differential systems の characteristics とその応用

垣江　邦夫

Introduction

古典的例1． n **変数** $x=(x_1,\ldots,x_n)$ **の関数** $u(x)$ **の1階の偏導関数を** $p_i=u_{x_i}$ **で表す．** u **を未知関数とする次の1階偏微分方程式系を考える．**

$$F_j(x_i,\ldots,x_n,u,p_1,\ldots,p_n)=0(j=1,\ldots,r).$$

(†)

$$[F_j,F_k]=\sum_{i=1}^{n}\left(\frac{d\,F_j}{d\,x_i}\frac{\partial F_k}{\partial p_i}-\frac{d\,F_k}{d\,x_i}\frac{\partial F_j}{\partial p_i}\right)$$

(**ここに** $\frac{d}{dx_i}=\frac{\partial}{\partial x_i}+p_i\frac{\partial}{\partial u}$) (bracket)**を用いる．**

Definition. (†) が包合系(system in involution) $\Longleftrightarrow [F_j, F_k] \equiv 0 \pmod{F_1, \dots, F_r}$.

・ (†)が包合系ならば，常微分方程式に帰着して解くことができる．(さまざまな方法が周知)

そうであることの一つの説明： 接触要素の空間 $J_1 = \{(x_1, \dots, x_n, u, p_1, \dots, p_n)\}$における contact 1-form $\theta = du - \sum_{i=1}^n p_i dx_i$ を　多様体 $M = \{P \in J_1; F_j(P) = 0\,(j = 1, \dots, r)\}$ に制限したものを　θ_M で表す．

- θ_M は Cauchy characteristic vectors

$$X_{F_j} = \sum_{i=1}^{n} \left(\frac{\partial F_j}{\partial p_i} \frac{d}{d\,x_i} - \frac{d\,F_j}{d\,x_i} \frac{\partial}{\partial p_i} \right)$$

$(j = 1, \ldots, r)$ を有する.

注. F's が　$F_j = \sum_{i=1}^{n} a^{(j)}(x) p_i$ の形であるときは、この理論のdual は　Frobeniusの理論に他ならない.

古典的例2．2変数x, yの関数$z(x, y)$の導関数を$p = z_x, q = z_y, r = z_{xx}, s = z_{xy}, t = z_{yy}$で表す．$z$を未知関数とする2階の偏微分方程式を考える．

$$F(x, y, z, p, q, r, s, t) = 0. \qquad (\ddagger)$$

この偏微分方程式は　一般に、例1のように常微分方程式に帰着させて解を求めることはできない．　その解法と解の性質は方程式の構造（型）により異なる．

$z=z(x,y)$を(‡)の解とする．これに対し (x,y,z,p,q,r,s,t)-空間の　曲面S:

$$z=z(x,y), p=z_x(x,y), q=z_y(x,y),$$
$$r=z_{xx}(x,y), s=z_{xy}(x,y), t=z_{yy}(x,y).$$

が対応する．S上に曲線$\mathcal{C}$をとる:

$$\mathcal{C}: x=x(\tau), y=y(\tau), z=z(\tau), p=p(\tau), q=q(\tau),$$
$$r=r(\tau), s=s(\tau), t=t(\tau)$$

問題：$\mathcal{C}$上において，x,y,z,p,qの値からr,s,tが一意的に定まるか？

(Cauchy問題に由来．一意に定まらないとき，　$\mathcal{C}$ を特性曲線と呼ぶ).

これは方程式系

$$F = 0, rdx + sdy = dp, sdx + tdy = dq$$

がただ一つの解r, s, tを有するかという問題となる．この方程式系のJacobian は

$$F_r(dy)^2 - F_s(dy)(dx) + F_t(dx)^2. \qquad (\sharp)$$

(以下この式を$\lambda = \frac{dy}{dx}$に関する2次多項式とも見なす.)
従って　$\mathcal{C}$上$(\sharp)$が0とならないならば, r, s, tは一意的に定まる．

仮定：$(\sharp)$ は $\mathcal{C}$ 上相異なる 2 実根 λ, μ を有する, $F_r \neq 0$, $\mathcal{C}$ 上 $dy - \lambda dx = 0$ が成り立つ.

このとき　接触条件と $dF = 0$ のほかに $\mathcal{C}$ 上成り立つもう一つの新しい関係式が導かれる:

$$dy - \lambda dx = 0,$$

$$dz - pdx - qdy = 0,$$

$$dp - rdx - sdy = 0, dq - sdx - tdy = 0,$$

$$dr + \mu ds + \frac{X}{F_r}dx = 0,\ ds + \mu dt + \frac{Y}{F_r}dx = 0.$$

ここに　$X = F_x + pF_z + rF_p + sF_q, Y = \cdots.$

この系を <u>(Monge) characteristic system</u> **と呼ぶ.**

上の状況において　$dy - \lambda dx$ と $dy - \mu dx$ は characteristic covector と呼ばれる.

２階偏微分方程式 $F = 0$ は, (♯) の 2 根の状況に応じて, 実で相異なるとき双曲型(hyperbolic), 実根を有さないとき楕円型(elliptic) であるといわれる.

上の例に3種のcharacteristicsが現れた．
・characteristic covectors:　自然に一般の differential systems へ一般化される．
・Cauchy characteristics: Cartanが一般的な理論を確立している．
・Monge characteristics: Darboux達が“求積論”などにおいて利用．しかし自然で明確な定義がまだ与えられていなかった．例えば　一般の differential systemへ拡張する方法・手順は　まだ与えられていなかった．

その後に発展したinvolutive differential systems **の理論**(E. Cartan, Kähler,etc., Kuranishi, Goldschmidt, Spencer, Matsuda,.... ）**に基づく議論を行うことにより　一般の**differential system **に対し一つの自然な定義を与えることができ**，　**それにより古典理論の一拡張ができる．**

・・・　この部分が講演の主要部分．

Terminologies, Notations, etc.

関数、多様体などは differentiable of class C^∞ (or real analytic) **であるとする.**

$\boxed{\clubsuit}$ Σ: an exterior differential system (EDS) on a manifold M

$\mathcal{F}(M)$ ={the ring of all functions on M},
$\wedge^k(M)$ ={the set of all exterior differential forms of degree k}, $\wedge^0(M) = \mathcal{F}(M)$.
$\wedge(M) = \sum_{k=0}^{\infty} \wedge^k(M)$: a graded $\mathcal{F}(M)$-module.

$\wedge(M)$ is equipped with exterior product $\wedge$ (ring), and exterior differentiation d.

Definition. An exterior differential system on M is a subset Σ of $\wedge(M)$ satisfying the conditions:

(i) Σ is an ideal of $\wedge(M)$,

(ii) Σ is generated by homogeneous elements,

(iii) $\phi \in \Sigma \Rightarrow d\phi \in \Sigma$.

(**その性質に因み**　a differential ideal **とも呼ばれる**)

$\Sigma^k \stackrel{def}{=} \Sigma \cap \wedge^k(M)$: Σ **の** k **次斉次部分**

condition (ii) $\Leftrightarrow \Sigma = \sum_{k=0}^{\infty} \Sigma^k$.

$(y^1, \ldots, y^m)$: a local coordinate system of M.
$\phi \in \wedge^k(M)$ **の局所座標表示：**

$$\phi = \sum_{1 \leq \alpha_1 < \ldots < \alpha_k \leq m} \phi_{\alpha_1, \ldots, \alpha_k}(y) dy^{\alpha_1} \wedge \ldots \wedge dy^{\alpha_k}$$

$$= \frac{1}{k!} \sum_{1 \leq \alpha_1, \ldots, \alpha_k \leq m} \phi_{\alpha_1, \ldots, \alpha_k}(y) dy^{\alpha_1} \wedge \ldots \wedge dy^{\alpha_k}.$$

A submanifold $\iota : N \hookrightarrow M$ is an
<u>integral manifold</u> of $\Sigma \Leftrightarrow \iota^* \phi = 0$, $\forall \phi \in \Sigma$.
(この条件は，　非線形微分方程式系で表される)

$\boxed{\clubsuit}$ $\mathcal{R}_l$: a system of non-linear partial differential equations of order l (SPDE)

$\mathcal{E}$: a fibered manifold with base manifold X and projection $\pi : \mathcal{E} \to X$. **即ち** $\pi : \mathcal{E} \to X$ is a surjetive submersion.
$(x^1, \ldots, x^n)$: a local coordinate system of X.
$(x^1, \ldots, x^n, y^1, \ldots, y^m)$: a fibered chart of $\mathcal{E}$.
$J_k(\mathcal{E})$: the space of k-jets of sections of $\mathcal{E}$.
$\pi^k_l : J_k(\mathcal{E}) \to J_l(\mathcal{E})(k \geq l)$: the canonical projection.

