Handle attaching in wrapped Floer homology

and

brake orbits in classical Hamiltonian systems.

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Wrapped Floer homology (HW)

(i) (M^{2n}, λ) : Liouville domain. $\underset{\text{def}}{\iff}$ $\begin{cases} \cdot\ (M, d\lambda) : \text{symplectic.} \\ \cdot\ \lambda \wedge (d\lambda)^{n-1} > 0 \text{ on } \partial M. \end{cases}$

M: cpt mfd, $\partial M \neq \phi$

$\lambda \in \Omega^1(M)$

(ii) $L^n \subset M$ $\begin{cases} \cdot\ L \pitchfork \partial M, \quad \partial L = L \cap \partial M. \\ \cdot\ \lambda|_L \equiv 0 \left(\begin{array}{l} \Rightarrow L : \text{Lagrangian of } (M, d\lambda) \\ \quad \partial L : \text{Legendrean of } (\partial M, \lambda) \end{array} \right) \end{cases}$

$\longrightarrow HW_*(\underbrace{M, \lambda, L}_{\text{Liouville triple.}})$

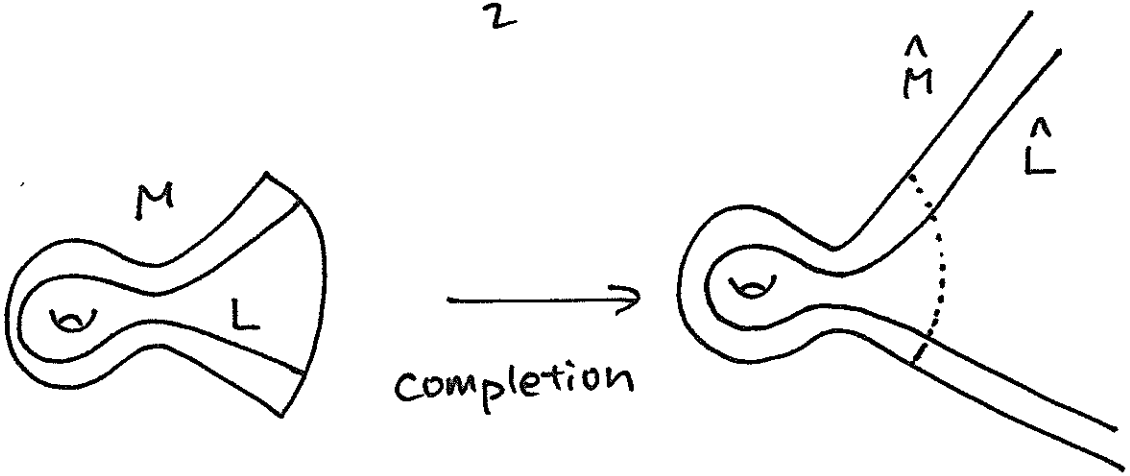
In the following, we assume $\pi_1(M, L) = \pi_2(M, L) = 0$.

Completion

$$\hat{M} := M \underset{i}{\cup} \partial M \times [1,\infty) \qquad \begin{array}{ccc} i : \partial M & \longrightarrow & \partial M \times \{1\} \\ \Psi & & \Psi \\ z & \longmapsto & (z, 1) \end{array}$$

$$\hat{\lambda} := \begin{cases} \lambda & (\text{on } M) \\ r \cdot \pi^* \lambda|_{\partial M} & (\text{on } \underset{r}{\partial M \times [1,\infty)}) \end{cases} \qquad \begin{array}{ccc} \pi : \partial M \times [1,\infty) & \longrightarrow & \partial M \\ \Psi & & \Psi \\ (z, r) & \longmapsto & z \end{array}$$

$$\hat{L} := L \underset{i}{\cup} \partial L \times [1,\infty)$$



$(\hat{M}, d\hat{\lambda})$: symplectic

$\hat{\lambda}|_{\hat{L}} \equiv 0.$

$$H \in C^\infty(M) \longrightarrow HW_*(H : M.\lambda.L)$$

H is assumed to satisfy the following:

