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FOLIATIONS ON THE OPEN 3-BALL BY COMPLETE SURFACES

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Abstract. When is a manifold a leaf of a complete closed foliation on the open unit ball? We give some answers to this question.

1. Introduction and statement of results. This paper is concerned with the topology of leaves of foliations. The concept of foliation appeared in the 1940's as a geometric approach to solutions of differential equations, and is now widespread in various areas such as complex analysis, exterior differential systems and contact topology (see e.g. [6, 11]). Recall that a codimension q C^r *foliation* $\mathcal{F}$ on an n-dimensional smooth manifold M is a decomposition $\{L_\lambda\}_{\lambda\in\Lambda}$ of M into a disjoint union of injectively immersed connected $(n-q)$-dimensional submanifolds L_λ satisfying the following local triviality: each point of M has a neighborhood U such that $\mathcal{F}$ restricted to U is C^r diffeomorphic to the family $\{\mathbb{R}^{n-q}\times\{y\}\}_{y\in\mathbb{R}^q}$ of parallel $(n-q)$-dimensional planes in $\mathbb{R}^n$. Each L_λ is called a *leaf* of $\mathcal{F}$. Note that, by collecting all the vectors tangent to leaves, the foliation can alternatively be defined as an integrable subbundle of TM.

In 1975, Sondow [16] posed a basic question: when is a manifold a leaf? This question is natural (because it generalizes the classical embedding problem in differential topology) and important (because it may be related to the study of the topology of integral manifolds of differential equations). Thus, it has been investigated extensively in various settings (see e.g. [7, 8, 9, 10, 12, 13]).

The purpose of this paper is to consider this question in an interesting new setting. Let $\mathcal{F}$ be a foliation on a Riemannian manifold (M,g). A leaf L of $\mathcal{F}$ is called *closed* if it is a closed subset of M (this is equivalent to saying that L is *properly embedded*), and *complete* if L is complete with respect to the induced Riemannian metric $g|L$. A foliation $\mathcal{F}$ is said to be *closed* (resp. *complete*) if all leaves of $\mathcal{F}$ are closed (resp. complete). Now, our setting is

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as follows: We fix, as the manifold which supports foliations, the open unit ball $\mathbb{B}^n$ of the Euclidean space $\mathbb{R}^n$ with the induced Euclidean metric—the simplest incomplete open manifold. And foliations we try to construct on $\mathbb{B}^n$ should be complete and closed. The novelty of this setting is to treat *complete* closed foliations on *incomplete* open manifolds. As one can imagine, in order to construct such foliations, one must "turbulize" all the leaves along the (ideal) boundary of $\mathbb{B}^n$.

The motivation of this work comes from recent deep works of Alarcón, Globevnik and Forstnerič [1, 2, 3]. They consider holomorphic foliations on the open ball of $\mathbb{C}^n$. Our work is, in a sense, a real smooth (C^∞) version of theirs. Since holomorphic objects are very "rigid", constructions of complete holomorphic foliations are much harder than those of real ones. The advantage of our approach is that, by forgetting holomorphic rigidity, one can concentrate on overcoming purely topological difficulties. In fact, on some topic we have thus succeeded in constructing infinitely many examples of foliations that are new in the literature (see Theorem 1.2 below).

Now, the first result of this paper is the following; its holomorphic version has been obtained by Alarcón and Globevnik [1, 3]. (Note that our result is independent of theirs, because of the difference of codimension. The codimension of our foliation is 1 (the most cramped codimension, see §6), while the *real* codimension of their foliations is at least 2.)

Theorem 1.1. *For any connected open orientable smooth surface Σ, there is a codimension 1 complete closed smooth foliation on $\mathbb{B}^3$ with a leaf diffeomorphic to Σ.*

Remark. In [9], Hector and Bouma showed the same statement on $\mathbb{R}^3$. In [10], Hector and Peralta-Salas generalized it in higher dimensions.

Remark. The corresponding result to Theorem 1.1 for Sondow's original question (i.e. the realization of manifolds as leaves of foliations on *compact* manifolds) was first obtained by Cantwell and Conlon [7]. For recent developments in this area, see e.g. [4, 13].

Remark. A non-orientable surface cannot be a leaf of a foliation on $\mathbb{B}^3$. In fact, if it can, the foliation must be transversely non-orientable. The existence of such a foliation contradicts the simple connectedness of $\mathbb{B}^3$.

Our next concern is a uni-leaf foliation. Here, we call a foliation $\mathcal{F}$ *uni-leaf* if all the leaves of $\mathcal{F}$ are mutually diffeomorphic.

Example. A complete closed uni-leaf foliation on $\mathbb{B}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ can easily be constructed as follows. Begin with the standard foliation $\mathcal{H}$ on $\mathbb{B}^2$ defined by $dy = 0$. Then, obviously, all leaves of $\mathcal{H}$ are diffeomorphic to the real line and closed in $\mathbb{B}^2$. Let h be a diffeomorphism of $\mathbb{B}^2$ defined by

$$h(r,\theta) = \left(r, \theta + \tan\frac{\pi r^2}{2}\right),$$

where (r,θ) are the polar coordinates. Then h sends any leaf ℓ of $\mathcal{H}$ to a complete curve $h(\ell)$ in $\mathbb{B}^2$, because each end of $h(\ell)$ spirals asymptotically on $\partial\mathbb{B}^2$. Hence, $h(\mathcal{H})$ is a foliation we have desired. Note that, since h is real-analytic (C^ω), so is $h(\mathcal{H})$.

So, let us consider uni-leaf foliations on $\mathbb{B}^3$. For a connected open orientable surface Σ, let $\mathcal{E}$ be the set of ends of Σ with the usual topology, and $\mathcal{E}^*$ the closed subset of $\mathcal{E}$ consisting of non-planar ends. It is known [15] that the pair $(\mathcal{E},\mathcal{E}^*)$ and the genus determine the homeomorphism type of Σ. It is also known that two smooth surfaces are diffeomorphic if and only if they are homeomorphic.

Now, we will introduce a new concept. We assume that the genus g of Σ is either 0 or ∞. Let e be a point of $\mathcal{E}$. Let Z be empty if $g=0$ and a countably infinite subset of $\mathcal{E}-\mathcal{E}^*-\{e\}$ if $g=\infty$. Suppose further that every point of Z is an isolated point of $\mathcal{E}$ and that the derived set of Z in $\mathcal{E}$ is $\mathcal{E}^*$. In this situation we say that the 4-tuple $(\mathcal{E},\mathcal{E}^*,Z,e)$ satisfies the *self-similarity property* if the following condition holds: there exist two copies $(\mathcal{E}^+,\mathcal{E}^{+*},Z^+,e^+)$, $(\mathcal{E}^-,\mathcal{E}^{-*},Z^-,e^-)$ of $(\mathcal{E},\mathcal{E}^*,Z,e)$ and a homeomorphism $h:\mathcal{E}^+\vee_{e^+=e^-}\mathcal{E}^-\to\mathcal{E}$ such that $h(e^\pm)=e$ and $h(Z^+\sqcup Z^-)=Z$ (hence, $h(\mathcal{E}^{+*}\vee_{e^+=e^-}\mathcal{E}^{-*})=\mathcal{E}^*$), where $\vee$ is the wedge sum.