- The space $J_k(\mathcal{E})$ is a fibered manifold over X with projection $\pi^k_{-1} = \pi \circ \pi^k_0$, and $(x^i, y^\alpha, p^\alpha_i, p^\alpha_{i_1 i_2}, \dots, p^\alpha_{i_1 \dots i_k};$
$1 \le i \le n, 1 \le \alpha \le m, 1 \le i_1 \le i_2 \le \dots \le i_k \le n)$
is its fibered chart. Here the functions $p^\alpha_{i_1 \dots i_q}$ are defined as follows: when u is described by $y^\alpha = u^\alpha(x) (1 \le \alpha \le m)$,
$p^\alpha_{i_1 \dots i_q}(j_l(u)(x)) = \partial^q u^\alpha / \partial x^{i_1} \dots \partial x^{i_q}(x)$.

Usually PDE is defined by a system of equations

$$F_\gamma(x^i, y^\alpha, p_i^\alpha, p_{i_1 i_2}^\alpha, \ldots, p_{i_1 \ldots i_l}^\alpha) = 0\,(\gamma = 1, \ldots, r).$$

We adopt the following definition.

Definition. A system of nonlinear partial differential equations of order l on $\mathcal{E}$ is a fibered submanifold $\mathcal{R}_l$ of $J_l(\mathcal{E})$; $\pi^l_{-1} : \mathcal{R}_l \to X$ is a surjective submersion.

A solution of $\mathcal{R}_l$ is a section u of $\mathcal{E}$ such that its l-jet $j_l(u)(x)$ at x belongs to $\mathcal{R}_l$ for each x.

involutiveneness of a symbol

T, V: real vector spaces,
$\dim T = n, \dim V = m$.
By a symbol of order l we mean a subspace $G_l \subset S^l T^* \otimes V$. Giving a symbol G_l of order l is equivalent to giving a surjective linear mapping $\sigma_l : S^l T^* \otimes V \longrightarrow W$, $(W$: a vector space$)$ such that $G_l = \ker \sigma_l$.

The k-th prolongation G_{l+k} of G_l:

$$G_{l+k} \stackrel{def}{=} (S^k T^* \otimes G_l) \cap (S^{l+k} T^* \otimes V).$$

The canonical injection
$\delta_{1,l-1} : S^l T^* \longrightarrow T^* \otimes S^{l-1} T^*$ **を用いると** a symbol G_l of order l **は**, a symbol $\hat{G}_1$ of order 1 **と見なせる：**

$$\hat{G}_1 = G_l \subset T^* \otimes \overline{V}, \text{ where } \overline{V} = S^{l-1} T^* \otimes V.$$

- $\hat{G}_{1+k} = G_{l+k} \, (k = 0, 1, \ldots)$ **が成り立つ**.

G_l: a symbol of order l, Denote $G = G_l = \hat{G}_1$.
$\{t_1, \ldots, t_n\}$: a basis of T,
$\{t^1, \ldots, t^n\}$: its dual basis of T^*.
$G_{(i)} \overset{def}{=} \{\zeta \in G; \zeta(t_i) = 0 (1 \leq i \leq k)\}$.
$(i = 1, \ldots, n)$, $G_{(0)} \overset{def}{=} G$.
<u>Lemma</u>. For any basis $\{t_1, \ldots, t_n\}$ of T,

$$\dim \hat{G}_{1+1} = \dim G_{l+1} \leq \sum_{i=0}^{n} \dim G_{(i)}.$$

Definition. $G_l \subset S^l T^* \otimes V$ is involutive
$\Leftrightarrow \exists\{t_1, \ldots, t_n\}$: a basis of T such that
$\dim G_{l+1} = \sum_{i=0}^{n} \dim G_{(i)}$.
A basis $\{t_1, \ldots, t_n\}$ satisfying this condition is called regular.
Assume that G_l is involutive.
The integers $g_i = \dim G_{(i)}$ $(i = 0, 1, \ldots, n)$ are defined independently of the choice of regular basis $\{t_1, \ldots, t_n\}$. $g_0 \geq g_1 \geq \ldots \geq g_n = 0$.

Definition. characters of G_l:
$s_i = g_{i-1} - g_i \ (i = 1, \ldots, n)$.
Proposition. If G_l is involutive, then its prolongations G_{l+k} are involutive, and the last non-vanishing characters $s_p = s_p(G_{l+k})$ are equal to that of G_l.

Examples: $m = \dim V, n = \dim T, l =$ order

(1) **∼**(4) are involutive symbols:

(1) $G_l \subset S^l T^*$ with $\operatorname{codim} G_l = 1 (m = 1)$.

$g_i = \binom{n+l-i-1}{l} - 1 \, (0 \leq i < n), g_n = 0;$

$\dim G_{l+1} = \binom{n+l}{l+1} - n$.

$s_i = \binom{n+l-i-1}{l-1} \, (0 \leq i < n), s_n = 0$.

(2) $G_1 \subset T^* \otimes V$ defined by ($\{v_\alpha\}$: a basis of V)
$G_1 = \{\sum_{1\le i\le n, 1\le\alpha\le m} \zeta_i^\alpha t^i \otimes v_\alpha$;

$$\sum_{1\le i\le n, 1\le\alpha\le m} \zeta_i^\alpha a_\alpha^{i\beta} = 0(\beta = 1, \ldots, m)\}$$

($\{a_\alpha^{i\beta}\}$ with $\det(a_\alpha^{n\beta}) \neq 0$ is given).

$g_i = (n-1-i)m\,(0 \le i < n)$, $g_n = 0$,$\dim G_{1+1} = \sum_{i=1}^{n-1} mi$. $s_1 = \cdots = s_{n-1} = m, s_n = 0$

(3) $G_1 \subset T^* \otimes V = T^*$, $(m = 1, l = 1)$.

Let $\dim G_1 = n - r$. $g_i = n - r - i\,(1 \le i \le n - r), g_i = 0\,(n - r < i \le n)$, $\dim G_{1+1} = \binom{n-r+1}{2}$.

$s_1 = \cdots = s_{n-r} = 1, s_{n-r+1} = \cdots = s_n = 0.$

Put $T^* = (\text{span}\{\xi, \eta\}); \dim T = 2$.

(4) $G_2 =$span$\{\xi^2, \xi\eta\} \subset S^2T^*$

$(m = 1, n = 2, l = 2)$.

$g_0 = 1, g_i = 0\,(i = 1, 2), \dim G_{1+1} = 1.\ s_1 = 1, s_2 = 0$.

The following is not an involutive symbol:

(5) $G_2 =$span$\{\xi^2, \eta^2\} \subset S^2T^*$

$(m = 1, n = 2, l = 2)$. $g_0 = 1, g_i = 0\,(i = 1, 2)$,

$\dim G_{1+1} = 0$.

- $G_l \subset S^lT^*$ is involutive $\Leftrightarrow \exists H(\xi, \eta)$: a homogeneous polynomial of degree $l + 1 - r$ s.t. $G_l =$ span$\{\xi^j\eta^k H(\xi, \eta)(j + k = r - 1)\}$.

differential systems の involutiveness (包合性)

$\boxed{1^\circ}$ In the case of EDS Σ

A fundamental problem: To construct all integral manifolds of Σ.

換言すれば

For each dimension n, construct all n-dimensional integral manifolds of Σ.

$\iota : N \hookrightarrow M$: an n-dimensional submanifold

- N: an integral manifold $\Leftrightarrow \iota^*\phi = 0, \forall\phi \in \Sigma^n$.

(Proof. Σ is an ideal)

N**が**$y^\beta = u^\beta(y^1, \ldots, y^n)\,(n < \beta \le m)$ **と局所座標表示されるとする**.

N **が** an integral manifold **であることは 関数** $u^\beta(y')\,(y' = (y^1, \ldots, y^n))$ **がある微分方程式系** $\Phi(n)$ **の解であることと表現される**.

Denote $q_i^\alpha = \frac{\partial u^\beta}{\partial x^i}$.

• N is an integral manifold
$\Leftrightarrow$ a system $\Phi(n)$ of differential equations consisting of $F_\phi\,(\phi \in \Sigma^n)$, where

$$F_\phi = \sum_{1 \le \alpha_1 < \ldots < \alpha_n \le m} \phi_{\alpha_1,\ldots,\alpha_n}(y^1, \ldots, y^n, u^{n+1}, \ldots, u^m) \frac{\partial(u^{\alpha_1}, \ldots, u^{\alpha_n})}{\partial(y^1, \ldots, y^n)},$$

$u^i(y') = y^i\ (1 \le i \le n)$ by convention,
$\frac{\partial(u^{\alpha_1},\ldots,u^{\alpha_n})}{\partial(y^1,\ldots,y^n)} = \det(q_i^\alpha; \alpha = \alpha_1, \ldots.\alpha_n; i = 1, \ldots, n)$.

