $C \subset \partial\Sigma$

Moreover, if ξ_α is over-twisted, then $S_C(\alpha) = 0$

[Proof]





$$-1 \leqq [\alpha]_x \leqq 0$$

$L \subset (M, \xi)$ is transverse link

$\Leftrightarrow$ L : link in M (oriented)

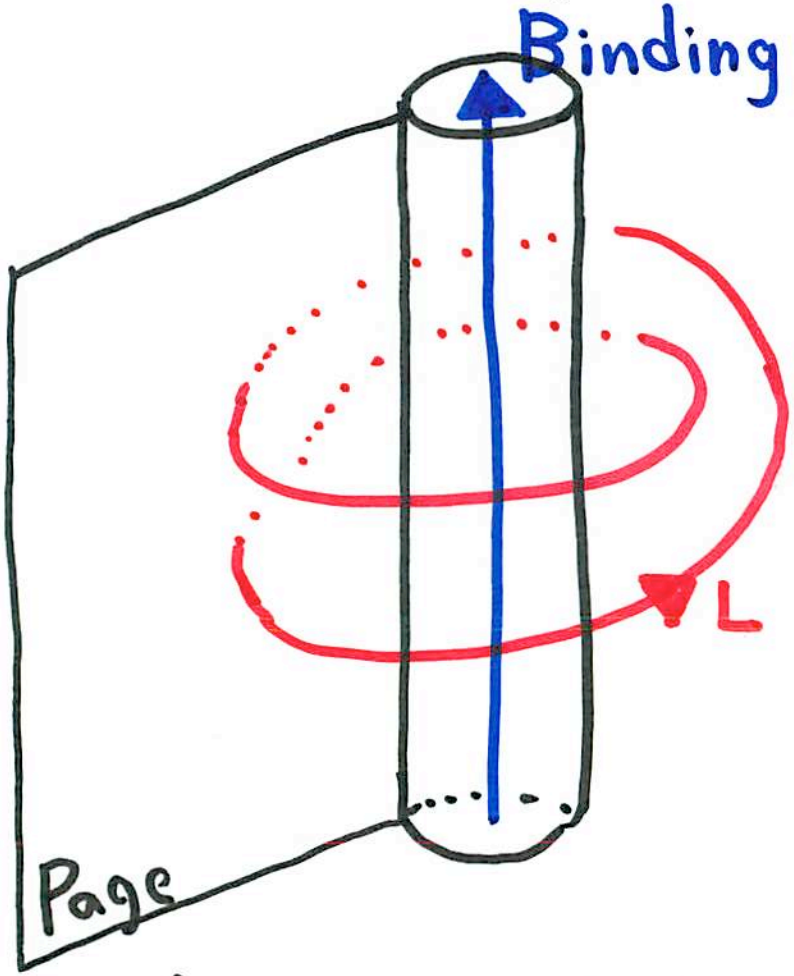
L transverse ξ positively

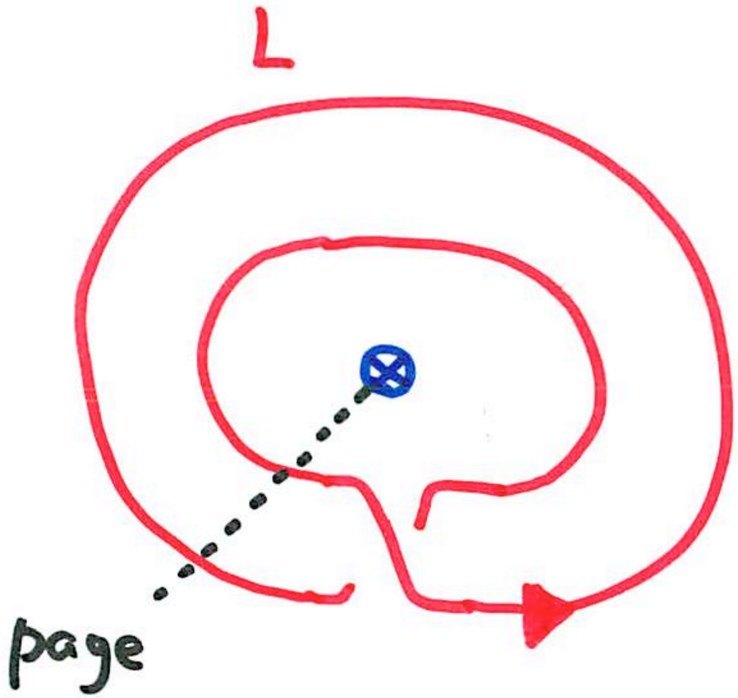
※ contact structure ξ $\leftrightarrow$ plane field $\begin{cases} \text{tangent to page} \\ \text{transverse to binding} \end{cases}$

