Modifications of 2-Dimensional Foliations on 4-Manifolds and Tautness

Yoshihiko MITSUMATSU (Chuo Univ., Tokyo)
and
Elmar VOGT (Freie Universität Berlin)

23rd January, 2011, in Kyoto

Aim

- introduce modifications of 2-dimensional foliations on 4-manifolds
- cohomological obstruction to turbulizations
- how to geometrically realize?
- geometric tautness of resultant foliations

Key Behind

- 3D (Geodisic) Anosov flow

PLAN

(*what I would like to talk about*)

§0. Introduction

§1. Turbulization : *a geometric construction*

§2. Turbulization : *a dynamical interpretation*

§3. Cohomology Equations for $\sum_g \times S^2$
& Exotic Solutions

§4. Geometric Realizations of Exotic Solutions

§5. Other Modifications

§6. Sullivan's Criterion on Tautness

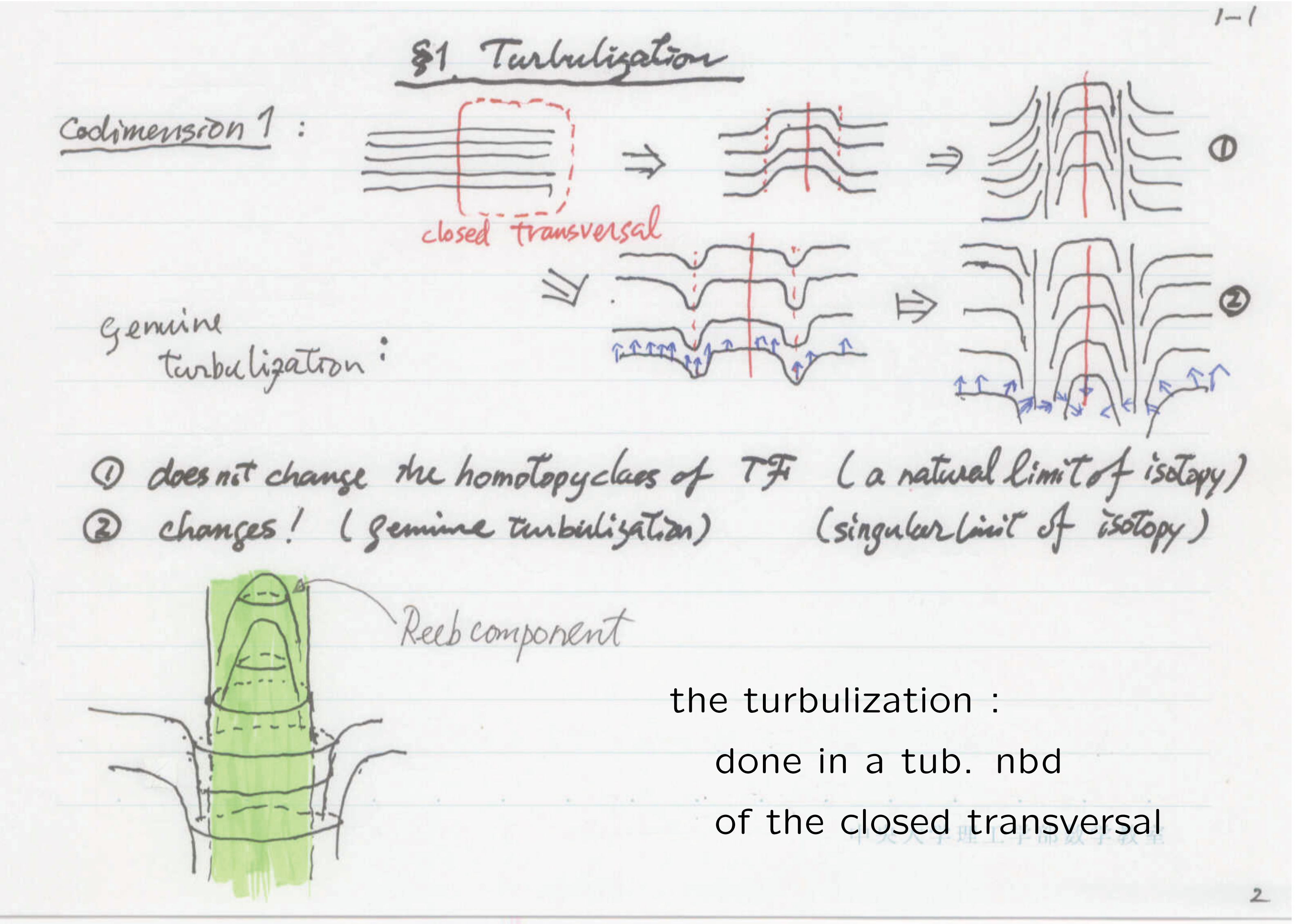
§7. Tautness of Modified Foliations

notations: (everything is oriented and smooth)