• F_ϕ is a polynomial function of the variables
$q_i^\beta (1 \le i \le n, n < \beta \le m)$.

・integral manifoldsの一つの構成手順：
Construct a 0-dim IM N^0; construct a 1-dim IM N^1 s.t. $N^1 \supset N^0$; $\ldots$; construct an n-dim IM N^n s.t. $N^n \supset N^{n-1}$.

• construction of $N^k \Leftrightarrow$ solution of a PDE $\Phi(k)$.

最終的に得られる $N = N^n$ がan integral manifold となることの証明が肝要！

この手順が実行できるための条件

" involuiveness(包合性) " が発見された：

E.Cartan: Pfaffian systems に対して　(1901)

E,Kähler: 一般の EDS に対して　(1934).

$\mathcal{G}_k(M)$: the manifold of all k-dim contact elements on M; $E_k \in \mathcal{G}_k(M) \Leftrightarrow E_k \subset T_y M$ for some $y \in M$.

Definition. An element $E_k \in \mathcal{G}_k(M)$ of origin y is an integral element of Σ
$\Longleftrightarrow \phi(y)|_{E_k} = 0$, $\forall \phi \in \Sigma$ (equivalently $\forall \phi \in \Sigma^k$)

$I^k\Sigma$: the set of all k-dim integral elements of Σ; $I^k\Sigma \subset \mathcal{G}_k(M)$, $I\Sigma = \cup_{k=0,1,2,\ldots} I^k\Sigma$.

- N is an integral manifold of $\Sigma \Leftrightarrow$ each tangent space T_yN is an integral element of Σ.

$E_k \in I_y^k\Sigma$ の極要素(polar element):

$H(E_k) = \{v \in T_yM ; \text{span}\{v, E_k\} \in I\Sigma\}$

・$H(E_k)$ contains E_k.

SPDE $\Phi(n)$ の階層化:

$\Phi(0) \subset \Phi(1) \subset \cdots \subset \Phi(n-1) \subset \Phi(n)$ (座標系 $\{y^\alpha\}$ に依存) が構成される.

Definition(Kähler). $E_k \in I_y^k\Sigma$ is regular

$\Longleftrightarrow$

(i) $\exists\{\phi_1, \ldots, \phi_r\} \subset \Phi(k)$ which gives a regular local equation of $I^k\Sigma$ around E_k in $\mathcal{G}_k(M)$.

(ii) $\dim H(E_k) =$constant on around E_k in $I^k\Sigma$.

A chain $E_0 \subset E_1 \subset \ldots \subset E_n$ of integral elements of Σ is called a regular chain of Σ if each E_k with $k < n$ is regular.

Definition. Σ is involutive at $E_n \in I^n\Sigma$

$\Leftrightarrow \exists E_0 \subset E_1 \subset \ldots \subset E_n$: a regular chain.

以下この状況を仮定：

$t(E_k) \stackrel{def}{=} \mathrm{codim}\, H(E_k)$ in T_yM

$t_k(E_n) \stackrel{def}{=} \max\{t(E_k); E_k \subset E_n\}$ $(k = 0, 1, \ldots, n)$.

$t_0(E_n) \geq t_1(E_n) \geq \ldots \geq t_n(E_n)$.

The integers $s_0(E_n), \ldots, s_n(E_n)$ defined by

$$s_0(E_n) = t_0(E_n), s_n(E_n) = m - n - t_{n-1}(E_n),$$
$$s_k(E_n) = t_k(E_n) - t_{k-1}(E_n)\,(1 \leq k < n)$$

are called <u>(Cartan) characters</u> of Σ at E_n.

Cartan-Kähler Theorem. Assume that
M, Σ: real analytic, $E_n \in I^n\Sigma$.
Σ is involutive at $E_n \Rightarrow \exists\ N$: an analytic n-dimensional integral manifold of Σ with $T_yN = E_n$. Parametrization of n-dim IM: the general integral manifold N with T_yN being near E_n depend on s_0 constants, s_1 functions of 1 variables, ..., s_n functions of n variables.

$\diamondsuit$ the symbol of Σ

Assume that Σ is involutive at $E_n \in I^n_y\Sigma$.

Definition. The symbol of Σ at E_n is the space
$C(E_n) = T_{E_n}(I^n_y \Sigma)$
$= \{\zeta \in T_{E_n}(\mathcal{G}_n(M)_y); < \omega(y), \zeta >= 0, \forall \omega \in \Sigma^n_y\}.$
considered as a subspace of $E^*_n \otimes (T_y M/E_n)$
in terms of the isomorphism
$\chi : T_{E_n}(\mathcal{G}_n(M)_y) \to E^*_n \otimes (T_y M/E_n)$;
$C(E_n) \subset E^*_n \otimes (T_y M/E_n)$.
Proposition. If Σ is involutive at E_n, then the symbol $C(E_n) \subset E^*_n \otimes (T_y M/E_n)$ is involutive.

$\boxed{2^\circ}$ In the case of SPDE $\mathcal{R}_l$

The $\underline{k\text{-th prolongation}}$ $\mathcal{R}_{l+k}$ of $\mathcal{R}_l$:

$\mathcal{R}_{l+k} = J_k(\mathcal{R}_l) \cap J_{l+k}(\mathcal{E})$(set!)$(k = 1, 2, \ldots)$

(**ここに**$J_{l+k}(\mathcal{E}) \hookrightarrow J_k(J_l(\mathcal{E}))$).

局所表示 $\mathcal{R}_l : F_\gamma = 0(\gamma = 1, \ldots, r)$**のとき**

$\mathcal{R}_{l+1} : F_\gamma = 0, \frac{dF_\gamma}{dx^i} = 0(1 \leq i \leq n, 1 \leq \gamma \leq r)$

($\frac{dF_\gamma}{dx^i}$:total differentiation with respect to x^i (**全微分**))

• $\{$ C^∞ solutions of $\mathcal{R}_{l+k}\}$=$\{$ C^∞ solutions of $\mathcal{R}_l\}$. (Provided $\mathcal{R}_{l+k}$ is a fibered manifold, that is, a SPDE)

$\diamondsuit$The <u>symbol</u> $G_l = \{G_{l,P}; P \in \mathcal{R}_l\}$ of $\mathcal{R}_l$:

$V(\mathcal{E}) \overset{def}{=} \ker \pi_* : T\mathcal{E} \to TX$ (vertical bundle).

- an exact sequence of vector bundles over $J_l(\mathcal{E})$:

$$0 \to S^l T^* \otimes_{J_l} V(\mathcal{E}) \xrightarrow{\epsilon_l} V(J_l) \xrightarrow{(\pi^l_{l-1})_*} (\pi^l_{l-1})^{-1} V(J_{l-1}) \to 0$$

(ここに $T^* = T^*X, J_k = J_k(\mathcal{E})$).

$G_l \overset{def}{=} S^l T^* \otimes_{J_l} V(\mathcal{R}_l) \cap (\epsilon_l)^{-1}(V(\mathcal{R}_l))$.

- $\mathrm{Ann}(G_l)$ =span of

$\sum_{i_1,\ldots,i_l,\alpha} F_{\gamma,p^\alpha_{i_1\ldots i_l}} \partial_{x^{i_1}} \cdots \partial_{x^{i_l}} \otimes dy^\alpha (1 \le j \le r)$.

Definition. $\mathcal{R}_l$ is involutive $\Leftrightarrow$
(i) G_{l+1} is a vector bundle over $\mathcal{R}_l$,
(ii) The symbol G_l is involutive,
(iii) $\pi_l^{l+1} : \mathcal{R}_{l+1} \to \mathcal{R}_l$: surjective.

The (Cartan) characters $s_i = s_i(P)\,(1 \leq i \leq n)$ are defined to be the characters $s_i = s_i(G_{l,P})\,(1 \leq i \leq n)$ of the symbol $G_{l,P}$.

- $s_1 \geq s_2 \geq \ldots \geq s_n$ (SPDE**だから**).

後の議論で特に(ii)**が重要である.**

解析解の存在定理：

The Cartan-Kähler theorem. If an analytic SPDE $\mathcal{R}_l$ is involutive, the $\mathcal{R}_l$ admits (local) analytic solutions. When $l = 1$, the general solution depends on s_0 constants, s_1 functions of 1 variables, . . ., s_n functions of n variables. When $l > 1$, the same statement is valid with s_i's being replaced by suitable ones.

補足： $s_n > 0 \Leftrightarrow$ underdetermined;

$s_n = 0 \Leftrightarrow$ determined,overdetermined.