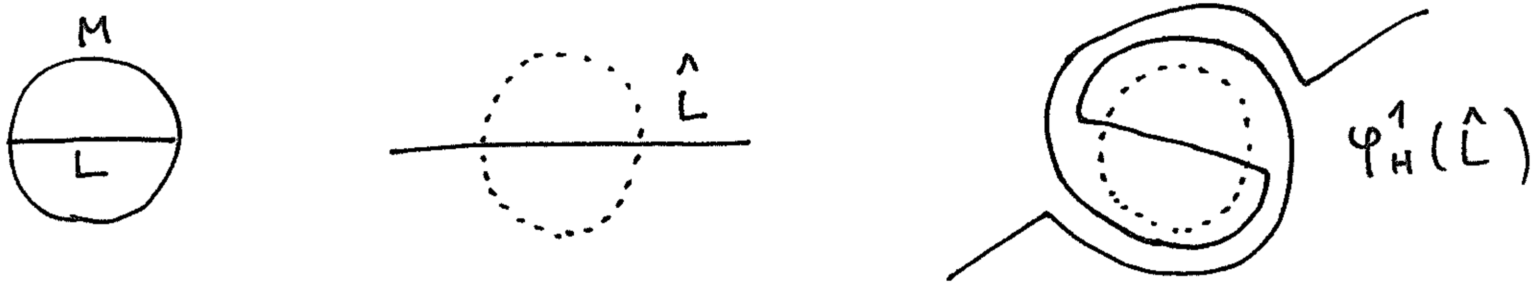
- $\exists r_0 > 1,\ H(z,r) = a_H \cdot r + b_H$ on $\underset{z}{\partial M} \times \underset{r}{[r_0,\infty)}$

  $\left(a_H \in \mathbb{R}_{>0} \setminus \left\{ \int_\gamma \lambda \;\middle|\; \begin{array}{l} \gamma : \text{Reeb chord of} \\ \partial L \text{ in } (\partial M, \lambda) \end{array} \right\} \right)$

- $\varphi_H^1(\hat{L}) \pitchfork \hat{L}$ (φ_H^1 is the time-1 map. gen'd by X_H).

$$HW_*(H : M.\lambda.L) := HF_*(\hat{L}, \varphi_H^1(\hat{L}))$$
$$:= H_*(CF_*(\hat{L}, \varphi_H^1(\hat{L})), \partial)$$

- CF_* : Free $\mathbb{Z}_2$-mod. gen'd over $\hat{L} \cap \varphi_H^1(\hat{L})$
- ∂ : counting J-hol. discs.



- $HW_*(H : M . \lambda . L)$ depends only on a_H. Hence we define.

$$HW_*^a(M.\lambda.L) := HW_*(H : M.\lambda.L) \quad (a_H = a)$$

- When $a \le b$, $\exists$ can. hom. $HW_*^a(M.\lambda.L) \to HW_*^b(M.\lambda.L)$

$$HW_*(M.\lambda.L) := \lim_{a \to \infty} HW_*^a(M.\lambda.L).$$

<u>Properties</u> (1) $(M.\lambda_s.L)_{0 \le s \le 1}$: smooth family of Liouville triple

$\Rightarrow HW_*(M.\lambda_s.L)$ does not depend on s.

(2) When $\delta > 0$ is suff. small, $\exists$ can. isom $H^{n-*}(L) \xrightarrow{\sim} HW_*^\delta(M.\lambda.L)$

(3) If $H^{n-*}(L) \simeq HW_*^\delta(M.\lambda.L) \to HW_*^a(M.\lambda.L)$

is not isom, then $\exists \gamma$: Reeb chord of ∂L in $(\partial M.\lambda)$ s.t

$\int_\gamma \lambda \le a$.

Cotangent bdl

N^n : C^∞ mfd

$q_1, \cdots, q_n$: loc. chart on N.

$p_1, \cdots, p_n$: chart on $T_q^* N$, w.r.t. $dq_1, \cdots, dq_n$.

