$f\bar{g}$-型特異点のミルナー束と両立する接触構造について

Masaharu ISHIKAWA

Tohoku University

24 January, 2011

Plan of this talk

§**1. Milnor fibrations and contact structures**

§**2. Milnor fibrations of type $f\bar{g}$**

§**3. Fibered, Seifert multilinks in homology 3-spheres**

§**4. PT graph links in S^3**

§1. Milnor fibrations and contact structures

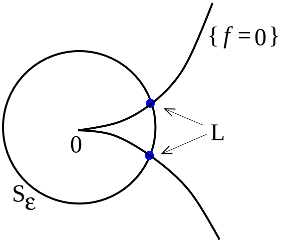
$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$: polynomial map

$S_\varepsilon := \{z \in \mathbb{C}^{n+1} \mid \|z\| = \varepsilon\} \quad 0 < \varepsilon << 1$

$L = \{f = 0\} \cap S_\varepsilon$: the link of singularity $(f, 0)$.

Milnor fibration Theorem ('68)

$\frac{f}{|f|} : S_\varepsilon^{2n+1} \setminus \{f = 0\} \to S^1$ is a locally trivial fibration.



M : an oriented, closed, smooth 3-manifold

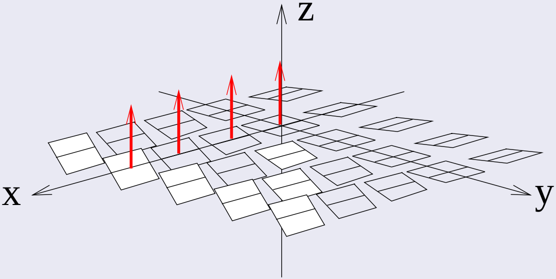
α : 1-form on M s.t. $\alpha \wedge d\alpha > 0$

$\xi = \ker \alpha$: positive contact structure

R_α : Reeb vector field of α $(\Leftrightarrow d\alpha(R_\alpha, \cdot) = 0,\ \alpha(R_\alpha) = 1)$

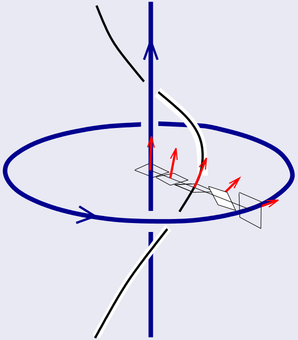
The standard contact structure on $\mathbb{R}^3$

$\alpha = dz + xdy$. $\alpha \wedge d\alpha = dx \wedge dy \wedge dz$. $R_\alpha = \frac{\partial}{\partial z}$

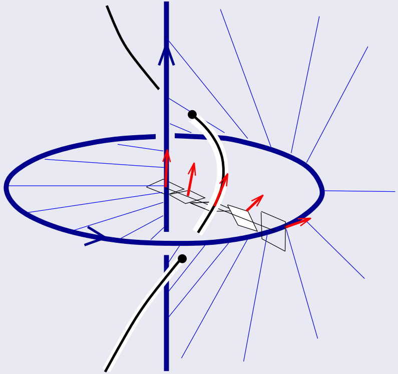


The standard contact structure on S^3

$$\alpha = \frac{1}{2}\sum_{i=1,2}(x_i dy_i - y_i dx_i) \text{ on } S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$$



Morse singularity and contact structure

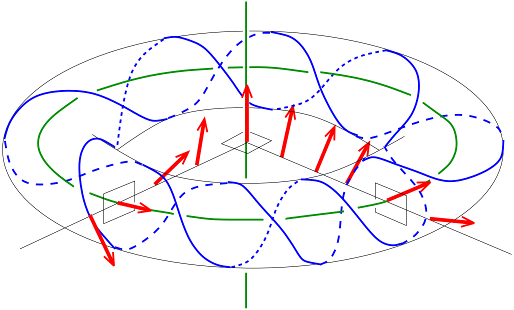


We will say that the Milnor fibration of the Morse singularity is compatible with the standard contact structure.

To define the notion of "compatible" for, for example, the Milnor fibration of $f(z,w) = z^p + w^q$, we need to allow a perturbation of the 2-plane field.