Remark. This concept is not the same as the usual self-similarity in fractal geometry. Ours is a kind of *pointed* self-similarity, meaning that we fix a basepoint once and for all and then only consider subspaces containing the basepoint and mappings preserving the basepoint.

Example. (1) Let C be a Cantor set embedded in S^2. Then all ends of the surface $\Sigma_C = S^2 - C$ are planar, and the endset $\mathcal{E}$ of Σ_C is identified with C and hence has the self-similarity property. In fact, $\mathcal{E}$ can be expressed as $\mathcal{E}=\mathcal{E}^+\vee_e\mathcal{E}^-$, where $\mathcal{E}^+$ and $\mathcal{E}^-$ are subsets of $\mathcal{E}$ both homeomorphic to C such that $\mathcal{E}^+\cap\mathcal{E}^-=\{e\}$ for some $e\in\mathcal{E}$. In this case, Z is empty.

(2) Let J be the orientable open surface with one end and infinite genus (the so-called Jacob's ladder). We take in J a discrete infinite subset S and put $\Sigma_J = J - S$. Then the endset $\mathcal{E}$ of Σ_J consists of isolated planar ends e_n ($n\in\mathbb{N}$) each of which corresponds to a point of S and one non-planar end e to which e_n's converge. Thus, $\mathcal{E}=\{e\}\cup\{e_n\}_{n\in\mathbb{N}}$ and $\mathcal{E}^*=\{e\}$. See Fig. 1. Set $Z=\{e_n\}_{n\in\mathbb{N}}$. Then the 4-tuple $(\mathcal{E},\mathcal{E}^*,Z,e)$ satisfies the self-similarity property. Indeed, it suffices to define: $\mathcal{E}^+=\{e\}\cup\{e_{2n-1}\}_{n\in\mathbb{N}}$, $\mathcal{E}^-=\{e\}\cup\{e_{2n}\}_{n\in\mathbb{N}}$, $\mathcal{E}^{+*}=\mathcal{E}^{-*}=\{e\}$, $Z^\pm=\mathcal{E}^\pm-\{e\}$, $e^+=e^-=e$, and h as the identity.

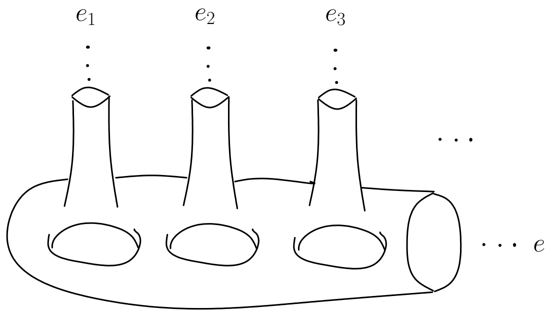

Fig. 1

We remark that there are many other surfaces whose endsets have the self-similarity property; we give examples at the end of §4.

The next theorem shows that infinitely many surfaces can be realized as leaves of uni-leaf foliations on $\mathbb{B}^3$.

Theorem 1.2. *Let Σ be a connected open orientable smooth surface with genus g either 0 or ∞, and let $(\mathcal{E}, \mathcal{E}^*)$ be its endset pair. Suppose that there exist a point e of $\mathcal{E}$ and a subset Z of $\mathcal{E} - \mathcal{E}^* - \{e\}$ such that*

(1) *Z is empty if $g = 0$ and countably infinite if $g = \infty$,*
(2) *every point of Z is an isolated point of $\mathcal{E}$,*
(3) *the derived set of Z in $\mathcal{E}$ is $\mathcal{E}^*$, and*
(4) *$(\mathcal{E}, \mathcal{E}^*, Z, e)$ satisfies the self-similarity property.*

Then there exists a codimension 1 complete closed smooth uni-leaf foliation of $\mathbb{B}^3$ having Σ as a leaf.

Remark. In the holomorphic situation, the existence of a uni-leaf foliation on the 2-dimensional holomorphic ball is known in the case where the leaf Σ is the disk $\{z \in \mathbb{C} \mid |z| < 1\}$ (see [2]). It seems that the problem remains open whether other Riemann surfaces can be leaves of some holomorphic uni-leaf foliations.

The following two arguments are crucial to proving our theorems:

(1) We build a kind of barrier in $\mathbb{B}^n$ (called a *labyrinth* in [2]) in order to force all leaves to become complete. The existence of a holomorphic labyrinth is a profound result. On the other hand, we find that a real labyrinth is quite easy to build (see §2).

(2) We show that the self-similarty property of the endset of a surface is a sufficient condition for the surface to be a leaf of some uni-leaf foliation. This is proved by a careful construction of a rather complicated submersive function on some domain of $\mathbb{R}^3$ (see §4).

We close this section with two more remarks.

REMARK. All foliations in this paper are C^∞. For foliations in Theorems 1.1 and 1.2 the authors have an idea of raising the differentiability to C^ω by using C^ω approximation of C^∞ Morse functions and diffeomorphisms. But, at present, they have not written up the proof in full precision yet.

REMARK. All foliations in this paper are closed, hence their holonomy pseudogroups are always trivial.

2. Constructing complete foliations on the ball. The content of this section is a real smooth version of the argument developed in [2].

Let $\{r_k\}$ and $\{s_k\}$ be sequences of real numbers satisfying $0 < s_1 < r_1 < s_2 < r_2 < \cdots \to 1$, and let B_k (resp. S_k) be the closed ball (resp. the sphere) in $\mathbb{R}^n$ centered at the origin with radius r_k (resp. s_k). Put $\Gamma_k = S_k - U_\varepsilon(p_k)$, where $0 < \varepsilon \ll s_1$, $p_k = (0, \ldots, 0, (-1)^k s_k)$ and $U_\varepsilon(p_k)$ is the open ε-neighborhood of p_k in $\mathbb{R}^n$. A path $\gamma : [0,\infty) \to M$ in an open manifold M is called *divergent* if $\gamma(t)$ leaves any compact subset of M as $t \to \infty$. Then the following is evident.

LEMMA 2.1. *Every divergent smooth path in $\mathbb{B}^n$ avoiding $\bigcup_{k\ge k_0} \Gamma_k$ (for some k_0) has infinite length.*

We denote $P_k = \mathbb{R}^{n-1} \times [-k,k]$. Let Ω be an open set of $\mathbb{R}^n$ diffeomorphic to $\mathbb{B}^n$ such that its image projected to the last coordinate of $\mathbb{R}^n$ is unbounded. Then we can choose an exhaustive sequence $\{C_k\}_{k\in\mathbb{N}}$ of subsets of Ω satisfying the following properties: (i) C_k is diffeomorphic to the closed ball, (ii) $C_k \subset \operatorname{Int} C_{k+1}$, (iii) $C_k \subset P_{k+1}$, and (iv) $C_k - P_k \neq \emptyset$.