$\mathcal{F}$, $\mathcal{G}$, ... : 2-dim foliation (mostly on M^4)
L : a cpt leaf or an embedded surface in M
$\tau\mathcal{F}$, $\nu\mathcal{F}$: tangent, normal bundle to $\mathcal{F}$
$\sum_g$: closed oriented surface of genus g
$e(\cdot)$: the euler class

§0. Introduction

0-1. Turbulization in Codimension 1 (review)



0-2. Turbulization in Higher Codimension

a formulation:

turbulization := a modification of a foiation $\mathcal{F}$ in a tub. nbd of a closed transversal $L \pitchfork\!\uparrow \mathcal{F}$ s.t. $L \pitchfork\!\downarrow \mathcal{F}^{\mathrm{new}}$

0-3. Impossibility of Turbulization

- codimension ≥ 2, rarely $\exists$ closed transversal

- foliated $\sum_g$-bundle $\Rightarrow$ Not Turbulizable ($g \neq 1$)

$\cdots\cdots$ why?

L_1, L_2 : neaby fibres $\pitchfork \mathcal{F}$

i.e., ... , $\nu L_i = \tau\mathcal{F}|_{L_i}$: trivial!

$$\therefore \chi(\textstyle\sum_g) = \langle e(\nu\mathcal{F}), [L_2]\rangle = \langle e(\nu\mathcal{F}^{\text{new}}), [L_2]\rangle$$
$$= \langle e(\nu\mathcal{F}^{\text{new}}), [L_1]\rangle = \langle -e(\nu\mathcal{F}), [L_1]\rangle = -\chi(\textstyle\sum_g)$$

0-4. When can we turbulize?

For $L \cong \sum_g \pitchfork \mathcal{F}^2$ on M^4 , ($g \geq 2$),

Answer: Turbulization is possible

$$\Leftrightarrow$$
$$[L]^2 \mid \chi(\textstyle\sum_g)(= 2 - 2g)$$
$$\Leftrightarrow$$

Turbulization is geometrically realizable

Method: 1st $\Leftrightarrow$: by h-principle

2nd $\Leftrightarrow$: $\exists$ (geodesic) Anosov flow,

the principal aim of the talk

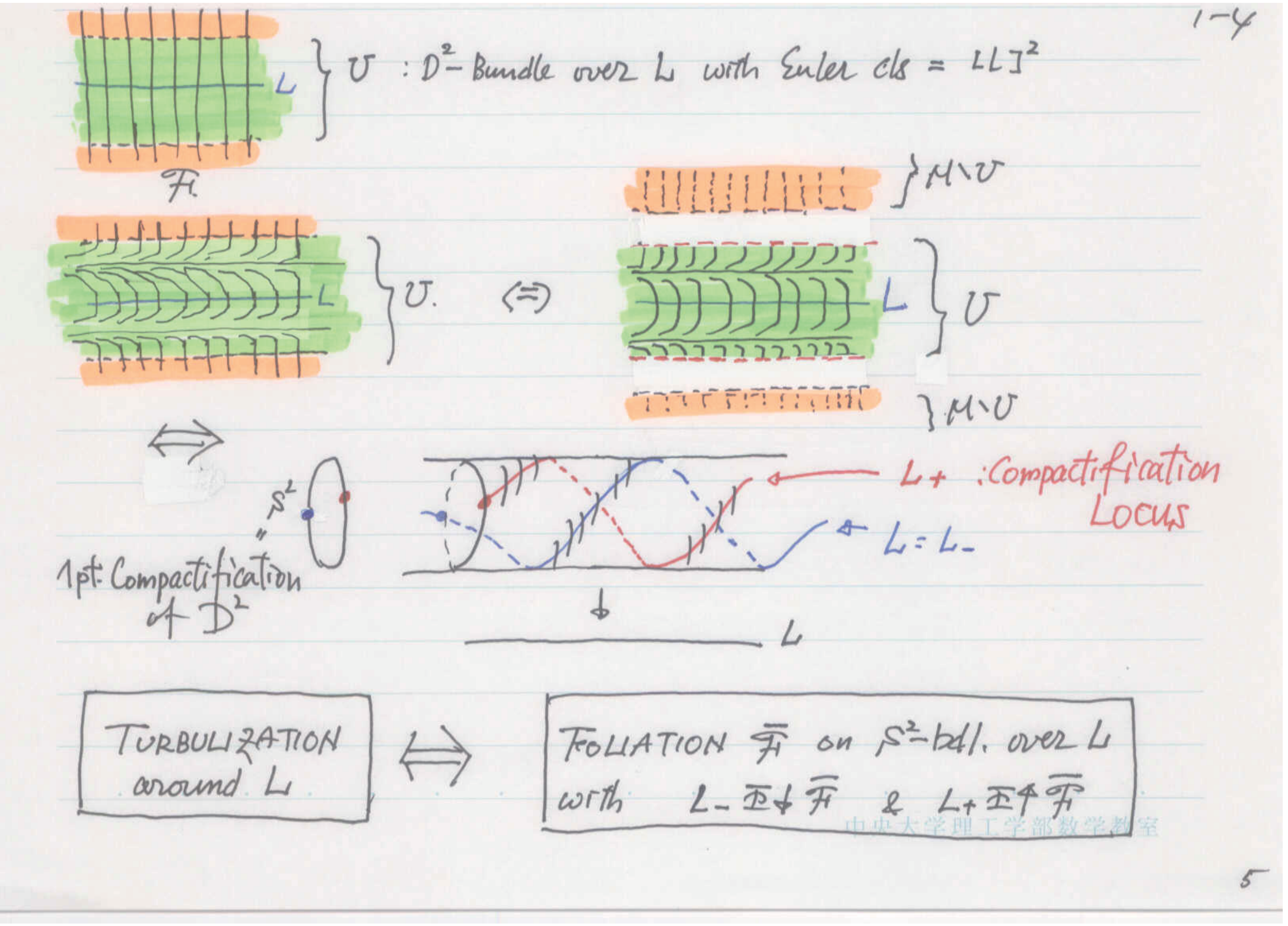
§1. Turbulization
– a geometric construction –