Definition

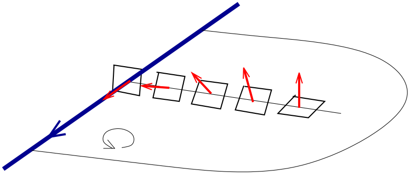
Two contact manifolds (M, ξ) and (M', ξ') are said to be **contactomorphic** if there exists a diffeomorphism $\varphi : M \to M'$ such that $d\varphi(\xi) = \xi'$.



Definition [Giroux '00]

A contact structure ξ is called compatible with a fibered link L in a 3-manifold M if there exist a contactomorphism $(M, \xi) \longrightarrow (M, \xi')$ and a contact form α with $\xi' = \ker \alpha$ such that

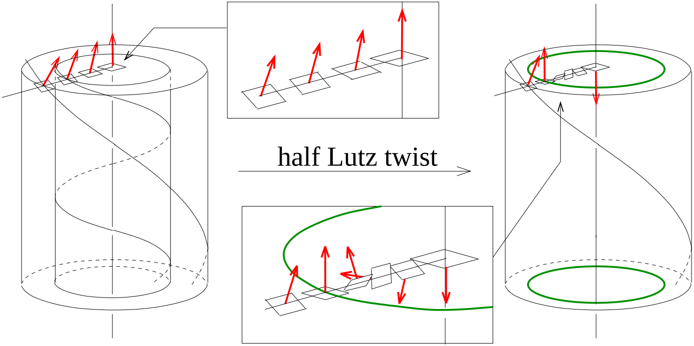
- **R_α is tangent to L in the same direction, and**
- **R_α is positively transverse to the interiors of the fiber surfaces of L.**



Theorem [Thurston-Winkelnkemper ('75)]

Any fibered link in a 3-manifold admits a compatible contact structure.

- **A disk D such that T_xD is tangent to ξ_x at each point $x \in \partial D$ is called an overtwisted disk.**
- **A contact structure is called overtwisted if it has an overtwisted disk.**
- **A contact structure is called tight if it has no overtwisted disk.**



Contact structures on S^3

Theorem [Eliashberg, etc...]

S^3 admits

- an overtwisted contact structure for each homotopy class of 2-plane fields, and
- a unique tight contact structure, called the standard contact structure.

Remark

The Milnor fibrations are always compatible with the standard contact structure on S^3.

Therefore, we cannot expect any contribution of "contact structures" to the study of the Milnor fibrations in low-dimensional topology.

There are several generalizations:

- **Plane curves in $\mathbb{C}^2$ [Rudolph]**
- **Real singularities of $(\mathbb{R}^4, 0) \longrightarrow (\mathbb{R}^2, 0)$**

  [Milnor, Perron, Rudolph, Seade, Pichon, Oka, etc...]
- **Milnor fibrations of higher-dimension**

  [Ustilovsky, van Koert, Caubel-Némethi-PopescuPampu, etc...]

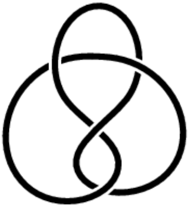
§2. Milnor fibrations of type $f\bar{g}$

Theorem [Milnor ('68)]

If $f : (\mathbb{R}^{2n+2}, 0) \to (\mathbb{R}^2, 0)$ has an isolated singularity at the origin then there exists a locally trivial fibration $S_\varepsilon \setminus \{f = 0\} \to S^1$.

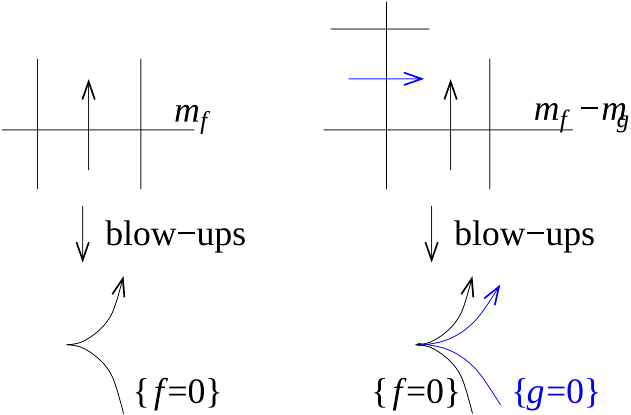
Theorem [Perron ('81)]

There exists a polynomial map $f : (\mathbb{R}^4, 0) \to (\mathbb{R}^2, 0)$ with an isolated singularity at the origin such that $L = \{f = 0\} \cap S_\varepsilon$ is the "figure eight knot".