So

transverse link $\leftrightarrow$ link transverse to page ex.) transverse link in (S^3, ξ_{std})

ξ_{std} is represented by an openbook (D^2, id)



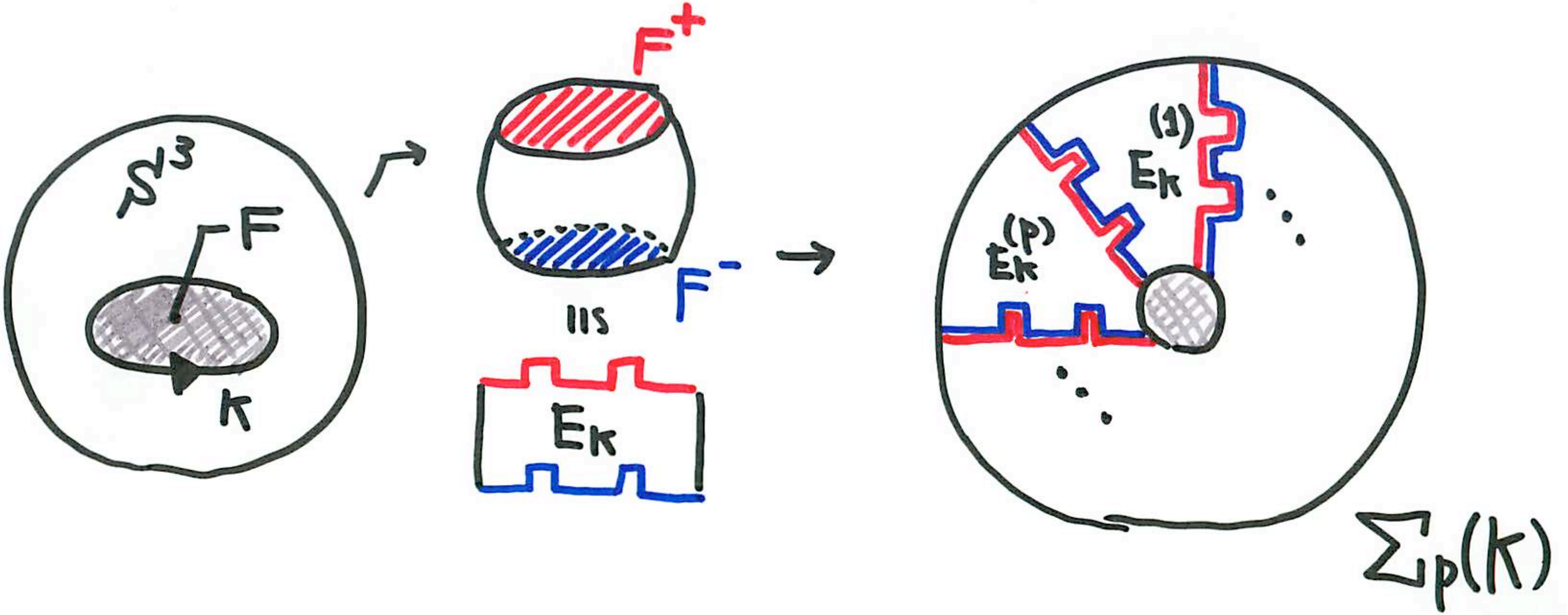


↳ transverse link is reprensented as a closed braid.