LEMMA 2.2. *There exists a diffeomorphism Φ from Ω to $\mathbb{B}^n$ such that for all $k \in \mathbb{N}$:*

(1_k) *$\Phi(C_k) = B_k$, and*
(2_k) *$\Phi(P_k \cap C_k) \cap \Gamma_k = \emptyset$.*

Proof. We will enlarge the domain of definition inductively. First, define Φ on C_1 so that it satisfies (1_1) and (2_1). This is possible because, by (iv) above, $C_1 - P_1$ is non-empty and $\Gamma_1 \subset \operatorname{Int} B_1$. Next, suppose Φ has already been defined on C_k so as to satisfy (1_ℓ) and (2_ℓ) for $\ell \le k$. Since $\Gamma_{k+1} \subset B_{k+1} - B_k$ and, by (iv) above, $C_{k+1} - P_{k+1}$ is non-empty, it is possible to extend the definition of Φ on C_{k+1} so that $\Phi(C_{k+1} - P_{k+1}) \supset \Gamma_{k+1}$ and that $\Phi(C_{k+1} - C_k) = B_{k+1} - B_k$. Then we see that the resulting Φ satisfies (1_{k+1}) and (2_{k+1}). Since $\{C_k\}_{k\in\mathbb{N}}$ is an exhausting sequence in Ω, this inductive procedure gives a diffeomorphism from Ω to $\mathbb{B}^n$, as desired. ∎

LEMMA 2.3. *Let Φ be as in Lemma 2.2. Then $\Phi(P_k \cap \Omega) \cap \Gamma_k = \emptyset$ for every $k \in \mathbb{N}$.*

Proof. Since $\Gamma_k \subset B_k = \Phi(C_k)$, a point p of Ω satisfies $\Phi(p) \in \Gamma_k$ only if $p \in C_k$. Hence, by Lemma 2.2(2_k), $\Phi(P_k \cap \Omega) \cap \Gamma_k = \Phi(P_k \cap C_k) \cap \Gamma_k = \emptyset$. ■

Let $\mathcal{G}$ be a closed foliation on Ω (that is, every leaf of $\mathcal{G}$ is a closed subset of Ω). Then the direct image $\mathcal{F} = \Phi(\mathcal{G})$ is a closed foliation on $\mathbb{B}^n$. Here, we consider the following property (P) for $\mathcal{G}$:

(P) for any leaf L of $\mathcal{G}$ there exists $k \in \mathbb{N}$ such that $L \subset P_k$.

We recall that a leaf F of $\mathcal{F}$ is complete if and only if every divergent smooth path in F has infinite length. The next lemma gives a sufficient condition for the completeness of the leaves of $\mathcal{F}$.

LEMMA 2.4. *If $\mathcal{G}$ satisfies* (P)*, then all leaves of $\mathcal{F}$ are complete.*

Proof. Suppose $\mathcal{G}$ satisfies (P); let F be any leaf of $\mathcal{F}$. Put $L = \Phi^{-1}(F)$. Since L is a leaf of $\mathcal{G}$, by (P) there exists $k_L \in \mathbb{N}$ such that $L \subset P_{k_L}$. Then, recalling Lemma 2.3, for any $k \geq k_L$ we have $F \cap \Gamma_k = \Phi(L) \cap \Gamma_k \subset \Phi(P_{k_L} \cap \Omega) \cap \Gamma_k \subset \Phi(P_k \cap \Omega) \cap \Gamma_k = \emptyset$. Therefore, F does not intersect $\bigcup_{k \geq k_L} \Gamma_k$, hence, in particular, neither does any smooth path on F. This together with Lemma 2.1 implies the completeness of the leaves of $\mathcal{F}$. ■

We summarize the result obtained in this section as follows.

PROPOSITION 2.5. *Let $\mathcal{G}$ be a closed foliation on an open subset Ω of $\mathbb{R}^n$ diffeomorphic to $\mathbb{B}^n$ such that*

(1) $\mathrm{pr}_n(\Omega)$ *is unbounded, where* $\mathrm{pr}_n : \mathbb{R}^n \to \mathbb{R}$ *is the projection to the nth coordinate, and*
(2) *each leaf L of $\mathcal{G}$ has property* (P)*:* $\mathrm{pr}_n(L)$ *is bounded.*

Then there exists a diffeomorphism Φ from Ω to $\mathbb{B}^n$ such that $\Phi(\mathcal{G})$ is a complete closed foliation on $\mathbb{B}^n$.

3. Realizing open surfaces as leaves. First, we prepare an elementary result:

PROPOSITION 3.1. *Let W be a smooth open manifold and $\mathcal{C}$ be a countable family of injective smooth paths $c_k : [0,\infty) \to W$ $(k \in \mathbb{N})$ such that*

(1) *c_k is divergent for each k, and*
(2) *they are pairwise disjoint and the family $\mathcal{C}$ is locally finite.*

Then $W - \bigcup_k c_k([0,\infty))$ is diffeomorphic to W.

Proof. The reasoning consists in "pushing to infinity" each point $c_k(0)$ along the path c_k. Let $\dim W = n$ and give W an arbitrary Riemannian metric. Put $N = D^{n-1} \times [-1,\infty)$ (where D^{n-1} is the closed unit disk in $\mathbb{R}^{n-1}$ centered at the origin). Take a non-negative bounded smooth function $\lambda : N \to \mathbb{R}$ such that $\lambda = 1$ near ∂N and that for $(p,t) \in N$, $\lambda(p,t) = 0$ if and only if $p = 0$ and $t \in [0,\infty)$. Define a smooth vector field V on N

by $V = \lambda\frac{\partial}{\partial t}$, and let $\varphi : N \times [0,\infty) \to N$ be the (local) flow generated by V. Then the map $g : N \to N - (\{0\} \times [0,\infty))$ defined as $g(p,t) = \varphi((p,-1), t+1)$ is a diffeomorphism which is the identity near ∂N. Now, for each k, take a neighborhood N_k of $c_k([0,\infty))$ and a diffeomorphism $u_k : N \to N_k$ so that (1) N_k's are pairwise disjoint and the family is locally finite, (2) $u_k(0,t) = c_k(t)$ for $t \in [0,\infty)$, and (3) the diameter of $u_k(D^{n-1} \times \{t\})$ tends to 0 as $t \to \infty$. We then obtain the desired diffeomorphism $h : W \to W - \bigcup_k c_k([0,\infty))$ by setting $h = u_k \circ g \circ {u_k}^{-1}$ on N_k for any k, and the identity everywhere else. ■

Using almost the same argument, we can also show the following

PROPOSITION 3.2. *Let M be a smooth manifold, P a subset of M, and N a neighborhood of P in M. Put $W = M \times \mathbb{R}$. Let $\ell_p = \{p\} \times (-\infty, a_p]$ ($p \in P$, $a_p \in \mathbb{R}$) be a family of vertical lines in W. If $\bigcup_{p\in P} \ell_p$ is closed in W, then $W - \bigcup_{p\in P} \ell_p$ is diffeomorphic to W by a diffeomorphism preserving the fibers ($\{\{m\} \times \mathbb{R}\}_{m\in M}$) and equal to the identity outside $N \times \mathbb{R}$.*

Proof. It suffices to push $\bigcup_p \ell_p$ to $-\infty$ with respect to the $\mathbb{R}$-factor. To be precise, take a non-negative bounded smooth function $\mu : W \to \mathbb{R}$ which vanishes exactly on $\bigcup_p \ell_p$ and is constantly 1 outside $N \times \mathbb{R}$. Consider the flow $\psi : W \times \mathbb{R} \to W$ on W generated by the vector field $\mu\frac{\partial}{\partial z}$, where z is the coordinate of $\mathbb{R}$. Take a smooth function $\lambda : M \to \mathbb{R}$ such that $\lambda(p) > a_p$ for $p \in P$. Then the map $h : W \to W - \bigcup_p \ell_p$ defined by $h(m,t) = \psi((m,\lambda(m)), t - \lambda(m))$ is the desired diffeomorphism. ■

Note that in this proposition, (i) the local finiteness need not be assumed and (ii) the pushing-to-infinity operation can be carried out for an arbitrary small neighborhood N of P.