1.1 Reduction to $\sum_g \times S^2$

- the problem reduces from $L \pitchfork\!\uparrow \mathcal{F}$ to
  - bdl-fol $\mathcal{F}_b$ on $\sum_g \times S^2$ modified into $\overline{\mathcal{F}}$
  - sections $L_\pm \pitchfork\!\uparrow \mathcal{F}_b \Rightarrow L_+ \pitchfork\!\uparrow \overline{\mathcal{F}}$ & $L_- \pitchfork\!\downarrow \overline{\mathcal{F}}$

where

$$[L_\pm] = (1, b) \in H_2(\textstyle\sum_g \times S^2; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z},$$
$$[L_\pm]^2 = \pm 2b = \pm [L]^2$$

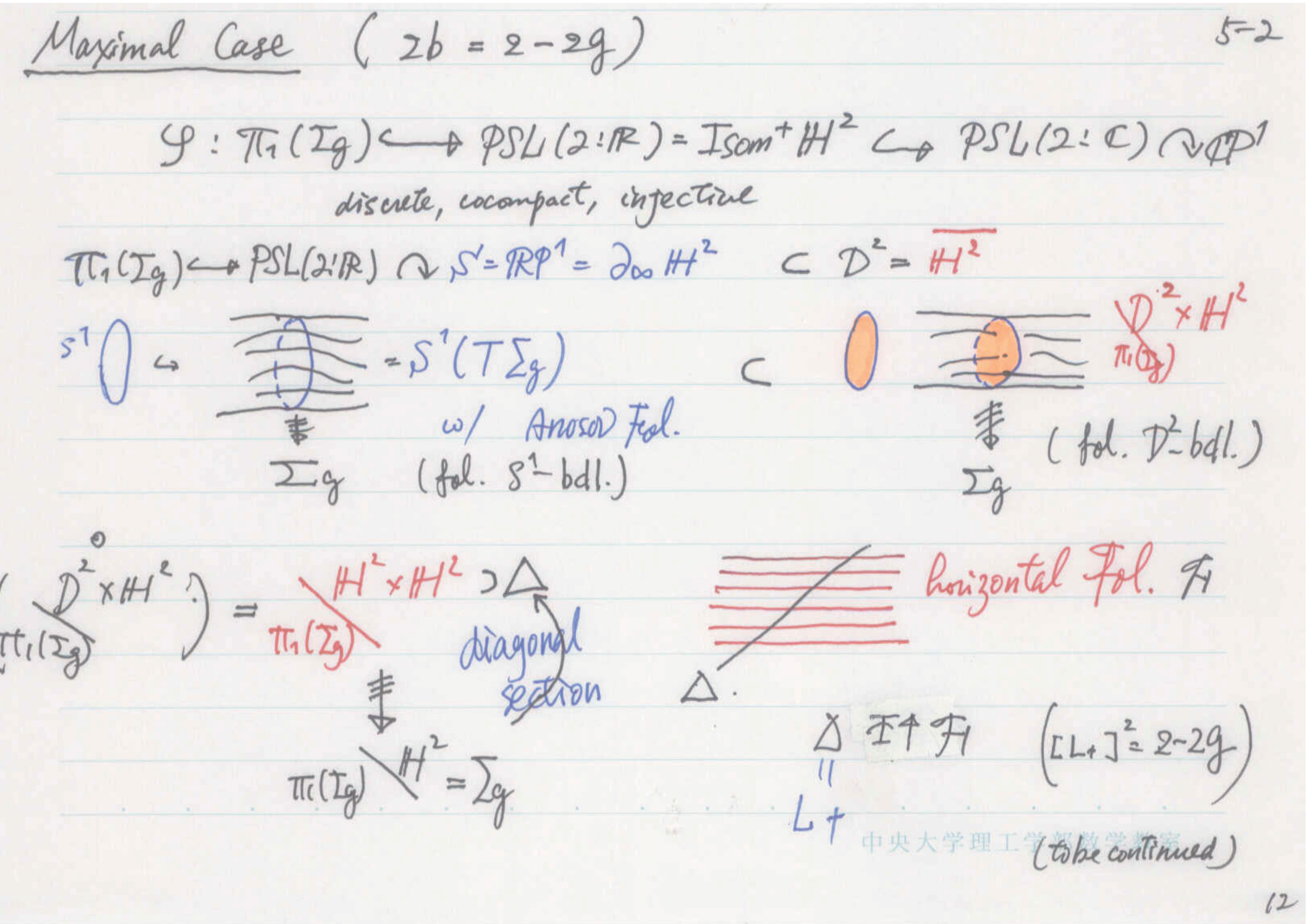


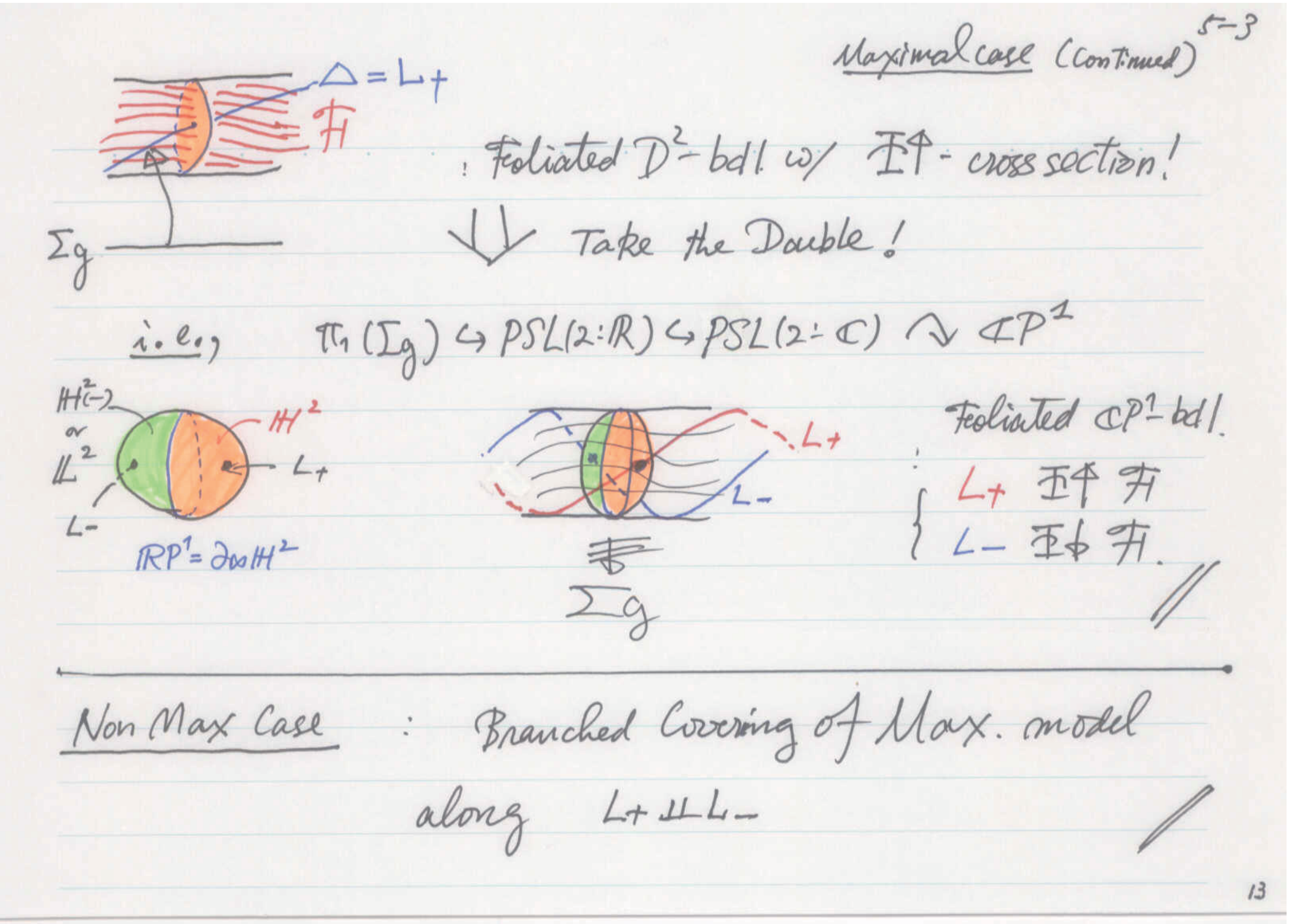
1-2. Maximal Case : A Well-Know Foliation

The case : $[L]^2 = \pm\chi(L) = \pm(2-2g)$
i.e., $\nu L \cong TL$

$$\varphi : \Gamma = \pi_1(L) \hookrightarrow Isom^{+}\mathbb{H} = PSL(2;\mathbb{R})$$
$$\subset PSL(2;\mathbb{C}) \text{ acting on } \mathbb{C}P^1$$

If you start from bdl-fol., a bit difficult to imagine how to modify it ... waite unitl next section