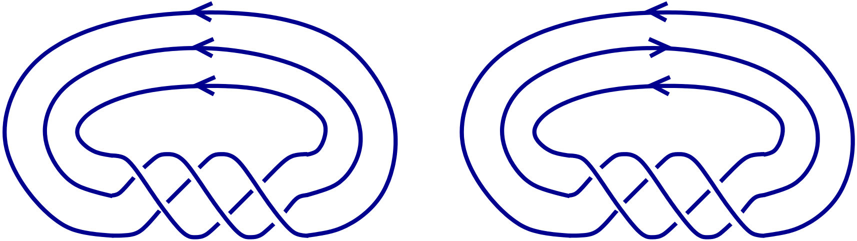


Theorem [Pichon-Seade (2008)]

Let $f, g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be polynomial maps. Then $\frac{f\bar{g}}{|f\bar{g}|} : S_\varepsilon \setminus \{f\bar{g} = 0\} \to S^1$ is a locally trivial fibration in most cases.



The link of $(f\bar{g}, 0)$:



Remark that $(\bar{g}, 0) \underset{\text{ori. pres.}}{\cong} (g, 0)$, hence its compatible contact structure is standard.

Theorem [I. (2010)]

$\frac{f\bar{g}}{|f\bar{g}|}: S_\varepsilon \setminus \{f\bar{g}=0\} \to S^1$ **is compatible with an overtwisted contact structure.**

Outline of the proof:

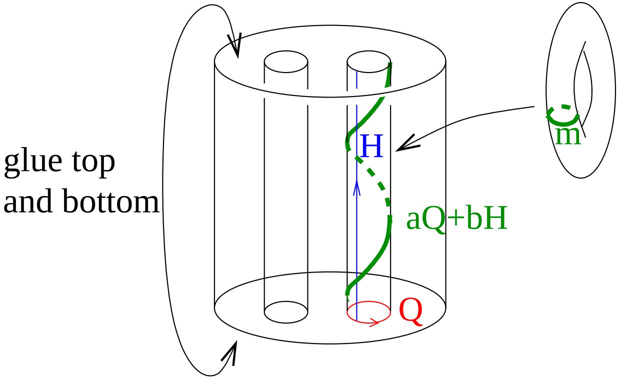
(1) **Prove the assertion for torus links with reversed orientation.**

(2) **Extend the proof for cablings.**

Note: In Step (2), we need the notion of multilinks.

§3. Fibered, Seifert multilinks in homology 3-spheres

$\mathcal{S} = S^2 \setminus \sqcup_{i=1}^{k} \mathrm{int}\, D_i^2$
$\Sigma = (\mathcal{S} \times S^1) \cup \bigcup_{i=1}^{k} (D^2 \times S^1)_i$ with $\mathfrak{m}_i \mapsto a_i Q_i + b_i H$.



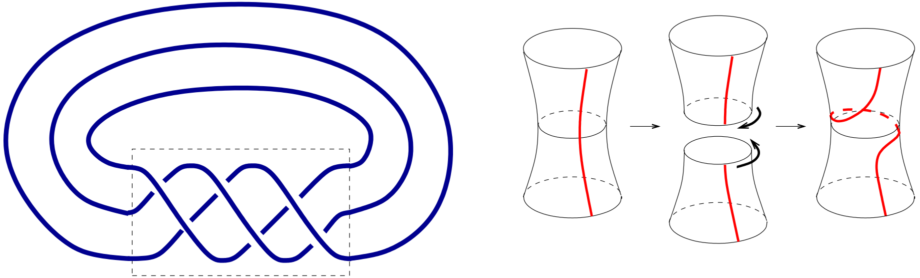
We assume that $\sum_{i=1}^{k} b_i a_1 \cdots a_{i-1} a_{i+1} \cdots a_k = 1$
s.t. Σ is a homology 3-sphere.

Definition

A link in Σ consisting of a union of orbits in Σ is called a Seifert link in Σ.

Definition

We say a Seifert link is positively-twisted (PT for short) if $a_1 a_2 \cdots a_n > 0$.

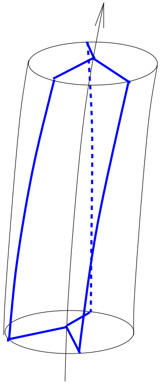


$L = L_1 \cup \cdots \cup L_k$, **where** L_i **is a connected component.**

$\underline{m} = (m_1, \cdots, m_k)$, **where** $m_i \in \mathbb{Z}$.