Proof of Theorem 1.1. Any connected open orientable surface Σ can be constructed as follows: First, remove from $\mathbb{R}^2$ a closed totally disconnected set X. (Then $X \cup \{\infty\}$ will be the endset $\mathcal{E}$ of Σ, where ∞ is the point at infinity of the one-point compactification of $\mathbb{R}^2$.) Next, take an at most countable set Z in $\mathbb{R}^2 - X$ in such a way that for any compact set K in $\mathbb{R}^2 - X$ the intersection $Z \cap K$ is finite. (The set of accumulation points of Z in $\mathbb{R}^2 \cup \{\infty\}$ will be $\mathcal{E}^*$.) Then, for each point q of Z, choose a small compact neighborhood U_q of q in $\mathbb{R}^2 - X$ so that they are pairwise disjoint, and in each U_q perform a surgery to attach a handle. The resulting surface is Σ. Observe that the entire construction above can be carried out in $(\mathbb{R}^2 - X) \times \mathbb{R}$. To do so, for each $q \in Z$ choose a small compact neighborhood V_q of $(q,0)$ in $(\mathbb{R}^2 - X) \times \mathbb{R}$, and perform ambient surgeries on $(\mathbb{R}^2 - X) \times \{0\}$ inside each V_q. Thus, we obtain Σ as a properly embedded submanifold of $(\mathbb{R}^2 - X) \times \mathbb{R}$. Note that Σ separates $(\mathbb{R}^2 - X) \times \mathbb{R}$ into two connected components.

We then take a Morse function $f : (\mathbb{R}^2 - X) \times \mathbb{R} \to \mathbb{R}$ such that

- $f(x, y, z) = z$ for $(x, y, z) \in (\mathbb{R}^2 - X) \times [(-\infty, -1] \cup [1, \infty)]$, and
- 0 is a regular value of f with $f^{-1}(0) = \Sigma$.

The existence of such f follows from the above construction of Σ. We let $\mathrm{Crit}(f)$ denote the set of critical points of f (which is a countably infinite set if Σ has non-planar ends). Now, we take an increasing sequence $\emptyset = K_0 \subset K_1 \subset \cdots$ of codimension 0 compact submanifolds in $\mathbb{R}^2 - X$ such that $\bigcup_{i=1}^{\infty} K_i = \mathbb{R}^2 - X$. For each $p \in \mathrm{Crit}(f)$, we can construct an injective smooth path $c_p : [0, \infty) \to (\mathbb{R}^2 - X) \times \mathbb{R}$ as follows. Suppose $p \in (K_i - K_{i-1}) \times \mathbb{R}$. Then

(1) $c_p(0) = p$,
(2) c_p intersects neither Σ nor $(\mathbb{R}^2 - X) \times \{\pm 1\}$,
(3) c_p does not intersect $K_{i-1} \times \mathbb{R}$,
(4) for each $j \geq i$, c_p intersects $\partial K_j \times \mathbb{R}$ transversely in exactly one point,
(5) $c_p(t)$ converges to a point in $(X \cup \{\infty\}) \times \{\pm 1/2\}$ as $t \to \infty$, and
(6) if $p \neq q$, then $c_p([0, \infty))$ and $c_q([0, \infty))$ are disjoint.

We denote by M the space obtained from $(\mathbb{R}^2 - X) \times \mathbb{R}$ by removing $\bigcup_{p \in \mathrm{Crit}(f)} c_p([0, \infty))$. Then applying Proposition 3.1 for $W = (\mathbb{R}^2 - X) \times \mathbb{R}$ and $\mathcal{C} = \{c_p\}$ we see that M is diffeomorphic to $(\mathbb{R}^2 - X) \times \mathbb{R}$.

Next, define an open subset Ω of $\mathbb{R}^3$ to be the union of M and $\mathbb{R}^2 \times (2, \infty)$. By the above argument, Ω is diffeomorphic to $[(\mathbb{R}^2 - X) \times \mathbb{R}] \cup [\mathbb{R}^2 \times (2, \infty)]$. Thus, by Proposition 3.2 we can push $X \times (-\infty, 2]$ to $-\infty$ with respect to the second coordinate and obtain Ω diffeomorphic to $\mathbb{B}^3$.

Now, we extend the domain of our Morse function f to Ω by defining f to be the projection to the second factor on $\mathbb{R}^2 \times (2, \infty)$. We let $\mathcal{G}$ denote the foliation on Ω whose leaves are connected components of the level sets of f. Then $\mathcal{G}$ has no singularities because all the critical points of f are removed from Ω. It is also obvious that all leaves of $\mathcal{G}$ are closed in Ω. By construction, $\mathcal{G}$ satisfies conditions (1) and (2) of Proposition 2.5. Therefore, $\mathcal{G}$ is diffeomorphic to a complete closed foliation on $\mathbb{B}^3$ containing Σ as a leaf. Theorem 1.1 is proved. ∎

4. Uni-leaf foliations. In this section we consider the question: which manifold is a leaf of a complete closed uni-leaf foliation on the open unit ball? This question was first asked by Alarcón and Forstnerič [2] in the holomorphic category. They showed that for any integer $n > 1$, there exists a complete closed holomorphic uni-leaf foliation of the open unit ball in $\mathbb{C}^n$ with disks as leaves. We work in the real smooth category and prove Theorem 1.2.

We first treat two simple cases: Σ_C and Σ_J given in §1. We will realize each of them as a leaf of a complete closed smooth uni-leaf foliation of $\mathbb{B}^3$. (Then the full proof of Theorem 1.2 will be understood as an elaboration of these cases.)