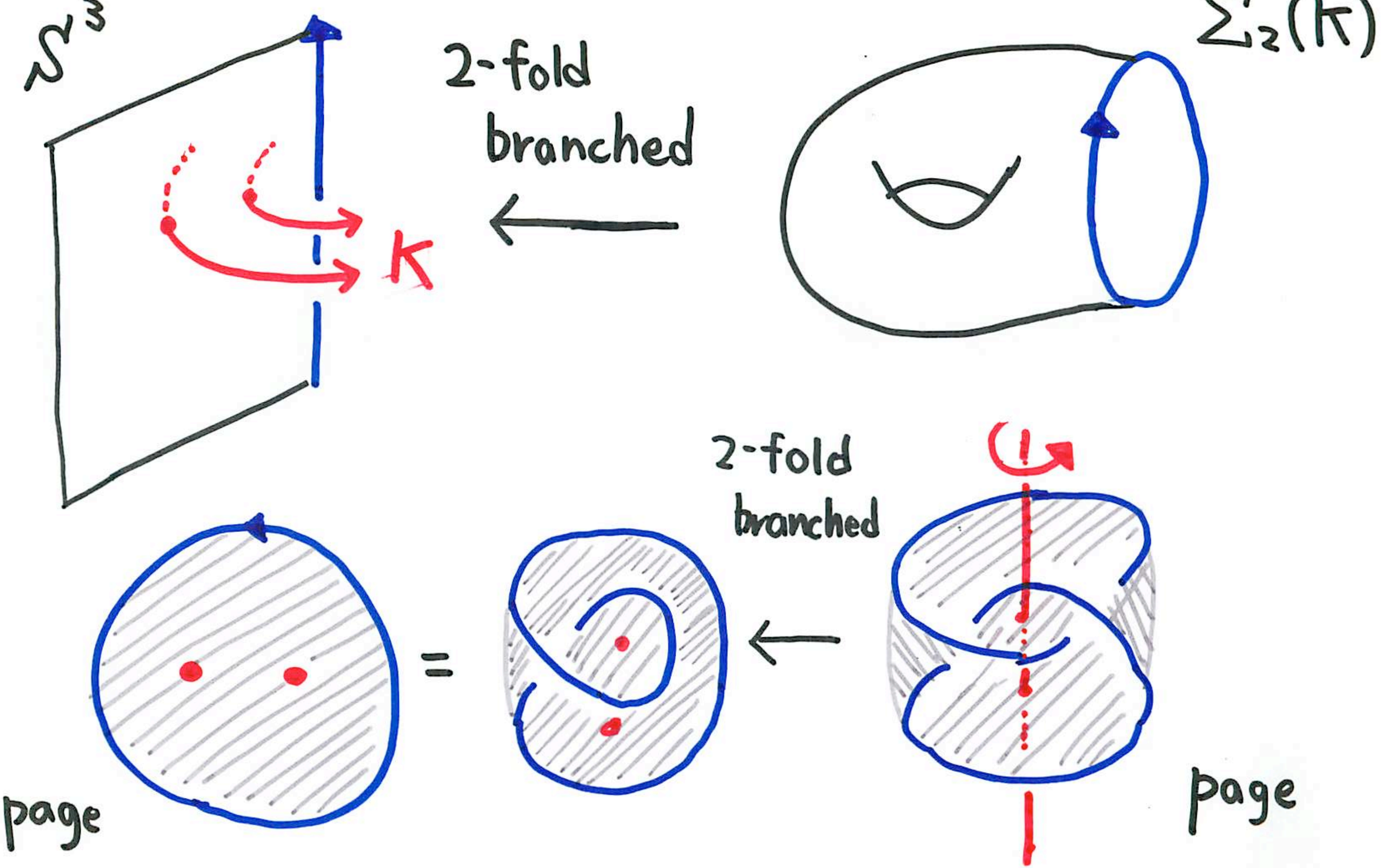
Branched cover

$K \subset S^3$: knot $\quad p \in \mathbb{Z}_{>0}$

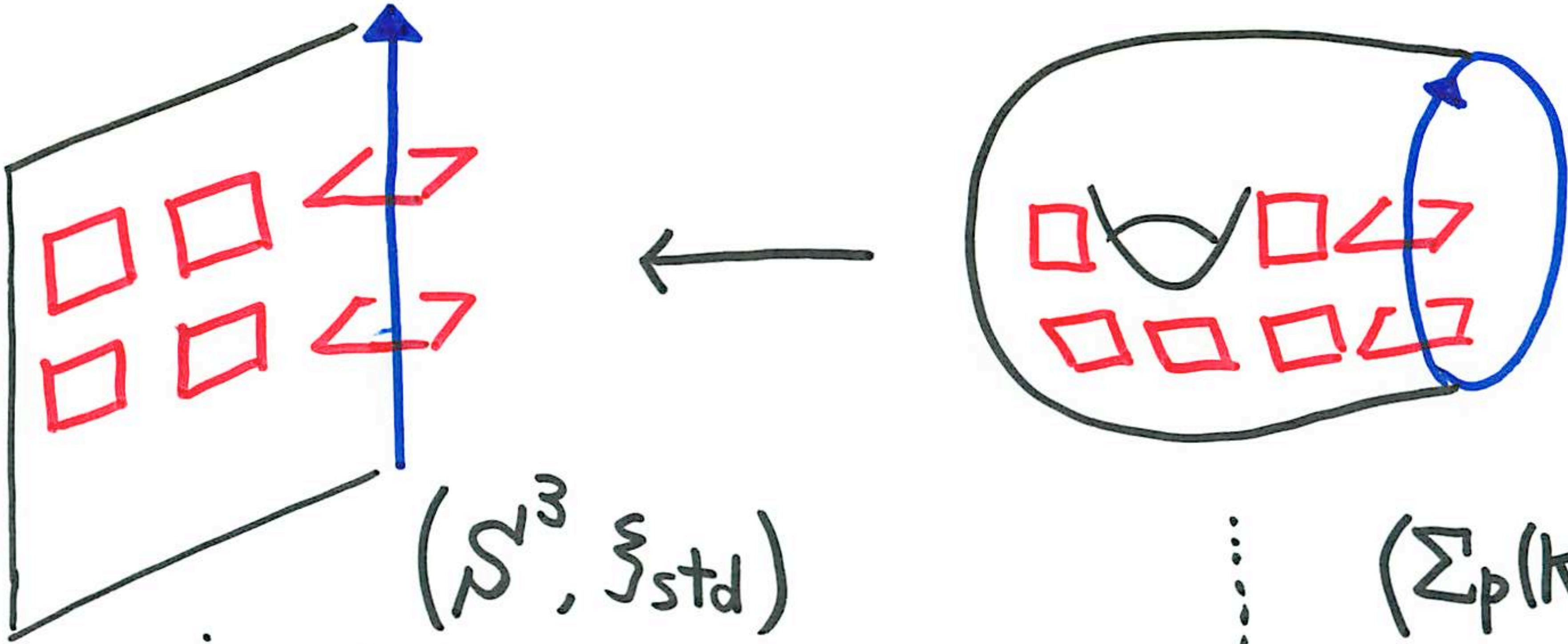
$\Sigma_p(K)$: p-fold cyclic branched cover



Openbook point of view



Contact branched cover



(S^3, ξ_{std})

$(\Sigma_p(K), \xi_p(K))$

perturb

perturb

ξ_{std}

$\xi_p(K)$

Theorem C (I)

$\beta \in B_n = MCG(\Sigma_{0,1,n} ; \partial\Sigma)$, $C = \partial\Sigma$

$K = \widehat{\beta} \subset (S^3, \xi_{std})$: transverse knot

$S_C(\beta) > 0 \Rightarrow \xi_p(K)$ is over-twisted