Deformations and stabilities of singular Legendre varieties 2011.1.21

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(1)

M : C^∞ mfd, $D \subset TM$ subbundle

$f : N \to M$ is D-integral $\overset{\text{def}}{\Longleftrightarrow}$ $f_*(TN) \subset D$

f : not nec. immersion

$\Phi : M \to M$ D-diffeom $\overset{\text{def}}{\Longleftrightarrow}$ $\Phi_* D = D$

$$\boxed{\text{Fund. Prob.: Classify } D\text{-integral maps by } D\text{-diffeoms}}$$

$$\begin{array}{ccc} N & \xrightarrow{f} & M \\ \sigma \downarrow \wr\!\! | & \circlearrowleft & |\!\!\wr \downarrow \Phi \\ N' & \xrightarrow{f'} & M \end{array}$$

Φ : D-diffeo

param. change

f, f' D-diffeomorphic

We take narrow class of recc.

$D = TM$ diffeomorphic 10

Ex. $D = TM$

Ex. $M = J^1(\mathbb{R}^n, \mathbb{R})$, D : contact structure $D = \{dy - \sum p_i dx_i = 0\}$ ($x_1, \dots, x_n, y, p_1, \dots, p_n$)

$J^1(\mathbb{R}^n, \mathbb{R}^k)$ $k \geq 2$ D : Grassmann str. ($x_1, \dots, x_n, y_1, \dots, y_k, p_{ij}$) $dy_i = \sum p_{ij} dx_j$

$J^2(\mathbb{R}^n, \mathbb{R})$ D : 2nd order contact str. ($x_1, \dots, x_n, y, p_1, \dots, p_n$, $q_{ij} = q_{ji}$)

$J^l(N, M)$ $\exists 1$ $D \subset T(J^l(N,M))$ s.t. D-integral sect. of $\pi : J^l(N,M) \to N$ is a jet section $j^l f : N \to J^l(N,M)$ for some $f : N \to M$

Ex.

$\mathbb{R}^3 = J^1(\mathbb{R}, \mathbb{R})$ x, y, p $D = \{dy - p\,dx = 0\}$ contact

$f : \mathbb{R} \to \mathbb{R}^3$
t

$p(t) = t + \varphi(t)$ ord $\varphi \geq 2$ $(t^2 + \cdots)dt$

$x(t) = \frac{1}{2}t^2 + \psi(t)$ ord $\psi \geq 3$ $\int (t + \varphi(t))(t + \psi')\,dt$

$y = \int p\,dx = \frac{1}{3}t^3 + \rho(t)$ ord $\rho \geq 4$ $= \int (t^2 + t\psi' + t\varphi + \varphi\psi')\,dt$

$= \frac{t^3}{3} + \int (t\psi' + t\varphi + \varphi\psi')\,dt$

《Homotopy method》

$f_s(t) = (\frac{1}{2}t^2 + s\psi, \frac{1}{3}t^3 + s^2\rho, t + s\varphi)$

family of integral maps

$\forall s_0 \in \mathbb{R}$, Find

$\sigma_s : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ diffeo

$\tau_s : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ D-diffeo

s.t. $f_s = \tau_s^{-1} \circ f_{s_0} \circ \sigma_s$ near s_0 $\Rightarrow$ $f_1 \sim f_0 = f$

$\sigma_{s_0} = id$, $\tau_{s_0} = id$ $\left(\dfrac{d\sigma_s}{ds} = \xi_s \circ \sigma_s \right)$

$\sigma_{s(0)} = 0$ $\tau_{s(0)} = 0$

Infinitesimal cond.

$$\frac{df_s}{ds} = (\xi_s)_* f_s - \eta_s \circ f_s$$

ξ_s : vect. field over $(\mathbb{R}, 0)$ 20

η_s 〃 $(\mathbb{R}^3, 0)$

Mather's equation

D-vector field

(e.g. , $M = J^1(\mathbb{R}^n, \mathbb{R})$)

(M^{2n+1}, D) contact mfd, $D = \{\alpha = 0\}$

$f: N \to M$ int f_s: integral deformation of f $f_s^*\alpha = 0$

$v = \dfrac{df_s}{ds} : N \to TM$ $v^*\tilde{\alpha} = 0$ $\tilde{\alpha}$: 1-form on TM

Ex. $\alpha = dy - p\,dx$ x, y, p; $\dot{x}, \dot{y}, \dot{p}$

$$\tilde{\alpha} = d\dot{y} - \dot{p}\,dx - p\,d\dot{x}$$
$$= d(\dot{y} - p\dot{x}) - \dot{p}\,dx - \dot{x}\,dp$$

$VI_f = \{ v: N \to TM \mid v^*\tilde{\alpha} = 0 \}$ inf. int. defom. sp.

$$\begin{tikzcd} N \arrow[r, "v"] \arrow[dr, "f"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