Example. Let C, Σ_C, $\mathcal{E}$, $\mathcal{E}^\pm$ and e be as in Example (1) of §1. Through the identification of $S^2 - \{e\}$ with $\mathbb{R}^2$, we regard Σ_C, $\mathcal{E} - \{e\}$ and $\mathcal{E}^\pm - \{e\}$ as subsets of $\mathbb{R}^2$. Now, put

$$\Omega = \mathbb{R}^3 - (\mathcal{E}^+ - \{e\}) \times [-1, \infty) - (\mathcal{E}^- - \{e\}) \times (-\infty, 1].$$

Then, by Proposition 3.2, Ω is diffeomorphic to $\mathbb{R}^3$. We denote by $\mathcal{G}$ the foliation on Ω obtained by restricting the foliation $\{\mathrm{pr}_3^{-1}(z)\}_{z\in\mathbb{R}}$ on $\mathbb{R}^3$. Then all leaves of $\mathcal{G}$ are diffeomorphic to Σ_C. In fact, this is obvious when $|z| \leq 1$. When $z > 1$ (resp. $z < -1$), see that $\mathrm{pr}_3^{-1}(z) \cap \Omega$ is diffeomorphic to $\mathbb{R}^2 - (\mathcal{E}^+ - \{e\})$ (resp. $\mathbb{R}^2 - (\mathcal{E}^- - \{e\})$). But, since we are assuming that $\mathcal{E}^\pm$ are homeomorphic to $\mathcal{E}$, $\mathrm{pr}_3^{-1}(z) \cap \Omega$ is diffeomorphic to Σ_C also in this case. Finally, since $\mathcal{G}$ satisfies conditions (1) and (2) of Proposition 2.5, $\mathcal{G}$ is diffeomorphic to a complete closed uni-leaf foliation on $\mathbb{B}^3$, as desired.

Example. We first embed Jacob's ladder J in $\mathbb{R}^3$. Let $H = \mathbb{R}^2 \times \{0\}$ and $0 < \varepsilon \ll 1$. For each $n \in \mathbb{Z} - \{0, \pm 1\}$, choose a small neighborhood U_n of $(n, 0)$ in $\mathbb{R}^2$. We put $W_n^+ = U_n \times (-1 - \varepsilon, n + \varepsilon)$ $(n \geq 2)$ and $W_n^- = U_n \times (n - \varepsilon, 1 + \varepsilon)$ $(n \leq -2)$. Inside each $W_n^\pm$ we perform an ambient surgery on H to attach a handle. Thus, we obtain a new surface embedded in $\mathbb{R}^3$ and diffeomorphic to J. Hereafter, we identify this surface with J. Next, we put $\ell_n^+ = \{(n, 0)\} \times [-1, \infty)$ $(n \geq 2)$ and $\ell_n^- = \{(n, 0)\} \times (-\infty, 1]$ $(n \leq -2)$. We can take a Morse function $f : \mathbb{R}^3 \to \mathbb{R}$ satisfying the following conditions (by isotoping J suitably in $\bigcup W_n^+ \cup \bigcup W_n^-$ if necessary):

(1) 0 is a regular value of f and $f^{-1}(0) = J$.
(2) $f(x, y, z) = z$ outside the union of W_n^+'s and W_n^-'s.
(3) The critical points of f are $A_n^+ = (n, 0, -1)$, $B_n^+ = (n, 0, n)$ $(n \geq 2)$ and $A_n^- = (n, 0, n)$, $B_n^- = (n, 0, 1)$ $(n \leq -2)$; their critical values are their z-coordinates. If we pass through A_n^+ or A_n^- (resp. B_n^+ or B_n^-), in the direction of increasing values of f, then the level set of f is modified so that a handle is created (resp. erased).
(4) Inside each $W_n^\pm$, f is a standard Morse function admitting a canceling pair of critical points.
(5) On each $\ell_n^\pm$, f is strictly increasing with respect to z.

Let $\mathcal{H}$ denote the foliation (with singularity) on $\mathbb{R}^3$ with the level sets of f as leaves. Then every regular leaf $f^{-1}(z)$ of $\mathcal{H}$ is diffeomorphic to J. In fact, if $|z| < 1$, this is obvious. If $|z| > 1$, then $f^{-1}(z)$ loses "half" of the infinitely many handles in comparison with $f^{-1}(0)$. But it still has infinite genus, hence, is diffeomorphic to J. We can also observe that any singular leaf of $\mathcal{H}$ has infinite genus. Now, denote by Ω the set obtained from $\mathbb{R}^3$ by removing the infinite family of half-lines $\ell_n^{\pm}$. Then it follows from Proposition 3.1 that Ω is diffeomorphic to $\mathbb{R}^3$. Since all the critical points are removed, $\mathcal{H}$ restricted to Ω becomes a non-singular foliation, say $\mathcal{G}$. We will check the topology of leaves of $\mathcal{G}$. First, we recall that the diffeomorphism type of Σ_J is characterized by being orientable and having infinite genus, one non-planar end (say e) and a countably infinite sequence of isolated planar ends converging to e. Note that each leaf $L = f^{-1}(z) \cap \Omega$ of $\mathcal{G}$ is obtained from the leaf $H = f^{-1}(z)$ of $\mathcal{H}$ by removing a countably infinite set of discrete points $H \cap \ell_n^{\pm}$. Therefore, if H is a regular leaf (hence diffeomorphic to J), then L is diffeomorphic to Σ_J. When H is a singular leaf, we have to notice that by the removal of one singular point from H, two punctures (i.e. planar ends) are produced on L. But, anyway, planar ends of L are countably infinite in number and each of them is isolated. Therefore, L is diffeomorphic to Σ_J also in this case. Consequently, all the leaves of $\mathcal{G}$ are diffeomorphic to Σ_J. As a final step, we take a diffeomorphism $\Psi : \mathbb{R}^3 \to V$, where $V = \{(x, y, z) \in \mathbb{R}^3 \mid (x^2 + y^2)|z| < 1\}$, such that Ψ preserves the one-dimensional foliation $dx = dy = 0$ leafwise. If we set $\widehat{\Omega} = \Psi(\Omega)$, we can see that the foliation $\Psi(\mathcal{G})$ on $\widehat{\Omega}$ has property (P) of §2, while $\widehat{\Omega}$ is unbounded in the direction of z. Consequently, by Proposition 2.5, we obtain a complete closed uni-leaf foliation on $\mathbb{B}^3$ with all leaves diffeomorphic to Σ_J.

Proof of Theorem 1.2. Let Σ be a connected open orientable smooth surface and $(\mathcal{E}, \mathcal{E}^*)$ the endset pair of Σ. We use the notation of §1. We assume that there exist e and Z as in Theorem 1.2 such that $(\mathcal{E}, \mathcal{E}^*, Z, e)$ has the self-similarity property. (Here, we should recall that if Σ has no genus, then $\mathcal{E}^*$ and Z are empty.) Put $X = \mathcal{E} - \{e\}$ and $X^{\pm} = \mathcal{E}^{\pm} - \{e^{\pm}\}$. Via h, we regard $\mathcal{E}^{\pm}$, $X^{\pm}$ and $Z^{\pm}$ as subsets of $\mathcal{E}$, X and Z respectively.

Now, we will start the construction of the uni-leaf foliation. We embed $\mathcal{E}$ into the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$ of $\mathbb{R}^2$ in such a way that e is mapped to ∞. From now on, we identify e with ∞ and regard X and Z as subsets of $\mathbb{R}^2$. We consider two cases separately.

The case where Z is empty. In this case, the construction is a verbatim repetition of the one in the case of Σ_C (S^2 minus a Cantor set): Namely, put

$$\Omega = \mathbb{R}^3 - X^+ \times [-1, \infty) - X^- \times (-\infty, 1]$$

and let $\mathcal{G}$ denote the foliation on Ω obtained by restricting the foliation $\{\mathrm{pr}_3^{-1}(z)\}_{z \in \mathbb{R}}$ on $\mathbb{R}^3$. Then, by Proposition 3.2, Ω is diffeomorphic to $\mathbb{R}^3$ and, by the self-similarity condition, all leaves of $\mathcal{G}$ are diffeomorphic to Σ. Finally, by Proposition 2.5, we can conclude that $\mathcal{G}$ is diffeomorphic to a complete closed uni-leaf foliation on $\mathbb{B}^3$.