Definition

The link $L(\underline{m}) = m_1 L_1 \cup \cdots \cup m_k L_k$ **is called a multilink.**



The figure is an example with multiplicity = 3

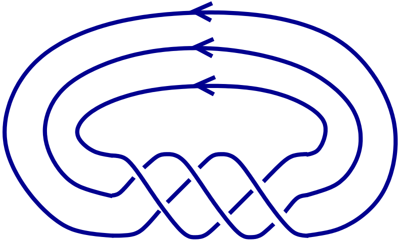
The multiplicities are used
when you consider the fiber surface.

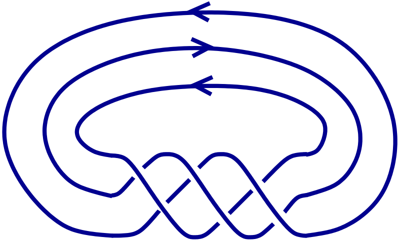
Such a fiber surface appears
when you study a fiber surface
with JSJ−decomposition.

Remark. "Rational open book" in [Baker-Etnyre-HornMorris])

Theorem 1

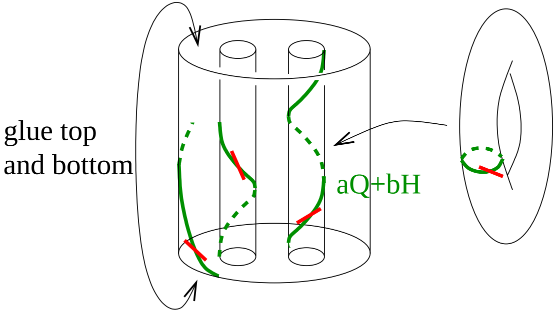
Let L be a fibered PT Seifert multilink in a homology S^3. Then, the contact structure compatible with $L(\underline{m})$ is tight if and only if either $m_1, \cdots, m_k > 0$ or $m_1, \cdots, m_k < 0$.





We may say that the orientatoin of $L(\underline{m})$ is canonical if either $m_1, \cdots, m_k > 0$ or $m_1, \cdots, m_k < 0$.

Sketch of the proof (1/3).



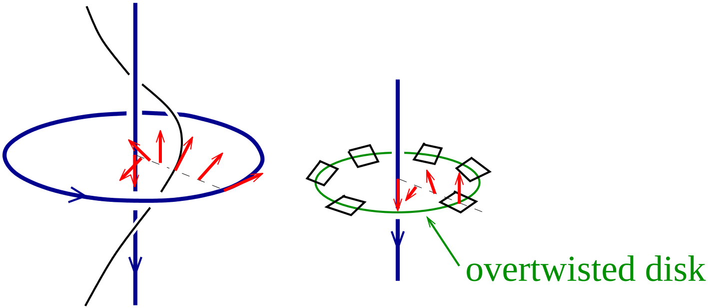
- **Not canonical $\Rightarrow$ OT**

(1) There exists a 1-form β on the holed sphere with $d\beta > 0$ such that $\ker \alpha = \ker(\beta + dt)$ has suitable slopes on $\partial(D^2 \times S^1)_i$

$$\Leftrightarrow \sum_{i=1}^{k} \frac{b_i}{a_i} = \frac{1}{a_1 \cdots a_k} > 0.$$

Sketch of the proof (2/3).

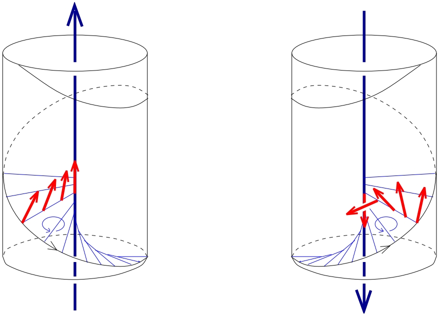
(2) Extend α into $(D^2 \times S^1)_i$ canonically if the orientation is the same as H and do it as shown in the figure if the orientation is reversed.



Thus, we can detect an overtwisted disk.

Sketch of the proof (3/3).

(3) To complete the proof, we need to check that the contact structure is actually compatible, and this is done as shown in the figures below.