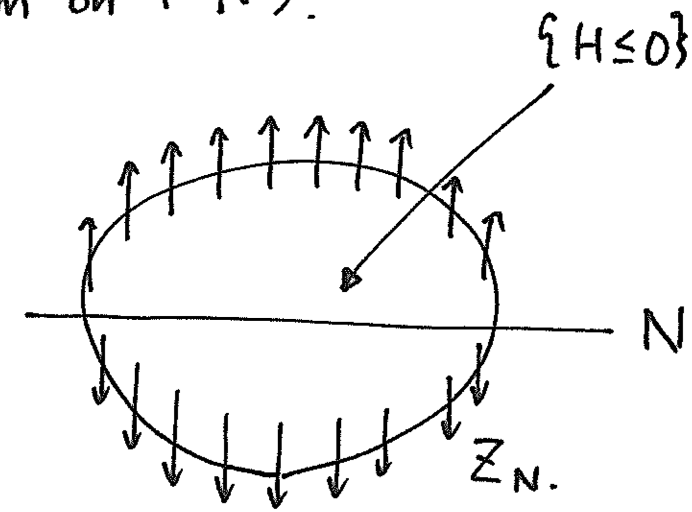
$\lambda_N := \sum_i p_i \, dq_i \in \Omega^1(T^*N)$

$Z_N := \sum_i p_i \frac{\partial}{\partial p_i} \in \mathfrak{X}(T^*N)$

$\omega_N := d\lambda_N$ (canonical symplectic form on T^*N).

Consider $H \in C^\infty(T^*N)$ s.t.

$$\begin{cases} \cdot \ \{H \le 0\} \subset T^*N \text{ is cpt.} \\ \cdot \ 0 \text{ is a reg. value of } H \ \& \ H|_N. \\ \cdot \ dH(Z_N) > 0 \text{ on } \{H = 0\} \setminus N. \end{cases}$$



<u>lem</u> (1) $\exists \lambda \in \Omega^1(T^*N)$ s.t.

$$\begin{cases} \cdot\ d\lambda = \omega_N \\ \cdot\ (\{H \leq 0\}, \lambda, \{H \leq 0\} \cap N) : \text{Liouville triple.} \end{cases}$$

(2) $HW_*(\text{———})$ depends only on diffeo type of $\{H \leq 0\} \cap N$ (in the following, denoted by $HW_*(\{H\leq 0\} \cap N)$).

<u>Thm</u> (Irie) $\{H \leq 0\} \cap N$ is connected & $\partial \neq \phi$

$\Rightarrow HW_*(\{H \leq 0\} \cap N) = 0$.

<u>Cor</u>. $\exists$ Reeb chord of $\{H = 0\} \cap N$ in $(\{H=0\}, \lambda)$ $(d\lambda = \omega_N)$

$\because$ $H^{n-*}(\{H=0\} \cap N) \to HW_*(\{H=0\} \cap N)$ is not isom. //

<u>Rem1</u>. The above cor. is very special case of the chord conj.

(cf. Mohnke, Holomorphic disks and the chord conjecture)

<u>Rem2</u>. The above cor was first shown by Bolotin in 1978.

as "existence thm of brake orbit".

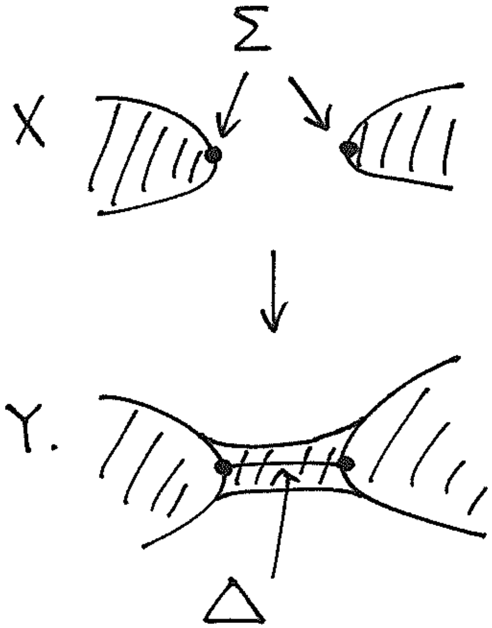
Pf. Step 1. $HW_*(D^n) = 0$ (easy)