The case where Z is countably infinite. Since $X - Z$ is closed in $\mathbb{R}^2$, similarly to §3, for each point q of Z, we can choose a small compact neighborhood V_q of $(q, 0)$ in the open 3-manifold $(\mathbb{R}^2 - (X - Z)) \times \mathbb{R}$ in such a way that they are pairwise disjoint. Then perform an ambient surgery on $(\mathbb{R}^2 - (X - Z)) \times \{0\}$ to attach a handle inside each V_q. Thus, we obtain a new surface as a properly embedded submanifold of $(\mathbb{R}^2 - (X - Z)) \times \mathbb{R}$. Let $\widehat{\Sigma}$ denote this surface. We see that the endset pair of $\widehat{\Sigma}$ is $(\mathcal{E} - Z, \mathcal{E}^*)$. We may assume that for each $q \in Z$ the intersection of $\widehat{\Sigma}$ and $\{q\} \times \mathbb{R}$ is a single point.

By the self-similarity property, X and Z can be expressed as disjoint unions $X = X^+ \sqcup X^-$ and $Z = Z^+ \sqcup Z^-$, where $X^{\pm}$ and $Z^{\pm}$ are homeomorphic to X and Z respectively. We put $A^+ = (X^+ - Z^+) \times [-1, \infty)$, $A^- = (X^- - Z^-) \times (-\infty, 1]$ and

$$O = \mathbb{R}^3 - A^+ - A^-.$$

Note that $X^+ - Z^+$ and $X^- - Z^-$ are closed in $\mathbb{R}^2$, and hence A^+ and A^- are closed in $\mathbb{R}^3$. We number the elements of Z arbitrarily: $Z^+ = \{q_n \mid n = 2, 3, 4, \ldots\}$ and $Z^- = \{q_n \mid n = -2, -3, -4, \ldots\}$. We then take a Morse function $f : O \to \mathbb{R}$ satisfying the following six conditions.

(1) $f^{-1}(0) = \widehat{\Sigma}$.
(2) $\mathrm{Crit}(f)$ consists of the following points:
- $(q_n, -1 - 1/n)$ and (q_n, n) for each $n = 2, 3, 4, \ldots$,
- (q_n, n) and $(q_n, 1 - 1/n)$ for each $n = -2, -3, -4, \ldots$.

(3) For each $p \in \mathrm{Crit}(f)$, the value $f(p)$ is the z-coordinate of p.

Let $W_q^+ = D_q^+ \times I_q^+$ $(q \in Z^+)$ be a compact product neighborhood of the segment $\{q\} \times [-1-1/n, n]$ in O, where D_q^+ is a closed disk in $\mathbb{R}^2$ centered at q such that $D_q^+ \cap X = \{q\}$ and $[-1 - 1/n, n] \subset I_q^+ \subset \mathbb{R}$. Similarly, let $W_q^- = D_q^- \times I_q^-$ $(q \in Z^-)$ be a compact product neighborhood of $\{q\} \times [n, 1 - 1/n]$ in O, where D_q^- is a closed disk in $\mathbb{R}^2$ centered at q such that $D_q^- \cap X = \{q\}$ and $[n, 1 - 1/n] \subset I_q^- \subset \mathbb{R}$. We choose the sets D_q^+ $(q \in Z^+)$ and D_q^- $(q \in Z^-)$ so as to be pairwise disjoint.

(4) Inside each W_q^+ $(q \in Z^+)$ or W_q^- $(q \in Z^-)$, f is conjugate to the standard Morse function which admits a standard canceling pair of critical points; the one with a smaller z-coordinate is of index 1 and the other is of index 2.
(5) The lines $\{q\} \times [-1-1/n, \infty)$ $(q \in Z^+)$ and $\{q\} \times (-\infty, 1-1/n]$ $(q \in Z^-)$ are transverse to the level sets of f everywhere except at critical points.
(6) $f(x, y, z) = z$ outside the union of W_q^+'s $(q \in Z^+)$ and W_q^-'s $(q \in Z^-)$.

Then the family of the level sets of f defines a singular foliation on O. The singularities are the critical points of f. We see that each level set contains at most one critical point. We can also observe that for each $z \in \mathbb{R}$ the endset pair $(\mathcal{E}_z, \mathcal{E}_z^*)$ of the level set $f^{-1}(z)$ is identified with: $(\mathcal{E} - Z, \mathcal{E}^*) \times \{z\}$ if $|z| \le 1$, $(\mathcal{E}^+ - Z^+, \mathcal{E}^{+*}) \times \{z\}$ if $z > 1$, and $(\mathcal{E}^- - Z^-, \mathcal{E}^{-*}) \times \{z\}$ if $z < -1$. By self-similarity, all of these are homeomorphic to $(\mathcal{E} - Z, \mathcal{E}^*)$.

As the next step, we define

$$C = \bigcup_{q_n \in Z^+} \{q_n\} \times [-1 - 1/n, \infty) \cup \bigcup_{q_n \in Z^-} \{q_n\} \times (-\infty, 1 - 1/n]$$

and

$$\varOmega = O - C.$$

By Propositions 3.1 and 3.2, $\varOmega$ is diffeomorphic to $\mathbb{R}^3$. Each level set $L_z = f^{-1}(z) \cap \varOmega$ of $f|_\varOmega$ is obtained from $f^{-1}(z)$ by deleting the points of intersection with C. Since all the critical points of f are removed by this deletion, every L_z is now a non-singular smooth surface. Let $\mathcal{G}$ be the foliation on $\varOmega$ thus obtained. Observe that if the point of intersection of $f^{-1}(z)$ and $\{q\} \times I$ ($q \in Z$, and I is either $[-1-1/n, \infty)$ or $(-\infty, 1-1/n]$) is not a critical point, then the deletion yields one puncture (or one planar end) on $f^{-1}(z)$, while if the point of intersection is a critical point, then the deletion yields two punctures (or two planar ends). Now, let Z_z be the set of all ends of L_z newly produced by these deletions. Then the endset pair of L_z can be expressed as $(\mathcal{E}_z \cup Z_z, \mathcal{E}_z^*)$, where $(\mathcal{E}_z, \mathcal{E}_z^*)$ is the endset pair of $f^{-1}(z)$. Since, as remarked above, each $f^{-1}(z)$ contains at most one critical point, it follows from property (2) of f that Z_z is identified with: Z if $|z| \le 1$, and the union of Z^+ and F_z if $z > 1$, and the union of Z^- and F_z if $z < -1$, where F_z is a (possibly empty) finite subset of $\mathbb{R}^2 - X$. (Supplementary explanation: If $z \ge 2$, then $f^{-1}(z)$ does not intersect $\{q\} \times (-\infty, 1-1/n]$ for any $q \in Z^-$. So, in this case, F_z is either a singleton or empty depending on whether there exists a critical point on $f^{-1}(z) \cap \{q\} \times [-1 - 1/n, \infty)$ for some $q \in Z^+$. If $1 < z < 2$, we see that $f^{-1}(z)$ intersects $\{q\} \times (-\infty, 1 - 1/n]$ for at most finitely many $q \in Z^-$.) Therefore, $(\mathcal{E}_z \cup Z_z, \mathcal{E}_z^*, Z_z, \infty)$ is identified with $(\mathcal{E}, \mathcal{E}^*, Z, \infty) \times \{z\}$ if $|z| \le 1$, $(\mathcal{E}^+ \cup F_z, \mathcal{E}^{+*}, Z^+ \cup F_z, \infty) \times \{z\}$ if $z > 1$, and $(\mathcal{E}^- \cup F_z, \mathcal{E}^{-*}, Z^- \cup F_z, \infty) \times \{z\}$ if $z < -1$.