- $[L]^2 =$ odd $\Rightarrow$ take a doube cover

- the Julia (limit) set = fin. covering of the unstable foliation of the geodesic Anosov flow

§2. Turbulization
– a dynamical interpretation –

In *turbulization* we must have a *turbulance*

Interprete the previous construction
- as a singular limit of an isotopy 'Anosov flow'
- starting from a bundle foliation $\mathcal{F}_b$ on $\sum_g \times S^2$

2.1. Regard the product bundle $\sum_g \times S^2$ over $\sum_g$ as $\mathbb{C}P^1$-bundle associated with $\Gamma = \pi_1(\sum_g) \hookrightarrow PSL(2;\mathbb{R}) \subset PSL(2;\mathbb{C})$

2.2. Decompose $\mathbb{C}P^1$-bdl into $\mathbb{H}$-, $\mathbb{R}P^1$-, & $\mathbb{L}$-bdl according to $\mathbb{C}P^1 = \mathbb{L} \cup \mathbb{R}P^1 \cup \mathbb{H}$

2.3. $\mathbb{R}P^1$-bundle carries the geodesic Anosov flow

2.4. $\mathbb{H}$[*resp.* $\mathbb{L}$]-bundle $\supset$ cross sectons $L_\pm$

$\mathbb{H}$-bundle $\cong T\sum_g \quad \supset \quad L_+ \cong$ 0-section
$= \bigcup_{0 \le r} S^1$-bdl of radius r

$\phi(t)$: the geodesic flow on $T\sum_g$
the Anosov flow $\varphi(t) = \phi(t)|_{r=1}$, $\phi(t) = \varphi(rt)$

2.5. Take a flow $\quad \Phi(t) = \phi(f(r^{-1} \cdot t)) \quad$ on $T\sum_g$
s.t. $\quad f(r) = r \ (r \le \frac{1}{2}), \quad f'(r) > 0 \ (r < 2)$
$\quad f(r) \equiv 1 \ (2 \le r)$

The same construction on $\mathbb{L}$-bdl

Proposition 2.6. On $T\sum_g \cong \mathbb{H}$ -bundle

$$\lim_{t\to\infty} \Phi(t)_* \mathcal{F}_b = \overline{\mathcal{F}}$$

($\overline{\mathcal{F}}$: the suspension of $\pi_1(\sum_g) \hookrightarrow PSL(2;\mathbb{C})$)

Proof On $\mathbb{H}$-bundle $\backslash L_+$

$$TM = \langle \frac{\partial}{\partial r} \rangle \oplus T[S^1(r)\text{bundle}] = \langle Y, S, U \rangle$$

$$T\mathcal{F}_b = \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \rangle, \quad \frac{\partial}{\partial \theta} = U \pm S.$$

By the geodesic flow $\phi(t)$,

$$\phi(t)_U = e^t U, \quad \phi(t)_* S = e^{-t} S$$

so that we have

$$\lim_{t\to\infty} \phi(t)_* \langle \frac{\partial}{\partial \theta} \rangle = \langle U \rangle.$$

Also from $\lim_{r\to 1-0} f'(r) = +\infty$ we see

$$\lim_{r\to 1-0} \lim_{t\to\infty} \langle \Phi^r(t)_* \frac{\partial}{\partial r} \rangle = \langle Y \rangle.$$

In total,

$$\lim_{r\to 1-0} \lim_{t\to\infty} \Phi^r(t)_* T\mathcal{F}_b = \langle Y, U \rangle = T\mathcal{F}^u.$$

§3. Cohomolgy Equations on $\sum_g \times S^2$

Come back to the original motivation for us.

3.1 <u>Basic question on the existence of a foliation</u>:

For given $L \subset M^4$,
$\exists$? a foliation $\mathcal{F}$ on M
s.t. L is a compact leaf [*resp.* $L \pitchfork \mathcal{F}$]

Combined with cohomological data e_1, $e_2 \in H^2(M;\mathbb{Z})$
for $e(\tau\mathcal{F})$ and $e(\nu\mathcal{F})$
$+$ homological data of $[L] \in H_2(M;\mathbb{Z})$

Thurston's h-principle,
Mather-Thurston's theory on $B\overline{\Gamma}_q$,
Dold-Whitney, Pontrjagin,
Milnor's inequality, and $+\alpha$
enables us to reduce the problem to
solving a system of (co-)homology equations for
e_1 and e_2.

Theorem(M.-Vogt) (a ver. for *cpt leaf*)

$\exists\, \mathcal{F}$ on M *s.t.* $L \in \mathcal{F}$, $e_1 = e(\tau\mathcal{F})$, & $e_2 = e(\nu\mathcal{F})$

$\Leftrightarrow$

(0) Milnor's ineq. $|[L]^2| \leq g - 1$
(1) $e_1^2 + e_2^2 = p_1(M) \in H^2(M;\mathbb{Z})$,
(2) $\langle e_1 \cup e_2, [M]\rangle = \chi(M)$,
(3) $e_1 + e_2 \equiv w_2(M) \in H^2(M;\mathbb{Z}/2)$,
(4) $\langle e_1, [L]\rangle = \chi(L)$,
(5) $\langle e_2, [L]\rangle = [L]^2$.

