Cup products on Morse homology of manifolds with boundary

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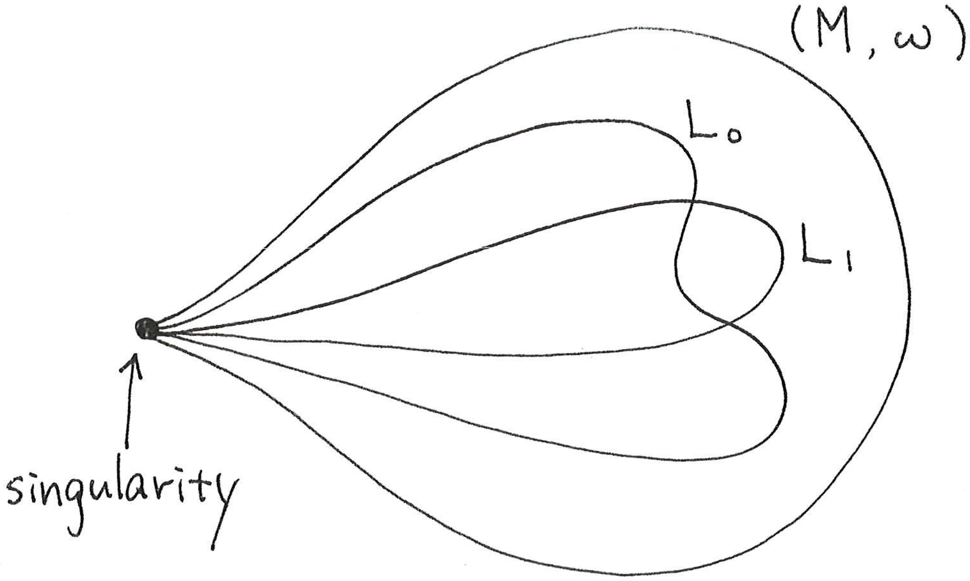
23 January 2011

Want : 1

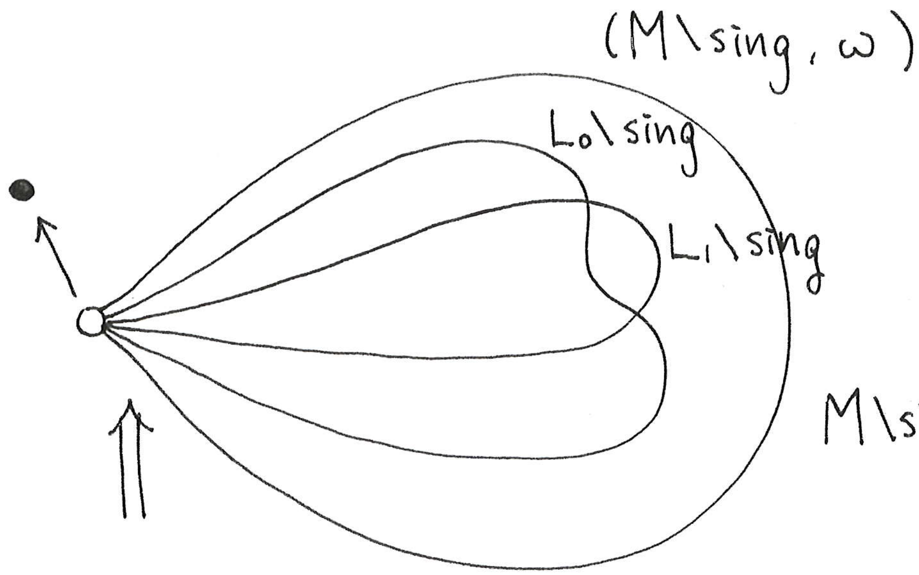


(M,ω) : symplectic

L_0, L_1 : Lagrangians

Floer theory ?