Lemma 4.1. *If F is a finite subset of $\mathbb{R}^2 - X$, then there is a homeomorphism $h : \mathcal{E} \cup F \to \mathcal{E}$ such that h is the identity on $\mathcal{E}^*$ and $h(Z \cup F) = Z$.*

Proof. Let F be $\{x_1, \ldots, x_r\}$. Take a point p in $\mathcal{E}^*$ and any sequence $\{p_i\}_{i=1}^\infty$ in Z converging to p. We define a bijection $h : \mathcal{E} \cup F \to \mathcal{E}$ by: $h(x_k) = p_k$ for $k = 1, \ldots, r$, $h(p_i) = p_{r+i}$ for $i \ge 1$, and h is the identity otherwise. Then the continuity of h easily follows. ■

By this lemma and the self-similarity property, the 4-tuple $(\mathcal{E}_z \cup Z_z, \mathcal{E}_z^*, Z_z, \infty)$ for the leaf L_z is homeomorphic to $(\mathcal{E}, \mathcal{E}^*, Z, \infty)$ for every $z \in \mathbb{R}$. Hence, all the leaves L_z of $\varOmega$ are diffeomorphic to $\varSigma$.

As the final step, we will transform the foliation $(\Omega, \mathcal{G})$ so that the resulting foliation has property (P). To do so, take any $(x_0, y_0) \in \mathbb{R}^2 - X$ and put $V = \{(x,y,z) \in \mathbb{R}^3 \mid ((x-x_0)^2 + (y-y_0)^2)|z| < 1\}$. Next, choose any diffeomorphism $\Psi : \mathbb{R}^3 \to V$ which preserves the vertical foliation $dx = dy = 0$ leafwise. Then we can see that the foliation $\Psi(\mathcal{G})$ on $\Psi(\Omega)$ has property (P) from §2, while $\Psi(\Omega)$ is unbounded in the direction of z. Therefore, by Proposition 2.5, we obtain a complete closed uni-leaf foliation on $\mathbb{B}^3$ with leaves diffeomorphic to Σ. This completes the proof of Theorem 1.2. ∎

Examples of surfaces with the self-similarity property. In the case of planar surfaces, $\mathcal{E}^*$ and Z are empty, hence, to check self-similarity, we only have to show that $\mathcal{E}_1 \vee_{e_1=e_2} \mathcal{E}_2$ is homeomorphic to $\mathcal{E}$ for some $e \in \mathcal{E}$, where $(\mathcal{E}_i, e_i)$, $i = 1, 2$, are copies of $(\mathcal{E}, e)$. The following surfaces have that property: $\mathbb{R}^2$, $\mathbb{R}^2$ minus a discrete closed infinite set, $\mathbb{R}^2$ minus a Cantor set, and S^2 minus a Cantor set.

In the case of non-planar surfaces, there are also many examples. Here, we give one family of surfaces $\Sigma(r)$, $r \in \mathbb{N}$ (Example (2) in §1 is $\Sigma(1)$).

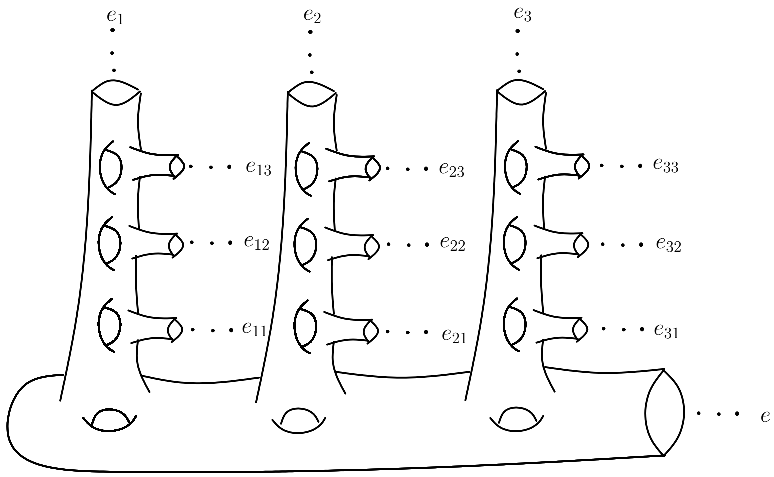

Fig. 2. $\Sigma(2)$

The endset pair $(\mathcal{E}, \mathcal{E}^*)$ of $\Sigma(r)$ is described as follows:

$$\begin{aligned} \mathcal{E} &= \{e, e_{i_1}, e_{i_1 i_2}, \ldots, e_{i_1 i_2 \cdots i_r} \mid i_k \in \mathbb{N},\, 1 \le k \le r\}, \\ \mathcal{E}^* &= \{e, e_{i_1}, e_{i_1 i_2}, \ldots, e_{i_1 i_2 \cdots i_{r-1}} \mid i_k \in \mathbb{N},\, 1 \le k \le r-1\}, \\ Z &= \{e_{i_1 i_2 \cdots i_r} \mid i_k \in \mathbb{N},\, 1 \le k \le r\}. \end{aligned}$$

Let $\mathcal{E}^{(\ell)}$ denote the ℓth derived set of $\mathcal{E}$. Then, for $1 \le \ell \le r-1$,

$$\mathcal{E}^{(\ell)} = \{e, e_{i_1}, e_{i_1 i_2}, \ldots, e_{i_1 i_2 \cdots i_{r-\ell}} \mid i_k \in \mathbb{N},\, 1 \le k \le r-\ell\},$$

and $\mathcal{E}^{(r)} = \{e\}$. For each $1 \le k \le r$, $e_{i_1 i_2 \cdots i_k}$ converges to $e_{i_1 i_2 \cdots i_{k-1}}$ as $i_k \to \infty$ while $i_1, \ldots, i_{k-1}$ are fixed, and e_{i_1} converges to e as $i_1 \to \infty$.