- **Canonical $\Rightarrow$ tight**

ξ is symplectically fillable (cf. [Lisca-Matić]). $\square$

§4. PT graph links in S^3

Σ_1, Σ_2 : homology S^3's.

L_i : a Seifert link in Σ_i.

S_i ; a link component of L_i.

$(\mathfrak{m}_i, \mathfrak{l}_i)$: a preferred meridian-longitude pair of $\Sigma_i \setminus \operatorname{int} N(S_i)$.

Definition

Glue $\Sigma_1 \setminus \operatorname{int} N(S_1)$ and $\Sigma_2 \setminus \operatorname{int} N(S_2)$ by the map

$$(\mathfrak{m}_1, \mathfrak{l}_1) \mapsto (\mathfrak{l}_2, \mathfrak{m}_2).$$

The obtained link $(L_1 \setminus S_1) \cup (L_2 \setminus S_2)$ is called a splice of L_1 a nd L_2 along S_1 and S_2.

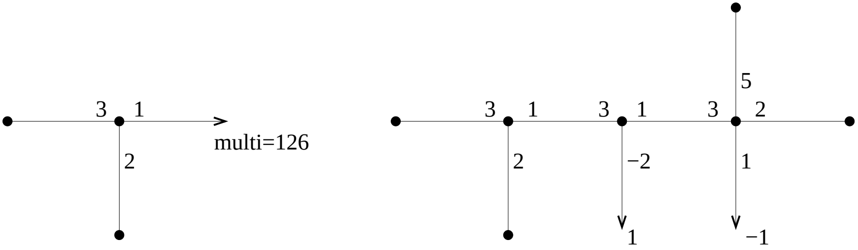
Remark. The ambient space of the splice is again a homology S^3.

Definition

A link obtained by iterating splicing of Seifert links is called a graph link

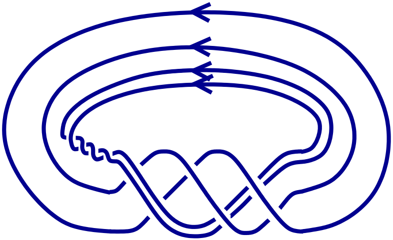
Definition

A link obtained by iterating splicing of PT Seifert links is called a PT graph link.

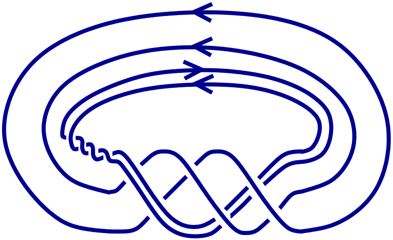


Theorem 2

Let L be a fibered PT graph link in S^3. Then, the contact structure compatible with L is tight if and only if the multiplicities are either all positive or all negative.



canonical



not canonical

Remark. The theorem about $f\bar{g}$ follows from Theorem 2.

Sketch of the proof.

We prepare two contact forms compatible with L.

α_1 : obtained by Thurston-Winkelnkemper's construction.

α_2 : obtained by splicing Seifert multilinks.

- Canonical $\Rightarrow$ tight

  Canonical $\Rightarrow$ each splice is like a positive cabling $\Rightarrow$ tight.

- Not canonical $\Rightarrow$ OT

  (1) Choose α_1 and let $N(S_i)$ be a solid torus for splicing. Then $(N(S_i), \ker\alpha_1)$ does not contain a half Lutz tube.

  (2) Choose α_2 then $\exists$ a half Lutz tube N' with OT disk D.

  (3) $\exists$ contactom. $\phi : (S^3, \ker\alpha_2) \to (S^3, \ker\alpha_1)$ s.t. $N(S_i) \subset \phi(N')$ and $\phi(\partial D) \subset \phi(N') \setminus N(S_i)$.

  (4) $\phi(D)$ remains in $(S^3, \ker\alpha_1)$ as an OT disk after the splicings. $\square$

Conjecture

Let L be a fibered link in S^3, with several components, compatible with tight contact structure. Let L' be a link obtained from L by reversing the orientations of some, but not all, link components. If L' is fibered then its compatible contact structure is overtwisted.

Further studies

- ? **Contact structures of other real singularities (e.g. mixed singularities of [Oka])**
- ? **Higher-dimensional quasipositive surfaces**
- ? **Contact structures and Seifert fibrations in higher-dimension.**

Thank you for your attention!