- Examples of involutive SPDE:

（既述のexamples of symbols (1)**〜**(4)**が対応）**

(1) a single eq. $F(x^i, y^\alpha, p^\alpha_{I^k}; 1 \leq k \leq l) = 0$

(2) a system of order 1:

$\sum_{1\leq i\leq n;1\leq\alpha\leq m} a^{i\beta}_\alpha(x)\frac{\partial u^\alpha}{\partial x^i} - b_\beta(x) = 0$

($\{a^{i\beta}_\alpha(x)\}$ with $\det(a^{n\beta}_\alpha)(x) \neq 0$ is given).

(3) 1**未知関数で**1**階の過剰決定系**:

$F_j(x_i, \ldots, x_n, u, p_1, \ldots, p_n) = 0 \quad (j = 1, \ldots, r)$

with $[F_j, F_k] \equiv 0\,(\mathrm{mod} F_1, \ldots, F_r)$.

(4) 2**変数関数**$z(x,y)$**に関する**2**階過剰決定系**:

$r - f(x,y,z,p,q) = 0, s - g(x,y,z,p,q) = 0$

with $\frac{df}{dy} = \frac{dg}{dx} \pmod{r - f, s - g}$.

- An example of a non-involutive system:

(5) $z = z(x,y)$**に関する**2**階過剰決定系**:

$r = 0, t = 0$. The symbol is not involutive.

$\boxed{3^\circ}$ EDS $\Sigma(\mathcal{R}_l)$

SPDEに自然にEDSが付随する.

$\diamondsuit$ the contact system on the space $J_l(\mathcal{E})$ (Gardner-Shadwick):

$P = j_l(u)(x) \in J_l(\mathcal{E})$. u defines a morphism $j_{l-1}(u) : U =$ (domain of u) $\to J_{l-1}(\mathcal{E})$.

Denote $\bar{P} = \pi^l_{l-1}(P)$.

$\Theta \overset{def}{=} \{\Theta(P); P \in J_l(\mathcal{E})\}$ where

$$\Theta(P) = (\pi^l_{l-1} - j_{l-1}(u) \circ \pi^l_{-1})^* T^*_{\bar{P}} J_{l-1}(\mathcal{E}).$$

• Θ is a vector subbundle of $T^*J_l(\mathcal{E})$ over $J_l(\mathcal{E})$.
$\Theta^\sharp$:the exterior differential system on $J_l(\mathcal{E})$ generated by all sections (1-forms) of Θ.
The system $\Theta^\sharp$ or the Pfaffian system $\Theta = \Theta^\sharp \cap \wedge^1(J_l(\mathcal{E}))$ is called the <u>contact system</u> on $J_l(\mathcal{E})$, and 1-forms in Θ are called <u>contact forms</u>.

局所座標による表示： the contact system Θ is generated by the following 1-forms:

$$\theta_\alpha^{i_1\dots i_k} = dp_\alpha^{i_1\dots i_k} - \sum_{i=1}^{n} p_\alpha^{i_1\dots i_k i} dx_i$$

$$(1 \le \alpha \le m, 1 \le i_1 \le \dots \le i_k \le n, 0 \le k < l).$$

Definition. $\Sigma(\mathcal{R}_l)$ is the differential ideal (EDS) generated by all contact forms on $J_l(\mathcal{E})$ and the functions F on $J_l(\mathcal{E})$ with $F|_{\mathcal{R}_l} = 0$.

$\Sigma(\mathcal{R}_l)$**の局所表示：**$\Sigma(\mathcal{R}_l)$ is generated by

$$F_\gamma, dF_\gamma, \theta_\alpha^{i_1\dots i_k}, d\theta_\alpha^{i_1\dots i_k}$$

$$(1 \leq \gamma \leq r, 1 \leq \alpha \leq m, 1 \leq i_1 \leq \ldots \leq i_k \leq n, 0 \leq k < l$$

as an algebraic ideal in $\wedge(J_l(\mathcal{E}))$.
Given a section u of $\mathcal{E}$, we have a subset $\mathcal{N}_u = \{j_l(u)(x); x \in X\} \subset J_l(\mathcal{E})$.

- $\mathcal{N}_u$ is an n-dimensional manifold, and the mapping $\pi^l_{-1} : \mathcal{N}_u \to X$ is a diffeomorphism.

Proposition. (a) u is a solution of $\mathcal{R}_l \Rightarrow \mathcal{N}_u \subset J_l(\mathcal{E})$ is an n-dimensional integral manifold of $\Sigma(\mathcal{R}_l)$.
(b) $\mathcal{N} \subset J_l(\mathcal{E})$ is an n-dimensional integral manifold of $\Sigma(\mathcal{R}_l)$ s.t. $\pi^l_{-1} : \mathcal{N} \to X$ is a diffeomorphism $\Rightarrow \exists$ a (unique) solution u of $\mathcal{R}_l$ such that $\mathcal{N} = \mathcal{N}_u$.

Theorem. $\mathcal{R}_l$ is involutive
$\Rightarrow \Sigma(\mathcal{R}_l)$ is involutive at $E_n \in I^n(\mathcal{R}_l)$.
($I^n(\mathcal{R}_l)$ is the one defined at p.58)

$\boxed{4^\circ}$ SPDE associated with Σ

Given a contact element $E_n^{(0)}$ on M.

$(y^1, \ldots, y^m)$: a local coordinate system of M around the origin y_0 of $E_n^{(0)}$ such that $dy^1, \ldots, dy^n$ are linearly independent on $E_n^{(0)}$.

Let V_0 be its coordinate neighborhood.

$\pi : V_0 \to \mathbb{R}^n$, $\pi(y^1, \ldots, y^m) \overset{def}{=} (y^1, \ldots, y^n)$.

$X_0 = \pi(V_0)$; an open subset of $\mathbb{R}^n$.

$\overset{\circ}{\mathcal{G}}_n(V_0) = \{E_n \in \mathcal{G}_n(V_0); \dim \pi_*(E_n) = n\}$.

$\mathcal{E}_0 \overset{set}{=} V_0$: a fibered manifold with base space X_0 and projection $\pi : \mathcal{E}_0 \to X_0$. ($(y^1, \ldots, y^m)$ is its fibered chart).

$\chi : J_1(\mathcal{E}_0) \to \overset{\circ}{\mathcal{G}}_n(V_0)$,

$\chi(j_1(u)(x)) \overset{def}{=} u_*(T_x X_0)$: a canonical diffeomorphism.

<u>Definition</u>. $\mathcal{S}_1 = \chi^{-1}(I^n\Sigma \cap \overset{\circ}{\mathcal{G}}_n(V_0))$.

If $I^n\Sigma \cap \overset{\circ}{\mathcal{G}}_n(V_0)$ is a manifold, then $\mathcal{S}_1$ defines SPDE of order 1 on $\mathcal{E}_0$.

- $\mathcal{S}_1$ coincides with $\Phi(n)$ appeared in $\boxed{1^\circ}$.

Theorem. If Σ is involutive at $E_n^{(0)}$, then $\mathcal{S}_1$ is involutive at $P_0 = \chi^{-1}(E_n^{(0)})$. The symbol G_{1,P_0} is canonically isomorphic to $C(E_n^{(0)})$. Thus e.g. $\mathcal{S}_1$ is elliptic if and only if Σ is elliptic.

involutive symbolsに付随するmodules

Notation:

$R_0 = \mathbb{R},\ R_q = S^q T\,(q = 1, 2, \ldots)$

$R = \sum_{q=0}^{\infty} R_q$: (isomorphic to) the polynomial ring in n variables.

$L = R \otimes V^* = \sum_{q=0}^{\infty} L_q,\ L_q = S^q T \otimes V^*$: a Noetherian graded R-module.

$\mathfrak{X}$ =the maximal ideal of R generated by R_1.

$G_l \subset S^l T^* \otimes V$: a symbol of order l.

$D_l = \mathrm{Ann} G_l \subset L_l = S^l T \otimes V^*$,
N =the (homogeneous) submodule of L generated by D_l.

Definition. The characteristic module $M = M(G_l)$ of a symbol G_l is the smallest (homogeneous) submodule M of L possessing the following two properties:
(i) $M \supset N$; (ii) $\mathfrak{X} z \subset M$ $(z \in L) \Rightarrow z \in M$.

注. **この定義は通常のもの** (e.g. Goldschmidt) **と異なる.** **性質** (ii) **が後述の** Monge characteristics **の議論で重要** (**少なくとも便利**) **となる.**

例えば 一つの利点は $M(G_{l+k}) = M(G_l)$ **が成り立つ.**

One can apply the elementary thory of Naetherian modules (cf. e.g. Zariski-Samuel) to the characteristic module M of a symbol G_l: M admits an irredundant primary decomposition in L

$$M = \cap_{j=1}^{\nu} Q_j, \quad Q_j \text{ being } \mathfrak{P}_j\text{-primary},$$

where $\mathfrak{P}_j$ are homogeneous prime ideals in R.