Theorem(M.-Vogt) (a version for *transversal*)

$\exists\, \mathcal{F}$ on M *s.t.* $L \pitchfork\!\uparrow \mathcal{F}$, $e_1 = e(\tau\mathcal{F})$, & $e_2 = e(\nu\mathcal{F})$

$\Leftrightarrow$

(1) - (5) with e_1 and e_2 exchanged,
(forget (0):Milnor's Ineq.)

3.2 Coh-Eq. for $\sum_g \times S^2 \supset$ section L

$L \subset \Sigma_g \times S^2 = M^4$

$\quad\downarrow$ section

$\quad\Sigma_g$

Look for $\mathcal{F} \ni L$

$H^2(\Sigma_g \times S^2 : \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$

$e_1 = (\alpha, \beta)$, $e_2 = (\gamma, \delta)$: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

$H_2(\Sigma_g \times S^2 : \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$

$[L] = (1, b) \quad (b>0)$

(p_1) : $e_1^2 + e_2^2 = 0$	$\alpha\beta + \gamma\delta = 0$ ... ①
(χ_M) : $e_1 \cdot e_2 = 4(1-g)$	$\alpha\delta + \beta\gamma = 4(1-g)$ ... ②
(w_2) : $e_1 \equiv e_2 \pmod 2$	$\alpha \equiv \gamma,\ \beta \equiv \delta \pmod 2$
(χ_L) : $\langle e_1, [1,b]\rangle = 2(1-g)$	$\alpha + b\beta = 2(1-g)$ ... ③
$([L]^2)$: $\langle e_2, (1,b)\rangle = 2b$	$\gamma + b\delta = 2b$ ... ④
(M) : $b \leq \frac{g-1}{2}$	$b \leq \frac{1}{2}(g-1)$

③④ ⇒ ① : $\left(\beta + \frac{g-1}{b}\right)^2 + (\delta-1)^2 = \left(\frac{g-1}{b}\right)^2 + 1$: circle

② : $\left(\beta + \frac{g-1}{b}\right) \cdot (\delta - 1) = \frac{g-1}{b}$: hyperbola

7

3-2

$$\boxed{\begin{aligned} &\left(\beta + \tfrac{g-1}{b}\right)^2 + (\delta-1)^2 = \left\{\tfrac{g-1}{b}\right\}^2 + 1 \\ &\left(\beta + \tfrac{g-1}{b}\right) \cdot (\delta - 1) = \tfrac{g-1}{b} \end{aligned}}$$

⊕ $\alpha \equiv \gamma,\ \beta \equiv \delta$, $b \leq \frac{g-1}{2}$

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = ?$

Standard Sol.

$$\begin{pmatrix} 2(1-g) & 0 \\ 0 & 2 \end{pmatrix}$$

Anti-Standard (AS)

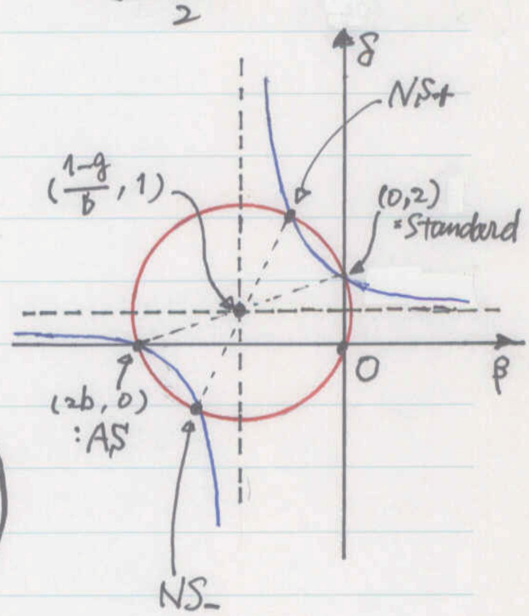
$$\begin{pmatrix} 0 & \frac{2(1-g)}{b} \\ 2b & 0 \end{pmatrix}$$

Non Standard +

$$\begin{pmatrix} (1-g)-b & \frac{1-g}{b}+1 \\ (1-g)+b & \frac{g-1}{b}+1 \end{pmatrix}$$

NonStandard −

$$\begin{pmatrix} (1-g)+b & \frac{1-g}{b}-1 \\ (g-1)+b & \frac{1-g}{b}+1 \end{pmatrix}$$



★ ∃ AS, NS± ⟺ $b \mid g-1$, $\left(b \leq \frac{g-1}{2}\right)$

★ For TURBULIZATION, AS without $b \leq \frac{g-1}{2}$ (necessarily $b \mid g-1$)

8

3.3 Anormalous Solutions and Geometric Interpretation

Standard Solution for L cpt leaf
$\Leftarrow$ foliated S^2-product

Standard Solution for L closed transversal
$\Leftarrow$ bundle foliation $\mathcal{F}_b$

Question: How about other anormalous solutions?