Idea : 2



$M \setminus sing$, $L_0 \setminus sing$, $L_1 \setminus sing$

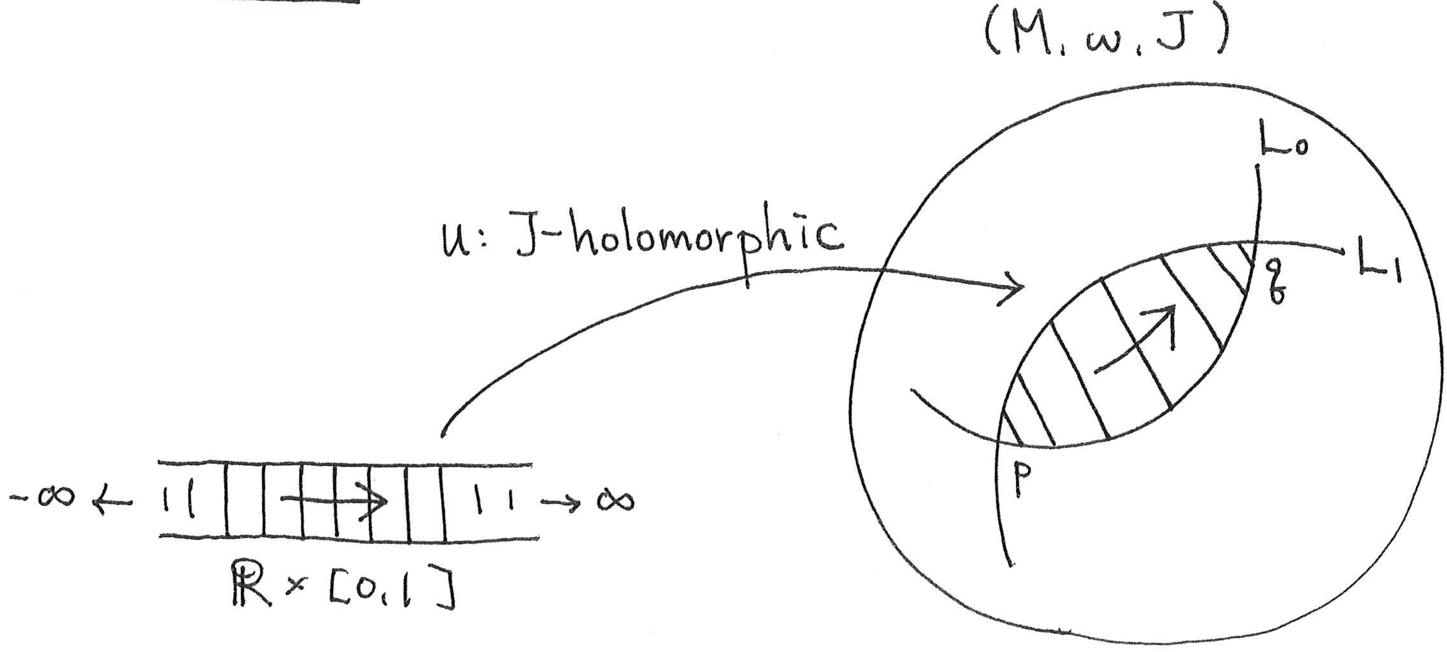
smooth

&

non-compact

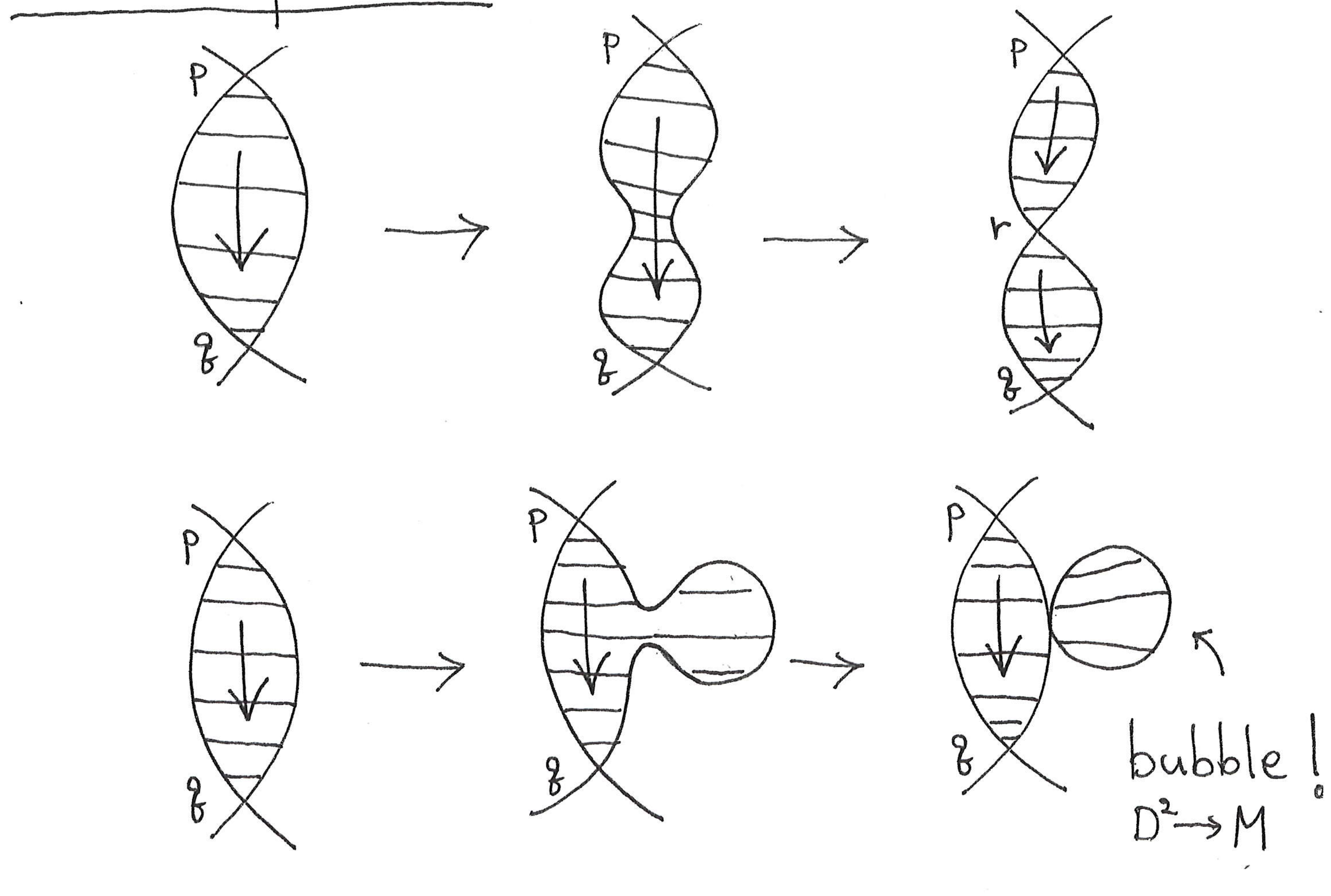
Suppose concave end !