• $\{\mathfrak{P}_1, \ldots, \mathfrak{P}_\nu\} = \{$the set of associated prime ideals of the quotient module $L/M\}$.

Lemma. Assume that G_l is an involutive symbol.
(i) There exists no $\mathfrak{X}$-primary component in the irredundant primary decomposition of the characteristic module M.
(ii) $M_q = N_q$ for any $q \geq l$.

Definition. A non-zero $\xi \in T^* \otimes \mathbb{C}$ is a characteristic covector for a symbol G_l
$\Leftrightarrow G_l \otimes \mathbb{C} \cap (\xi^l \otimes V \otimes \mathbb{C}) \neq \{0\}$
$\Leftrightarrow \sigma_{l,\xi} : V \otimes \mathbb{C} \to W \otimes \mathbb{C}$ defined by
$\sigma_\xi(v) = \sigma_l(\xi^l \otimes v)$ is not injective.
Otherwise ξ is non-characteristic for G_l.
$\Xi(G_l) \stackrel{def}{=} \{$all characteristic covectors for $G_l\}$.
$\Xi(G_l)$ is an algebraic variety in the complex projective space $\mathbb{P}(T^*)$. We shall call $\Xi(G_l)$ the characteristic variety of a symbol G_l.

<u>Theorem</u>. Assume that G_l is involutive. Let p be the non-negative integer determined by $s_p > 0, s_{p+1} = \ldots = s_n = 0$. Then the following are valid:

(i) $\operatorname{proj}\dim \mathfrak{P}_j \leq p-1$ and $\exists \mathfrak{P}_j$ s.t. $=$ holds.

(ii) $s_p = \displaystyle\sum_{\operatorname{proj}\dim \mathfrak{P}_j = p-1} \mu(Q_j)$

($\mu(Q_j)$ =the multiplicity of the component Q_j).

(iii) $\Xi(G_l) = \cup_{j=1,\ldots,\nu}$ {the variety of $\mathfrak{P}_j$}.

(iv) $\Xi(G_{l+k}) = \Xi(G_l)$.

$\diamondsuit$ involutive subsymbols

$G_l \subset S^l T^* \otimes V$: an involutive symbol of order l.

$D_l = \mathrm{Ann} G_l \subset L_l = S^l T \otimes V^*$.

Assume $s_1 > 0, s_2 = \cdots = s_n = 0$.

Consider the problem: construct involutive subsymbols $G_l' \subset G_l$.

既述の記号を用いる

Lemma. Let $z \in L_l$. $G_l' \overset{def}{=} \mathrm{Ann}\,(D_l, z)$ is involutive $\Leftrightarrow \exists \mathfrak{P}_j$ with the zero of $\mathfrak{P}_j$ being real such that $\mathfrak{P}_j\, z \subset M$.

Define $\kappa_e : T^* \otimes S^l T^* \otimes V \to S^l T^* \otimes V$ by $\kappa_e(\zeta) = \zeta(e), \zeta \in \mathrm{Hom}(T, S^l T^* \otimes V)$. This induces a morphism $\kappa_e : G_{l+1,P} \to G_{l,P}$.

Introduce a subspace of G_l:

$c(\mathfrak{P}_j) = T \cap \mathfrak{P}_j$(**１次部分**),

$C(\mathfrak{P}_j) =$ span$\{\kappa_e(G_{l+1}); e \in c(\mathfrak{P}_j)\}$.

Proposition. $G_l' \subset G_l$ is involutive $\Leftrightarrow$ $\exists \mathfrak{P}_j$ with the zero of $\mathfrak{P}_j$ being real such that $G_l' \supset C(\mathfrak{P}_j)$.

注. **上の議論は** $s_1 = \cdots = s_p > 0, s_{p+1} = \cdots = s_n = 0$ with $1 \le p < n$ **の場合に一般化することができる.**

The characteristic covectors of differential systems

$\boxed{1^\circ}$ The characteristic covectors of EDS Σ

Fix an integer(dimension) $n \geq 2$. Let
$E_{n-1} \in I^{n-1}\Sigma$. Assume $E_{n-1} \subset \exists E_n \in I^n\Sigma$.
$\Rightarrow \dim H(E_{n-1}) \geq n$.

<u>Definition</u>. E_{n-1} is a <u>characteristic</u> (resp. <u>non-characteristic</u>) element
$\Leftrightarrow \dim H(E_{n-1}) > n$ (resp. $= n$).

Assume that Σ is involutive at $E_n \in I^n\Sigma$.
A characteristic element can be characterized by using the symbol $C(E_n)$ of Σ;
$C(E_n) \subset E_n^* \otimes (T_yM/E_n)$ ($y =$ origin of E_n).
Lemma. Let $\xi \in E_n^*$ s.t. $E_{n-1} =<\xi>^\perp$. Then E_{n-1} is characteristic

$$\Leftrightarrow \quad C(E_n) \cap \{\xi \otimes (T_yM/E_n)\} \neq \{0\}.$$

In other words, a non-zero $\xi \in E_n^*$ defines a characteristic element $E_{n-1} \subset E_n$
$\Leftrightarrow \xi$ is a real characteristic covector of $C(E_n)$.

Definition. The characteristic variety $\Xi(E_n)$ of Σ at E_n is the characteristic variety of $C(E_n)$. A covector $\xi \in \Xi(E_n)$ is a characteristic covector of Σ at E_n. $\Xi(E_n)$ is an algebraic variety in the projective space associated with the complexfication of E_n^*.

Remark. If $s_n(E_n) > 0$, then $\forall \xi \in E_n^*$ is characteristic.

$\boxed{2^\circ}$ The characteristic covectors of SPDE $\mathcal{R}_l$

We shall always assume that $\mathcal{R}_l$ is involutive.

Around $x_0 \in X$ we regard X as the Euclidean $(x^1, \ldots, x^n)$- space.

$H \subset X$: a hyperplane with $x_0 \in H$.

$\xi = \sum_{i=1}^n \xi_i \, dx^i$: a covector with $< \xi >^\perp = T_{x_0} H$.

$\frac{\partial}{\partial \nu^i}$ $(i = 1, \ldots, n)$; a basis of $T_{x_0} X$ such that $\{\frac{\partial}{\partial \nu^i}; 1 \leq i < n\} = T_{x_0} H$ ($\frac{\partial}{\partial \nu^n}$ is transversal to H).

Let u be a section u of $\mathcal{E}$ described by $y^\alpha = u^\alpha(x) \, (\alpha = 1, \ldots, m)$.

The l-jet $j_l(u)(x_0)$ is determined by the values:

$$(\frac{\partial}{\partial \nu^1})^{\mu_1} \dots (\frac{\partial}{\partial \nu^n})^{\mu_n} u^\alpha(x_0)\,(\mu_1 + \cdots + \mu_n \leq l).$$

値 $(\frac{\partial}{\partial \nu^1})^{\mu_1} \dots (\frac{\partial}{\partial \nu^n})^{\mu_n} u^\alpha(x_0)$ with $\mu_1 + \cdots + \mu_n \leq l, \mu_n < l$ **と条件「(微分方程式系)**$j_l(u)(x_0) \in \mathcal{R}_l$**」から値** $(\frac{\partial}{\partial \nu^n})^l u(x_0)$ **が一意に定まる為の** H **即ち** ξ **に対する条件を追求すると　次の条件へ導かれる：**

$$G_{l,P_0} \cap (\xi^l \otimes V_{y_0}\mathcal{E}) \neq \{0\}.$$

（前述の条件は　Cauchy 問題に由来する条件である．$l=1$ の場合の説明：　関数 $u(x)$ の H 上のデータと微分方程式系を満たすという条件から，H 上における u の all derivatives (including of higher orders) が一意的に定まるという条件となる)

Definition. Let $\xi \in T^*_{x_0}X \otimes \mathbb{C}$.

ξ is characteristic (resp. non-characteristic) for $\mathcal{R}_l$ at $P_0 \Leftrightarrow$

$$G_{l,P_0} \otimes \mathbb{C} \cap (\xi^l \otimes V_{y_0}\mathcal{E} \otimes \mathbb{C}) \neq (\text{resp.} =)\{0\}.$$

$\Xi_P \stackrel{def}{=} \{$all characteristic covectors for $\mathcal{R}_l$ at $P\}$.

$\Xi_P \subset \mathbb{P}(T_x^* \otimes \mathbb{C})$: a complex algebraic variety.

The family $\Xi = \{\Xi_P; P \in \mathcal{R}_l\}$ is called the <u>characteristic variety</u> of $\mathcal{R}_l$.

上の条件からわかるように

- $\Xi_P = \Xi(G_{l,P})$.