Geometric Characterization (Interpretation) of Anormalous Sol's. 3-4

Standard $\begin{pmatrix} 2(1-g) & 0 \\ 0 & 2 \end{pmatrix}$ AS $\begin{pmatrix} 0 & \frac{2(1-g)}{b} \\ 2b & 0 \end{pmatrix}$ NS+ $\begin{pmatrix} (1-g)-b & \frac{1-g}{b}+1 \\ (1-g)+b & \frac{g-1}{b}+1 \end{pmatrix}$ NS− $\begin{pmatrix} (1-g)+b & \frac{1-g}{b}-1 \\ (g-1)+b & \frac{1-g}{b}+1 \end{pmatrix}$

Standard : $L_\pm$: oriented Leaves $\in \mathcal{F}$.

AS : L_+, $-L_- \in \mathcal{F}$

NS+ : $L_+ \in \mathcal{F}$, $L_- \pitchfork^{\uparrow} \mathcal{F}$

NS− : $L_+ \in \mathcal{F}$, $L_- \pitchfork^{\downarrow} \mathcal{F}$.

TURBULIZATION : $\left(\text{AS w/ } e_1 \leftrightarrow e_2 \text{ i.e. } \alpha \leftrightarrow \gamma,\ \beta \leftrightarrow \delta\right)$ $L_+ \pitchfork^{\uparrow} \mathcal{F}$. $L_- \pitchfork^{\uparrow} \mathcal{F}$.

10

Our Motivation: Realize these solutions
by geometric and visible constructions,
without relying on h-principle!

§4. Geometric Realizations of Exotic Solutions

WHY ? : $\exists$ Exotic Solutions $\Leftrightarrow$ $[L]^2 \mid \chi(\sum_g)$

[StdL]: foliated S^2-bdl

[StdT]: bdl fol. $\mathcal{F}_b$

[ASL] = Anti-Standard Sol for cpt leaf

[AST] = Anti-Standard Sol for closed $\overline{\pitchfork}$'l
: realized by $\mathbb{C}P^1$-model

[NS+] Non-Standard$_+$

[NS-] Non-Standard$_-$: [NS+] + [ASL]
or [NS+] + [AST]

Method: Construct $(P)SL(2;\mathbb{R})$-actions on S^2

Basic Pieces(orbits):

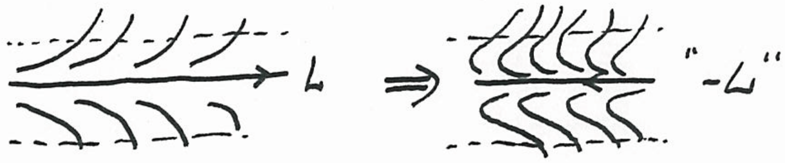
- $\mathbb{R}P^1$ or $S^1 = \mathbb{R}\tilde{P}^1$
- $\mathbb{R}^2(\setminus\{0\})$ = standard rep.
$= SL(2;\mathbb{R})/\{\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\}$:parabolic orbit
- $\mathbb{H} = SL(2;\mathbb{R})/SO(2;\mathbb{R})$: elliptic orbit

Ex[StdL]:
$\pi_1(\sum_g) \hookrightarrow SL(2;\mathbb{R}) \hookrightarrow SL(3;\mathbb{R})$ acts on S^2
= [parabolic orbit] $\cup$ [$\mathbb{R}P^1$] $\cup$ [parabolic orbit]

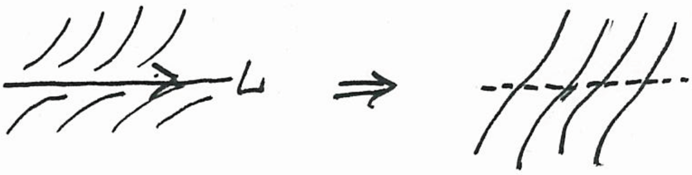
§5 Other Modifications

(1) Turbulization. [AST]

(2) Reversing the orientation of cpt. Leaf. [ASL]



(3)$_\pm$ cpt Leaf ⇒ Closed transversal



L 丕⋔ $\mathcal{F}$ [NS-]

([NS+] + [AST],
[NS+] + [ASL])

L 丕⋔ $\mathcal{F}'$ [NS+]

$$(4)_\pm = \{(3)_\pm\}^{-1}$$

	Cohomological Obstr. h-principle	Geometric Realization by M.-V.
(1) (4)	$[L]^2 \mid \frac{\chi(L)}{2}$	$[L]^2 \mid \frac{\chi(L)}{2}$
(2) (3)	$[L]^2 \mid \frac{\chi(L)}{2}$	$[L]^2 = \pm \frac{\chi(L)}{2}$

5.1 Reversing Orientaion of Cpt Leaf

- **a general principle**

  $M^4 = [0,1] \times V^3$

  $(V, \mathcal{G})$: a foliation of codimension 1

  Assumption : $\tau\mathcal{G} = \langle X, Y \rangle, \quad X,\ Y \in \mathcal{X}^\infty(V),$
  $[X, Y] \in \langle X \rangle$