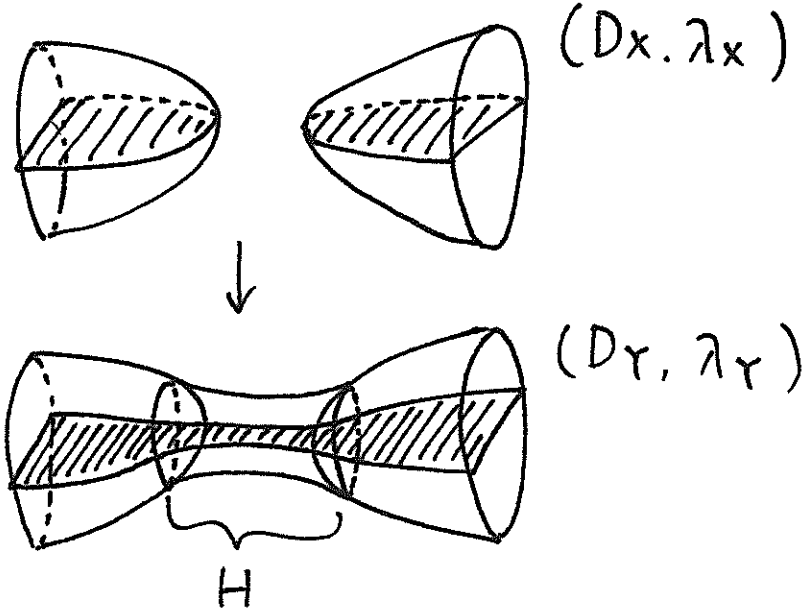
Step 2. If Y^n is obtained from X^n by attaching k-disc $(k<n)$, then $HW_*(Y) \simeq HW_*(X)$.

Step 2 is based on.

K. Cieliebak, Handle attaching in symplectic homology and the chord conjecture (2002).

$k=1$.





① We may assume (by perturbation of λ_X):

$\forall \gamma : [0, l] \to \partial D_X \in C(\partial X, \partial D_X, \lambda_X)$, ($\leftarrow$ Reeb chords of ∂X in $(\partial D_X, \lambda_X)$)

$\gamma(\{0, l\}) \cap \Sigma = \phi$.

② Consider seq. of handles $H = H_1 \supset H_2 \supset \cdots$ s.t. $\bigcap_{n=1}^{\infty} H_n = \Delta$.

$D_n := D_X \cup H_n \subset D_Y$. $Y_n := D_n \cap Y$. (D_n, λ_Y, Y_n): Liouville triple.

Then, $\forall a > 0$, $\exists n(a)$ s.t.

$$n \geq n(a) \Rightarrow C^{\leq a}(\partial Y_n, \partial D_n, \lambda_Y) = C^{\leq a}(\partial X, \partial D_X, \lambda_X)$$

$$\left(:= \{\gamma \in C(-) \mid \textstyle\int_\gamma \lambda_Y \leq a\}\right) \quad \cup \{\text{Reeb chords contained in } H_n\}.$$

③ From ②, we deduce the following:

$\forall a, N > 0$, $\exists n(a, N)$ s.t.

$$n \geq n(a, N) \Rightarrow HW_*^a(D_n, Y_n) \simeq HW_*^a(D_X, X)$$

$$(* \leq N)$$

④ By 'taking limit', we finally prove

$$HW_*(D_Y, Y) \simeq HW_*(D_X, X)$$

Viterbo functoriality

$m \le n \Longrightarrow \exists$ can. hom.

$(D_m \supset D_n)$ $\quad HW_*^a(D_m, Y_m) \to HW_*^a(D_n, Y_n)$

- $a \le b \Rightarrow$

$$\begin{tikzcd} HW_*^a(D_m, Y_m) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & HW_*^a(D_n, Y_n) \arrow[d] \\ HW_*^b(D_m, Y_m) \arrow[r] & HW_*^b(D_n, Y_n) \end{tikzcd}$$

- $HW_*(D_m, Y_m) \xrightarrow{\simeq} HW_*(D_n, Y_n)$

$$\begin{aligned} HW_*(D_Y, Y) &= \lim_{n\to\infty} HW_*(D_n, Y_n) \\ &= \lim_{n\to\infty}\left(\lim_{a\to\infty} HW_*^a(D_n, Y_n)\right) \\ &= \lim_{a\to\infty}\left(\lim_{n\to\infty} HW_*^a(D_n, Y_n)\right) \end{aligned}$$

③ + α $\longrightarrow$ $\overset{}{=} \lim_{a\to\infty} HW_*^a(D_X, X) = HW_*(D_X, X)$ //