$VH_M =$ (contact Hamiltonian vector fields over M)
$= \{ X: M \to TM \mid L_X \alpha = {}^{\exists}\mu\alpha \}$

$$\begin{tikzcd} M \arrow[r, "X"] \arrow[dr, equal] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

f: inf. contact stable $\overset{def}{\iff}$ $VI_f = f_*(V_N) + VH_M \circ f$

$M = J^1(\mathbb{R}^n, \mathbb{R}) \xrightarrow{\pi} J^0(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^{n+1}$

$VL_M = \{ X \in VH_M \mid X$ is π-lowerable $\}$

f: inf. Legendre stable $\overset{def}{\iff}$ $VI_f = f_*(V_N) + VL_M \circ f$

Th N: compact $f: N \to (M^{2n+1}, D)$ contact integral map of corank ≤ 1

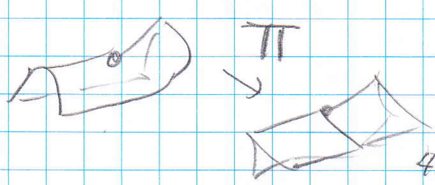
f: inf. contact stable $\iff$ f: contact stable

f: inf. Legendre stable $\iff$ f: Legendre stable

There exist exactly $[\frac{n}{2}]$-equiv. classes of contact stable integral map-germs of corank ≤ 1

$f_{n,k}: (\mathbb{R}^n, 0) \to (\mathbb{R}^{2n+1}, 0)$ $0 \leq k \leq [\frac{n}{2}]$

$f_{n,0}$ Legendre immersion, $f_{2,1} \to$



$f_{n,k} \underset{diffeo}{\not\sim} f_{n,k'}$ if $k \neq k'$

— module structure of VI_f —

$\mathcal{E}_M = (C^\infty$ functions on $M)$

$H \rightsquigarrow X_H$: contact Hamiltonian vector field char. by $i_{X_H}\alpha = H$ | $i_{X_1}\alpha = 1$

$X_{HK} = H\cdot X_K + K\cdot X_H - (HK)\cdot X_1$ $\quad X_1$: Reeb | $i_{X_1} d\alpha$

Define

$H * X_K := H\cdot X_K + K(X_H - H\cdot X_1)$

$X_K \circ f \in VI_f$

$H*(X_K\circ f) := (H\circ f)\cdot(X_K\circ f) + (K\circ f)(X_H - H\cdot K_1)\circ f$

Define in general for $v \in VI_f$ $\quad X_1 = \frac{\partial}{\partial y}$

$$\boxed{H * v := (H\circ f)\cdot v + (i_v\alpha)(X_H - H\cdot K_1)\circ f}$$

$\boxed{\text{Ex. } i_v\alpha = \dot{y} - p\dot{x}}$

Lemma VI_f is an $\mathcal{E}_M$-module

The $\mathcal{E}_M$-module structure is indep. of the choice of α

$v \in VI_f \qquad i_v\alpha \overset{\in \mathcal{E}_N}{:}$ generating func^n of v

$R_f := \{ h \in \mathcal{E}_N \mid dh \in \mathcal{E}_N\, d(f^*\mathcal{E}_M)\}$: $\mathcal{E}_M$-module

$$0 \to VI_f' \longrightarrow VI_f \longrightarrow R_f \to 0 \qquad \mathcal{E}_M\text{-exact}$$
$$v \longmapsto i_v\alpha$$

Ex $\quad i_v\alpha = \dot{y} - p\dot{x}$

$VI_f' = \{ v \in VI_f \mid i_v\alpha = 0,\ i_v\, d\alpha = 0\}$

$J^1(\mathbb{R}^n, \mathbb{R}^k)$

$\alpha_i = dy_i - \sum p_{ij}\, dx_j$

$1 \le i \le k$

$H * v =$

$(H\circ f)v + \sum_{i=1}^{k} (i_v\alpha_i)(X_H^i - HX_1^i)\circ f$

$H : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}$

$\exists$ similar structure for $J^1(\mathbb{R}^{n}, \mathbb{R}^k)$ $k \geqq 2$ (unpublished)

— Open problems

① Givental's conj: Any contact integral map $f : N^n \to M^{2n+1}$ is approximated by integral f' of corank $\leqq 1$.

② Generalization of Zhitomirskii th.

$f, g : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^{2n+1}, 0)$ contact integral

$n = 1$ $\underline{\underline{\text{Zhit}}}$ $f \underset{\text{diffeo}}{\sim} g \Rightarrow f \underset{\text{contact}}{\sim} g$ (with infinite codimensional exception)

$n \geqq 2$ $\quad f \underset{\text{diffeo}}{\sim} g \underset{?}{\Longrightarrow} f \underset{\text{cont}}{\sim} g$

③ Generalization of relative Darboux th.

$f, g : (\mathbb{R}^m, 0) \to (\mathbb{R}^{2n+1}, 0)$ C^∞ map

$f \underset{\text{diffeo}}{\sim} g,\ f^*\alpha \underset{\text{diffeo}}{\sim} g^*\alpha \underset{?}{\Longrightarrow} f \underset{\text{cont}}{\sim} g$