  Conclusion : $Z = f(t)\frac{\partial}{\partial t} + g(t)Y,\ (f, g) \neq (0, 0)$

  $$\boxed{\tau\mathcal{F} = \langle X, Z \rangle \text{ is integrable.}}$$

  e.g., $\varphi(t) : [0,1] \to [0,\pi]$,
  s.t. $\varphi(0) = 0,\ \varphi(1) = \pi,$
  $\varphi(t)$: flat at $t = 0, 1$
  $f(t) = \sin\varphi(t),\ g(t) = \cos\varphi(t).$

  $\Rightarrow$

  $$\boxed{\mathcal{F}|_{\{1\}\times\mathcal{G}} = -\mathcal{F}|_{\{0\}\times\mathcal{G}}}$$

- **an application**

  $(V, \mathcal{G}) = (S^1T\sum_g,$ Anosov (un)stable foliation)
  or its (double) covering
  $Y =$ the geodesic flow generator,
  $\langle X \rangle =$ strong (un)stable direction

§6 Sullivan's Criterion on Tautness

- **general case**

$$\boxed{\text{Geometric Tautness} \Leftrightarrow \text{Homological Tautness}}$$

Geometrically Taut $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ Riemannian metric
s.t. every leaf is a minimal submanifold

Homologically Taut $\overset{\text{def}}{\Leftrightarrow}$
$\{\text{foliation cycle}\} \cap \overline{\partial\{\text{tangential current }\}} = \emptyset$

- **codimension 1 case**

$$\boxed{\text{two tautness} \Leftrightarrow \text{Toplogical Tautness}}$$

Topologically Taut
$\overset{\text{def}}{\Leftrightarrow}$ $\nexists$ dead-end component
$\Leftrightarrow$ any leaf meets a closed transversal
$\Rightarrow$ **Thurston's Inequality** (convexity)

The tubulization destroys the convexity,
while *the milder* turbulization does not!!

§7 Modifications and Tautness

5.1 <u>Turbulization</u>

$(M, \mathcal{F})$, $L \subset M$, $L \pitchfork \mathcal{F}$
$\mathcal{F}^t$: the turbulization of $\mathcal{F}$ along L

<u>Theorem'</u>

(1) foliation cycle of $\mathcal{F}^t \overset{1:1}{\longleftrightarrow}$
foliation cycle of $\mathcal{F}$ supported away from L

(2?) any tangential 3-current of $\mathcal{F}^t$ descends to one of $\mathcal{F}$.

(3) $\mathcal{F}$: taut $\Rightarrow$ $\mathcal{F}^t$: taut

<u>Remark</u> $\mathcal{F}^t$ can be taut even if $\mathcal{F}$ is not!!

5.2 Reversing the Orientation of Cpt Leaf

- **Review**

  $V^3 = S^1T\sum_g$, $\mathcal{G} =$ Anosov unstable fol.

  $TV = \langle Y, S, U\rangle$, exp τY: Anosv flow ϕ_τ,
  $\langle S\rangle = \mathcal{G}^{ss}$, $\langle U\rangle = \mathcal{G}^{uu}$

  $[Y, U] = -U$, $[Y, S] = S$, $[S, U] = Y$

  $\tau\mathcal{G} = \langle Y, U\rangle$,

  dual 1-forms : Y^*, S^*, U^*;

  $dY^* = U^* \wedge S^*$, $dS^* = S^* \wedge Y^*$, $dU^* = Y^* \wedge U^*$

  $\tau\mathcal{G} = \ker S^*$

Construct $\mathcal{F}$ on $W = [0,1] \times V$ from $(V, \mathcal{G})$.

$\tau\mathcal{F} = \langle Z, U\rangle$, $\quad Z = f(t)\frac{\partial}{\partial t} + g(t)Y$,

$f|_{[\frac{1}{3},\frac{1}{3}]} \equiv 0$, $\quad f|_{(0,1)} > 0$; $\quad g(t) = f'(t)$

f : flat (of 2nd generation) at 0, 1

One More Construction !

Starting from the **stable foliation** $\overline{\mathcal{G}}$;

$\tau\overline{\mathcal{G}} = \langle Y, S\rangle = \ker U^*$,

$\tau\overline{\mathcal{F}} = \langle Z, S\rangle$, $\quad Z = f(t)\frac{\partial}{\partial t} + g(t)Y$,

we obtain a completely different foliation $\overline{\mathcal{F}}$!!

5.3 Tautness of Modified Foliations

Theorem A closed 2-form

$$\Omega = S^* \wedge (g(t)dt - f(t)Y^*) = d(-fS^*)$$

defines a foliation cycle (= ⋔ 'ly inv. measure) for $\mathcal{F}$.

Corollary $(W, \mathcal{F})$ is not taut.

Theorem $(W, \overline{\mathcal{F}})$ admits no foliation cycles and is taut.

Theorem' The 2nd modification of a taut foliation $(M, \mathcal{F})$ is again taut.

THANK YOU FOR YOUR ATTENTIONS