$\Diamond$ $\mathcal{R}_l$ **と** $\Sigma(\mathcal{R}_l)$ **の** char. covectors **の対応関係**:
$P \in \mathcal{R}_l$, $E_n \in I^n(\mathcal{R}_l)$ with origin P.
Lemma. The canonical injection

$$\beta_* : S^{l+1}T_x^*X \otimes V_y(\mathcal{E}) \longrightarrow V_{E_n}(\mathring{\mathcal{G}}_n(M)),$$

induces an isomorphism

$$\beta_* : G_{l+1,P} \longrightarrow C(E_n).$$

$\Xi(G_{l+1,P}) = \Xi(G_{l,P})$ **だから次が導かれる：**

Proposition. $\xi \in T_x^* X$: a char. covector of $\mathcal{R}_l$ at $P \Leftrightarrow$

$\hat{\xi} = (\pi^l_{-1})^* \xi$: a char. covector of $\Sigma(\mathcal{R}_l)$ at E_n

♠ 応用例

SPDE $\mathcal{R}_l$ is elliptic(**楕円型**) $\overset{def}{\Leftrightarrow}$ $\mathcal{R}_l$ admits no real characteristic covector.

Theorem(A C^∞ Cartan-Kähler theorem).
Assume $\dim X = 2$. If $\mathcal{R}_l$ is involutive and elliptic, then $\mathcal{R}_l$ admits C^∞ (local) solutions.

この存在定理はもちろんelliptic involutive EDS Σ**に適用できる**; 2**次元の**C^∞**積分多様体の存在が導かれる**.

注．楕円型で$\dim X \geq 3$**の場合の**C^∞**解の存在問題は未解決**. **楕円型線形系の場合は**Spencer**予想と呼ばれる予想がある．線形系であっても極めて難しい！**

D. Yang **は** $\Xi(\mathcal{R}_l)$**を介して**, involutive hyperbolic systems(**双曲型包合系**)**という概念を導入し** A C^∞ Cartan-Kähler theorem**を得た：**

If $\mathcal{R}_l$ is involutive hyperbolic, then $\mathcal{R}_l$ admits C^∞ (local) solutions.

Monge characteristics in differential systems

$\boxed{1^\circ}$ **微分方程式系**$\mathcal{R}_l$**の**Monge characteristics

$\Sigma(\mathcal{R}_l)$**：** $\mathcal{R}_l$**に付随する**EDS **（**$J_l(\mathcal{E})$ **上の系)**.

$\mathcal{R}_l$**に適合する**integral elements**は：** $P \in \mathcal{R}_l$,

$$I^n_P(\mathcal{R}_l) = \{E_n \in I^n_P(\Sigma(\mathcal{R}_l)); \dim (\pi^l_{-1})_* E_n = n\}.$$

$y = \pi^l_0(P) \in \mathcal{E}, x = \pi^l_{-1}(P) \in X$**と記す．**

$T_x = T_x X, T^*_x = T^*_x X$ **と略記**.

$E_n \in I^n_P(\mathcal{R}_l)$ **のとき**, $(\pi^l_{-1})_* : E_n \to T_x$ **は同型**.

$$v_e(E_n) \overset{def}{=} v \in E_n, \text{where}\, (\pi^l_{-1})_*(v) = e.$$

$e \in T_x$ **が与えられたとし、次の空間を導入する.**

$$B(e) = \operatorname{span}\{v_e(E_n); E_n \in I^n_P(\mathcal{R}_l)\} \subset T_P(J_l(\mathcal{E})),$$
$$D(e) = \mathrm{Ann}(B(e)) \subset T^*_P(J_l(\mathcal{E})).$$

・$D(e)$ は自明に空間

$$\Sigma^1_{(P)} = \{\phi(P) \in T^*_P(J_l(\mathcal{E})); \phi \in \Sigma(\mathcal{R}_l)^1\},$$

と空間 $(\pi^l_{-1})^* H_e$ $(H_e = \{\xi \in T^*_x; <\xi, e>= 0\})$ を含む． $(\dim H_e = n-1)$.

$D(e)$ がこれらの和空間より真に大きくなるときがあるか？

これを論ずるために

$$\kappa(\zeta) = \zeta(e), \zeta \in \mathrm{Hom}(T_x, S^l T_x^* \otimes V_y \mathcal{E})$$

で定義されるmorphism

$$\kappa_e : T_x^* \otimes S^l T_x^* \otimes V_y \mathcal{E} \longrightarrow S^l T_x^* \otimes V_y \mathcal{E}$$

を用いる. **これは**morphism $\kappa_e : G_{l+1,P} \to G_{l,P}$ **を誘導する**. (symbol **の項参照**)

上述の問題の回答：

Lemma. Assume that $\mathcal{R}_l$ is involutive. For each $P \in \mathcal{R}_l$, the following are valid:

(i) For any $e \in T_x$, $D(e) \supset \Sigma^1_{(P)} \oplus (\pi^l_{-1})^* H_e$.

(ii) $\dim \dfrac{D(e)}{\Sigma^1_{(P)} \oplus (\pi^l_{-1})^* H_e} = \dim \dfrac{G_{l,P}}{\kappa_e(G_{l+1,P})}$.

(iii) $\exists e \in T_x$ s.t. $D(e) = \Sigma^1_{(P)} \oplus (\pi^l_{-1})^* H_e$.

証明へのメモ： $B(e) = \{\text{span}\, v_e(E^0_n)\} \oplus \epsilon_l \kappa_e(G_{l+1,P})$,

$E^0_n \in I^n_P(\mathcal{R}_l)$ being fixed.

Definition. A vector $e \in T_x$ is a characteritic vector of $\mathcal{R}_l$ at $P \in \mathcal{R}_l$ $\Leftrightarrow$

$$\dim D(e) > \dim\{\Sigma^1_{(P)} \oplus (\pi^l_{-1})^* H_e\} = s_0(E_n) + n - 1$$

a characteristic vector e **の意義：**

$\exists$ a covector $\omega \in D(e)$, $\omega \notin \Sigma^1_{(P)} \oplus (\pi^l_{-1})^* H_e$.

Such a covector $\omega \in D(e)$ is not a covector in $(\pi^l_{-1})^* T^*_x$. We have a new covector ω not belonging to $\Sigma^1_{(P)} \oplus (\pi^l_{-1})^* T^*_x$ which has the property that ω annihilates all vectors $v_e(E_n)$ $(E_n \in I^n_P(\mathcal{R}_l))$.

Let $P \in \mathcal{R}_l$, and M_P be the characteristic module of the symbol $G_{l,P}$ (We call it the characteristic module of $\mathcal{R}_l$ at P.); M_P is a submodule of the $R_P = \sum_{q=0}^{\infty} S^q T_x^*$-module $L_P = \sum_{q=0}^{\infty} S^q T_x^* \otimes V_y(\mathcal{E})$. Let

$$M_P = \cap_{j=1}^{\nu(P)} Q_{j,P} \quad (Q_{j,P} : \mathfrak{P}_{j,P}\text{-primary modules}) \tag{$\dagger$}$$

be an irredundant primary decomposition in the R_P-module L_P.

The above Lemma implies "e is characteristic $\Leftrightarrow$ $\kappa_e : G_{l+1,P} \longrightarrow G_{l,P}$ is not surjective." The dual operator of κ_e is related the morphism in L_P/M_P defined to be the multiplication by e. Thus we obtain the following

<u>Theorem</u>. Assume that $\mathcal{R}_l$ is involutive.

$\{$The set of the characteristic vectors of $\mathcal{R}_l$ at $P\} = \cup_{j=1,\ldots,\nu}\, c(\mathfrak{P}_j)$, where $c(\mathfrak{P}_j) = \mathfrak{P}_j \cap T_x$.

Remark. When $n = 2$, If there is a real characteristic covector ξ of $\mathcal{R}_l$, then there is a characteristic vector v of $\mathcal{R}_l$ such that $< \xi, v >= 0$, and the converse is valid.
In the case $n > 2$, there is no such simple relations between characteristic covectors and characteristic vectors.

<u>Definition</u>. A vector $v \in T_P(J_l(\mathcal{E}))$ is a <u>Monge characteristic vector of $\mathcal{R}_l$ at P</u> $\Leftrightarrow v$ belongs to some element $E_n \in I^n_P(\mathcal{R}_l)$ and $(\pi^l_{-1})_* v$ is a characteristic vector of $\mathcal{R}_l$ at P.

♣ Monge characteristic systems

Applications of Monge characteristics are made by using what is called Monge charactreristic systems, which are Pfaffian systems on (an open set of) $J_l(\mathcal{E})$ containing $\mathcal{R}_l$ constructed from the primary decomposition (†) of the characteristic module $M = \{M_P; P \in \mathcal{R}_l\}$.