Now, put

$$\mathcal{E}^+ = \{e, e_{i_1}, e_{i_1 i_2}, \ldots, e_{i_1 i_2 \cdots i_r} \mid i_1 \text{ is even and } i_2, \ldots, i_r \text{ are arbitrary}\},$$
$$\mathcal{E}^- = \{e, e_{i_1}, e_{i_1 i_2}, \ldots, e_{i_1 i_2 \cdots i_r} \mid i_1 \text{ is odd and } i_2, \ldots, i_r \text{ are arbitrary}\},$$
$$\mathcal{E}^{+*} = \mathcal{E}^+ \cap \mathcal{E}^*,\ \mathcal{E}^{-*} = \mathcal{E}^- \cap \mathcal{E}^*,\ Z^+ = \mathcal{E}^+ \cap Z,\ Z^- = \mathcal{E}^- \cap Z,\ e^+ = e^- = e,$$

and $h = \mathrm{id} : \mathcal{E}^+ \vee_{e^+ = e^-} \mathcal{E}^- \to \mathcal{E}$. Then the 4-tuple $(\mathcal{E}, \mathcal{E}^*, Z, e)$ of the surface $\Sigma(r)$ has the self-similarity property.

QUESTION. List up all the open orientable surfaces whose endsets have the self-similarity property.

QUESTION. Can a surface which does not satisfy the self-similarity property be realized as a leaf of a uni-leaf foliation on $\mathbb{B}^3$?

5. Higher-dimensional leaves. In this section we consider the case of higher-dimensional leaves. We give two results. The first of these was kindly communicated to the authors by the referee, and is the following:

THEOREM 5.1. *Let M be a simply connected open n-manifold, $n \ge 3$, with a smooth foliation $\mathcal{F}$ by leaves diffeomorphic to $\mathbb{R}^{n-1}$. Then $\mathcal{F}$ is smoothly conjugate to a complete closed* (*and necessarily uni-leaf*) *foliation on $\mathbb{B}^n$.*

Proof. Let M and $\mathcal{F}$ be as in the hypothesis. Then, it follows from a deep result of Palmeira [14] that there are an open set Ω of $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^n$ and a diffeomorphism $h : M \to \Omega$ such that $h(\mathcal{F})$ coincides with the restriction to Ω of the foliation of $\mathbb{R}^n$ by horizontal hyperplanes. Here, we may assume that Ω satisfies (1) of Proposition 2.5. In fact, if $\mathrm{pr}_n(\Omega)$ is bounded, it suffices to replace Ω with $(\mathrm{id}_{\mathbb{R}^{n-1}} \times \varphi)(\Omega)$, where φ is an arbitrary diffeomorphism of the open interval $(\inf \mathrm{pr}_n(\Omega), \sup \mathrm{pr}_n(\Omega))$ onto $\mathbb{R}$. Clearly, $h(\mathcal{F})$ satisfies (2) of Proposition 2.5. Thus, by that proposition, there exists a diffeomorphism Φ from Ω to $\mathbb{B}^n$ such that $\Phi(h(\mathcal{F}))$ is a complete closed foliation on $\mathbb{B}^n$. ■

Let $\overline{\mathbb{B}}^n$ denote the closed unit n-ball, and pr_i the projection from a product space to its ith factor. Our next result in this section is

THEOREM 5.2. *Let $n \ge 3$. Suppose that F is a connected compact $(n-1)$-dimensional smooth submanifold of $\overline{\mathbb{B}}^{n-1} \times \mathbb{R}$ such that $F \cap (\partial\overline{\mathbb{B}}^{n-1} \times \mathbb{R}) = \partial\overline{\mathbb{B}}^{n-1} \times \{0\} = \partial F$ and F is transverse to $\partial\overline{\mathbb{B}}^{n-1} \times \mathbb{R}$ at ∂F. Let E be a closed subset of F such that*

(1) *$F - E$ is connected,*
(2) *E contains ∂F, and*

(3) *there exists a neighborhood U of E in F such that* $\mathrm{pr}_1 : \overline{\mathbb{B}}^{n-1} \times \mathbb{R} \to \overline{\mathbb{B}}^{n-1}$ *maps U diffeomorphically to* $\mathrm{pr}_1(U)$ *and* $\mathrm{pr}_1^{-1}\,\mathrm{pr}_1(U) \cap F = U$.

Then there is a codimension 1 *complete closed smooth foliation of* $\mathbb{B}^n$ *with a leaf diffeomorphic to* $F - E$.

Proof. The proof is essentially the same as in the surface case. Let n, F and E be as above. We take a Morse function $f : (\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)) \times \mathbb{R} \to \mathbb{R}$ such that

- $f = \mathrm{pr}_2$ on $(\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)) \times [(-\infty, -1] \cup [1, \infty)]$, and
- 0 is a regular value of f with $f^{-1}(0) = F - E$.

Next, we take an exhausting sequence $\{K_i\}$ of codimension 0 compact submanifolds in $\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)$, and a family of injective smooth paths $c_p : [0, \infty) \to (\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)) \times \mathbb{R}$, $p \in \mathrm{Crit}(f)$, satisfying the same six conditions as in §3. Then $M = (\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)) \times \mathbb{R} - \bigcup_{p \in \mathrm{Crit}(f)} c_p([0, \infty))$ is diffeomorphic to $(\overline{\mathbb{B}}^{n-1} - \mathrm{pr}_1(E)) \times \mathbb{R}$, and $\Omega = M \cup (\mathbb{B}^{n-1} \times (2, \infty))$ is diffeomorphic to $\mathbb{B}^n$. Finally, by exactly the same argument as in §3 we complete the proof. ∎

REMARK. For example, we may take for $E - \partial F$ the Whitehead continuum, the Menger sponge, and so on.

6. Higher codimensions

PROPOSITION 6.1. *Let q and q' be positive integers such that $1 \le q < q'$. Given a connected p-dimensional manifold L, if there is a codimension q complete closed smooth foliation on $\mathbb{B}^{p+q}$ with a leaf diffeomorphic to L, then there is a codimension q' complete closed smooth foliation on $\mathbb{B}^{p+q'}$ with a leaf diffeomorphic to L.*

Proof. Suppose $\mathcal{F}$ is a codimension q complete closed smooth foliation on $\mathbb{B}^{p+q}$ with a leaf diffeomorphic to L. Then the foliation on $\mathbb{B}^{p+q} \times \mathbb{B}^{q'-q}$ defined by $F \times \{z\}$ ($F \in \mathcal{F}$, $z \in \mathbb{B}^{q'-q}$) as leaves is a codimension q' complete closed smooth foliation and has a leaf diffeomorphic to L. Since $\mathbb{B}^{p+q} \times \mathbb{B}^{q'-q}$ is diffeomorphic to $\mathbb{B}^{p+q'}$ by a quasi-isometric diffeomorphism, the assertion follows. ∎

Combining Proposition 6.1 with Theorems 1.1 and 5.2, we obtain

THEOREM 6.2. *Let L be Σ in Theorem* 1.1 *or $F - E$ in Theorem* 5.2, *and let $p = \dim L$. Then, for any positive integer q, there is a codimension q complete closed smooth foliation on the open unit ball $\mathbb{B}^{p+q}$ having L as a leaf.*

Similarly, by Proposition 6.1 and Theorem 1.2 we have

THEOREM 6.3. *Let L be Σ in Theorem 1.2. Then, for any positive integer q, there is a codimension q complete closed smooth uni-leaf foliation on the open unit ball $\mathbb{B}^{3+q}$ having L as a leaf.*

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