Corresponding to each $\mathfrak{P}_{j,P}$, define

$$B(\mathfrak{P}_{j,P}) = \text{span}\ \{v_e(E_n); E_n \in I^n(\mathcal{R}_l), e \in c(\mathfrak{P}_{j,P})\} \subset T_P J_l(\mathcal{E})$$

$$D(\mathfrak{P}_{j,P}) = \mathrm{Ann}(B(\mathfrak{P}_{j,P})) \subset T^*_P J_l(\mathcal{E}).$$

Introduce: $x = \pi^l_{-1}(P), y = \pi^l_0(P)$

$$C(\mathfrak{P}_{j,P}) = \mathsf{span}\{\kappa_e(G_{l+1,P}); e \in c(\mathfrak{P}_{j,P})\},$$
$$\mathrm{Ann}\, C(\mathfrak{P}_{j,P}) \subset S^l T_x X \otimes V_y^* \mathcal{E}.$$

We have a direct sum decomposition:

$$B(\mathfrak{P}_{j,P}) = \{E_n \cap (\pi^l_{-1})_*^{-1} c(\mathfrak{P}_{j,P})\} \oplus \epsilon_l(C(\mathfrak{P}_{j,P})),$$

where E_n is a fixed element in $I^n(\mathcal{R}_l)$.

Lemma. (i) $D(\mathfrak{P}_{j,P}) \supset \Sigma^1_{(P)} + (\pi^l_{-1})^* c(\mathfrak{P}_{j,P})^\perp$.

(ii) $\dfrac{D(\mathfrak{P}_{j,P})}{\Sigma^1_{(P)} \oplus (\pi^l_{-1})^* c(\mathfrak{P}_{j,P})^\perp} \cong \dfrac{\mathrm{Ann}\, C(\mathfrak{P}_{j,P})}{M_{l,P}}$.

(iii)

Ann$C(\mathfrak{P}_{j,P}) = \{z \in L_l; c(\mathfrak{P}_{j,P})\, z \subset M_{l+1,P}\}$.

(iv) $[\mathrm{Ann}\, C(\mathfrak{P}_{j,P})] \overset{set}{=} \{\mathrm{Ann}\, C(\mathfrak{P}_{j,P})\}/M_{l,P}$. If each $c(\mathfrak{P}_{j,P})$ is not contained any one of the other $c(\mathfrak{P}_{k,P})\,(k \neq j)$, then the sum

$$[\mathrm{Ann}\, C(\mathfrak{P}_{1,P})] + \cdots + [\mathrm{Ann}\, C(\mathfrak{P}_{\nu,P})]$$

is a direct sum in the vector space$L_{l,P}/M_{l,P}$.

We assume that the following regularity conditions hold:
(a) The number $\nu = \nu(P)$ is constant on $P \in \mathcal{R}_l$, (b) For each $j = 1, \ldots, \nu$, the family $\{c(\mathfrak{P}_{j,P}); P \in \mathcal{R}_l\}$ defines a smooth subbundle $\mathbf{c}(\mathfrak{P}_j)$ of $(\pi^l_{-1}|_{\mathcal{R}_l})^{-1}TX$, (c) $\dim C(\mathfrak{P}_{j,P})$ is constant on $\mathcal{R}_l$.

- the family $\{D(\mathfrak{P}_{j,P}); P \in \mathcal{R}_l\}$ is a smooth vector bundle $\mathbf{D}(\mathfrak{P}_j)$ over $\mathcal{R}_l$.

Definition. The Monge characteristic system $\Delta(\mathfrak{P}_j)$ of $\mathcal{R}_l$ corresponding to $\mathfrak{P}_j$ is the Pfaffian system on $J_l(\mathcal{E})$ generated by all (smooth) sections ω of the cotangent bundle $T^*J_l(\mathcal{E})$ which satisfy $\omega(P) \in \mathbf{D}(\mathfrak{P}_j)$ for any point $P \in \mathcal{R}_l$.

- Each MCS $\Delta(\mathfrak{P}_j)$ contains the contact 1-forms on $J_l(\mathcal{E})$ and 1-forms dF with F being any function vanishing on $\mathcal{R}_l$, it contains other linearly independent 1-forms of which number can be calculated by using above Lemma.

♠ Application(**応用例**)

MCS**が最も自然に応用できる**SPDE$\mathcal{R}_l$ **は条件**

(H-1) $\mathcal{R}_l$ is involutive, and

$$s_1 > 0, s_2 = \cdots = s_n = 0$$

を満たすものである．具体的な結果を導くためには，　さらに次の(H-2) **を仮定する必要がある**:

(H-2) (a) The number $\nu = \nu(P)$ is constant on $P \in \mathcal{R}_l$, (b) $\mathfrak{P}_{j,P} L_P \subset Q_{j,P}$ $(j = 1, \ldots, \nu), P \in \mathcal{R}_l$,

(c) For each $j = 1, \ldots, \nu$, the family $\{c(\mathfrak{P}_{j,P}); P \in \mathcal{R}_l\}$ defines a smooth subbundle $\mathbf{c}(\mathfrak{P}_j)$ of $(\pi^l_{-1}|_{\mathcal{R}_l})^{-1}TX$ with rank $n-1$.

Note. Ξ_P is consists of s_1 real distinct points $\Rightarrow$ $\nu = s_1$, (H-2)-(i),(iii). (**この条件は** $\mathcal{R}_l$ **が** '**strictly hyperbolic**' **であることを意味する**)
これらの条件が満たされる $\mathcal{R}_l$ **に対し次の定理を**, Monge characteristics **を利用して証明することができる.**

<u>Theorem</u>(Existence of smooth solutions). $\mathcal{R}_l$ admits smooth local solutions.

The Proof consists of two steps:
(1) Using the Monge characteristic systems $\Delta(\mathfrak{P}_j)$, we construct a determined SPDE Φ, which is (non-linear) symmetrizable hyperbolic.
(ここに　定義を与える為に取り出したM.C.**の性質が使われる**)
(2) To show that solutions of $\mathcal{R}_l$ can be constructed by using solutions of Φ.

Application to the method of integration

(Extention of Darboux's method):
If $\nu - 1$ Monge characteritic systems $\Delta(\mathfrak{P}_j)\,(j = 1, \ldots, \nu - 1)$ admit sufficiently many functionally independent integrals, then solutions of $\mathcal{R}_l$ can be obtained by solving ordinary differential equations. (Here Monge characteritic systems $\Delta(\mathfrak{P}_j)\,(j = 1, \ldots, \nu - 1)$ are allowed to use those of prolongations $\mathcal{R}_{l+k}$. Darbuoux's method !)

This method is based on the following fact:
For a set $\mathcal{F}$ of function on $J_l(\mathcal{E})$, denote
$\mathcal{R}_l[\mathcal{F}] = \{P \in \mathcal{R}_l; F(P) = 0\,(F \in \mathcal{F})\}$.
Choose $k(\leq \nu)$ Monge characteristic systems, say, $\Delta(\mathfrak{P}_j)$ $(j = 1, \ldots, k)$. Let $\mathcal{F}$ be the union of $\{$a set of integrals $F_1^{(j)}, \ldots, F_{\mu_j}^{(j)}$ of $\Delta(\mathfrak{P}_j)\}$ $(j = 1, \ldots, k)$. Then $\mathcal{R}_l' = \mathcal{R}_l[\mathcal{F}]$ is an involutive system.

If there exist sufficiently many integrals of the $\nu - 1$ Monge characteristic systems, then for any initial condition (IC), we can consruct a SPDE $\mathcal{R}_l^\sharp$ contained in $\mathcal{R}_l$, satisfying the IC, and admitting Cauchy characteristics of dimension $n-1$. Solving $\mathcal{R}_l^\sharp$ gives a solution of $\mathcal{R}_l$ satisfying the given IC. If there do not exist enough integrals so as to apply this process, one can try the same procedure to the 1-th prolongation $\mathcal{R}_{l+1}$; and so on.

$\boxed{2^\circ}$ Monge characteristics of EDS Σ

残念ながら一般のEDS Σ**に対し**$\boxed{1^\circ}$**における同様の意味での**Monge characteristics**を導入することはできない.　しかし　次元**n**を指定し**Σ**を**prolongation**した**EDS $p\Sigma^n$**に対しては**　Monge characteristics **が導入できる．実際**　$p\Sigma^n$**は**$\Sigma(\mathcal{R}_l)$ **と同様の構造を持つからである．**

Cauchy characteristics of EDS Σ

Set $I_y\Sigma = \cup_{k=0}^{m} I_y^k\Sigma$.

Defintion. A tangent vector $v \in T_yM$ is a C-characteristic vector (C- means Cauchy and Cartan) of $\Sigma \Leftrightarrow$

$y \in I^0\Sigma$ and ⌈$E \in I_y\Sigma \Rightarrow$ span $\{v, E\} \in I_y\Sigma$⌋ .

In other words, $v \in T_yM$ is a C-characteristic vector if v belongs to all the spaces $H(E)$ where E runs through $E \in I_y\Sigma$.

Let $\mathcal{C}_y$ be the set of all C-characteristic vectors of origin y;

$$\mathcal{C}_y = \cap_{E \in I_y \Sigma} H(E).$$

- (a) The space $\mathcal{C}_y \subset T_y M$ is an integral element of Σ.

(b) $E \in I_y \Sigma \quad \Rightarrow$ span$\{\mathcal{C}_y, E\} \in I_y \Sigma$.
(c) If $E' \in I_y \Sigma$ satisfies span $\{E', E\} \in I_y \Sigma$ for $\forall E \in I_y \Sigma$, then $E' \subset \mathcal{C}_y$.

$\diamondsuit$ the notion of le systeme associe:
$\Sigma_{(y)} = \{\phi(y); \phi \in \Sigma\} \subset \wedge^* T_y^* M$ (an ideal)
<u>Definition</u>. Let $y \in I^0\Sigma$.

$$\mathcal{D}_y = \{v \in T_y M; v \lrcorner \phi \in \Sigma_{(y)} \text{ for all } \phi \in \Sigma_{(y)}\}.$$

<u>Lemma</u>. Let V^* be the smallest subspace of $T_y^* M$ such that $\Sigma_{(y)}$ is generated by $\Sigma_{(y)} \cap \wedge^*(V^*)$. Then the annihilator of V^* in $T_y M$ coincides with $\mathcal{D}_y$.

Theorem(E. Cartan). Assume that the family $\mathcal{D} = \{\mathcal{D}_y ; y \in M\}$ forms a subbundle of TM, equvivalently, $\dim \mathcal{D}_y$ is constant on M. Then the distribution $\mathcal{D}$ is completely integrable (in involution); that is, if X, Y are two (smooth) sections of $\mathcal{D}$, then so is the bracket $[X, Y]$.

• The inclusion $\mathcal{C}_y \subset \mathcal{D}_y$ holds without any assumption.

Lemma. If $\Sigma_{(y)}$ be generated by homogeneous elements of degree ≤ 2 as an algebraic ideal, then $\mathcal{C}_y = \mathcal{D}_y$

Theorem. Assume that $I^0\Sigma$ is a submanifold of M, and that, for any $y \in I^0\Sigma$, $\Sigma_{(y)}$ is generated by homogeneous elements of degree ≤ 2, and if the family $\mathcal{C} = \{\mathcal{C}_y; y \in M\}$ forms a subbundle of TM, then the distribution $\mathcal{C}$ is completely integrable.

♠ $\Sigma(\mathcal{R}_l)$ の C-characteristics

Assumption: $\mathcal{R}_l$ is involutive. $P \in \mathcal{R}_l$.

$M_P \subset L_P$: the characteristic module of $\mathcal{R}_l$.

Set $\mathfrak{m}_P = \{f \in R_P; fL_P \subset M_P\}$ (ideal of R_P).

$\mathcal{C}_P$=(the space of C-char. vectors of origin P).

Theorem. Assume that $\mathcal{R}_l$ is involutive.

$\mathcal{C}_P = E_n \cap (\pi^l_{-1})_*^{-1}(\mathfrak{m}_P \cap R_{1,x})$ ($E_n \in I^n(\mathcal{R}_l)$).

Corollary. $M_P = \cap_{j=1}^{\nu} Q_j$ ($Q_j : \mathfrak{P}_j-$primary): an IPD. Then $\mathcal{C}_P \subset \cap_{j=1,\dots,\nu} c(\mathfrak{P}_j)$. If exponent of $Q_j = 1\,(\forall j)$, $\mathcal{C}_P = \cap_{j=1,\dots,\nu} c(\mathfrak{P}_j)$.

Remark. $\Sigma(\mathcal{R}_l)$ に対し, v: C-char. $\Rightarrow$ v:Monge char.

Examples:(1) (**古典的例１）**１**未知関数**１**階の包合系**$\mathcal{R}_1$:

$F_j(x_i, \ldots, x_n, u, p_1, \ldots, p_n) = 0 (j = 1, \ldots, r)$ (**ここに** $F_j\,(j = 1, \ldots, r)$ **は　関数的に独立**)) **に付随する**$\Sigma(\mathcal{R}_1)$**は**, r **次元の**C-characteristics **を有する**. Cauchy characteristic vectors:

$$X_{F_j} = \sum_{i=1}^{n} (\frac{\partial F_j}{\partial p_i} \frac{d}{d\,x_i} - \frac{d\,F_j}{d\,x_i} \frac{\partial}{\partial p_i})\,(j = 1, \ldots, r).$$

(2) 2**変数の**1**未知関数に関する**2**階の包含系**
$\mathcal{R}_2 : r = 0, s = 0$ **に付随する** $\Sigma(\mathcal{R}_1)$ **は** 1**次元の**
C-characteristics **を有する**.

References

♠ Clasical theory of partial differential equations:

[1] Forsyth, A.R.; Theory of Diffferential Equations, Part IV, Vol. VI, Cambridge Univ. Press, London,1906.

[2] Goursat, E.; Leo̧ns sur l'intégrations des équations aux dérivées partielles du scond ordre *a* deux variables indépendentes, Tom.II, Hermann, Paris, 1898.

♠ The thory of involutive differential systems:

[3] Bryant, R.L., Chern, S.S., Gardner, R.B., Goldschmidt, H., and Griffiths, P.A.; Exterior Differential Systems, Springer, New York, 1991.

[4] Cartan, É.; Sur l'intégration des syst*e*mes d'équations aux différentielles totale, Ann. Sci. École Norm. Sup¿, 3^{e} série, 18(1901), 241-311.

[5] Cartan, É.; Les syst*e*mes différentiels extérieurs et leurs applications géométrique, Hermann, Paris, 1945.

[6] Goldschmidt, H.; Existence theorems for analytic linear partial differential equations, Ann. of Math., (2) 86 (1967), 246-270.
[7] Goldschmidt, H.; Integrability criteria for systems of nonlinear partial differential equations, J. Differential Geom., 1(1967), 269-307.
[8] Kähler, E.; Einführung in die Theorie der Systeme von Differentialgleichungen, Teubner, Leipzig, 1934.
[9] Kuranishi, M.; Lectures on involutive systems of partial differential equations, Publ. Soc. Mat. São Paulo, 1967.
[10] Matsuda, M.; The theory of exterior differential systems (in Japanese), Iwanami, Tokyo, 1976.
[11] Pommaret, J.F.; Systems of Partial Differential Equations and Lie Pseudogroups, Gordon and Breach, New York, 1978.
[12] Spencer, D.C.; Overdetermined systems of linear partial differential equations, Bull. of AMS, 75 (1969), 179-239.

♠ The contact system:

[13] Gardner R.B. and Shadwick W.F.; A simple characterization of the contact system on $J_k(E)$, Rocky Mount. J. Math., 17(1987), 19-21.

♠ Monge characteristics and their applications:

[14] Kakié, K.; On involutive systems of partial differential equations in two independent variables, J. Fac. Sci. Univ. Tokyo, Sect.IA, 21(1974), 405-433.

[15] Kakié, K.; The Cauchy problem for an involutive system of partial dfferential equations in two independent variables, J. Math. Soc. Japan, 27 (1975), 517-532.

[16] Kakié, K.; A fundamental property of Monge characteristics in involutive systems of non-linear partial differential equations and its application, Math. Ann., 273 (1985), 89-114.

[17] Kakié, K.; The Monge characteristics in involutive Pfaffian systems and its application to the Cauchy problem, Japanese J. of Math., 13 (1987), 127-162.

[18] Kakié, K.; On the integration of involutive systems of non-linear partial differential equations, J. Fac. Sci. Univ. Tokyo, Sect.IA, 36(1989), 537-569.

♠ Cauchy characteristics of $\Sigma(\mathcal{R}_L)$:

[19] Kakié, K.; Cauchy's characteristics of involutive systems of non-linear partial differential equations, Comm. Math. Univ. St. Pauli, 28(1979), 87-92.

♠ Existence of C^∞ solutions; [15],[16],[17], and

[20] Kakié, K.; Existence of smooth solutions of overdetermined elliptic differential equations in two independent variables, Comm. Math. Univ. St. Pauli, 48(1999), 181-210.

[21] Kakié, K.; Overdetermined elliptic systems of nonlinear differential equations in two independent variables, Comm. Math. Univ. St. Pauli, 57(2008), 91-136.

[22] MacKichan, B.; A generalization to overdetermined systems of the notion of diagonal operators, I. Elliptic operators, Acta Math., 126 (1971), 83-119; II. Hyperbolic operators, Acta Math., 134 (1975), 239-274.
[23] Yang, D.; Involutive Hyperbolic Differential Systems, Mem. of AMS, Vol.68, No.370, 1987.