Closed case : ③



- $J \circ du = du \circ i$
- $u(\mathbb{R},0) \subset L_0, \quad u(\mathbb{R},1) \subset L_1$
- $u(-\infty,[0,1]) = p, \quad u(\infty,[0,1]) = q$

Codim = 1 phenomena : ④

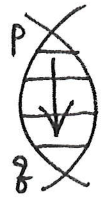


Floer homology : (5)

$$C(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2\, p$$

$$\partial : C(L_0, L_1) \longrightarrow C(L_0, L_1)$$

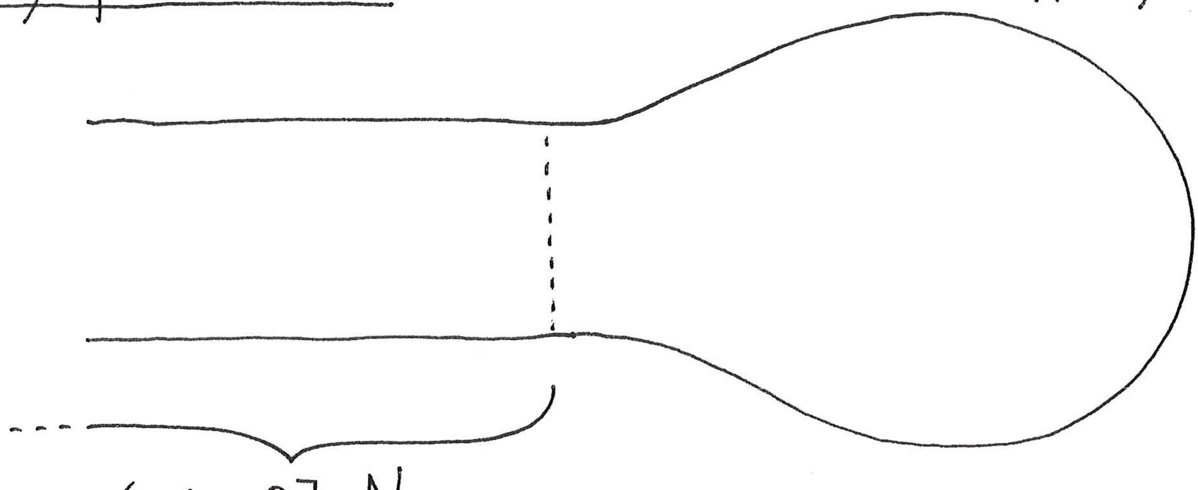
$$\partial p := \sum_{q \in L_0 \cap L_1} \# \left\{ \quad \text{J-hol} \;\&\; \text{vdim} = 0 \right\} q$$



Thm (Floer) Suppose no bubble $\Rightarrow \partial \circ \partial = 0$

$$HF(L_0, L_1) := \frac{\operatorname{Ker} \partial}{\operatorname{Im} \partial} \quad \text{Floer homology}$$

Our symplectic mfds (M, ω) (6)

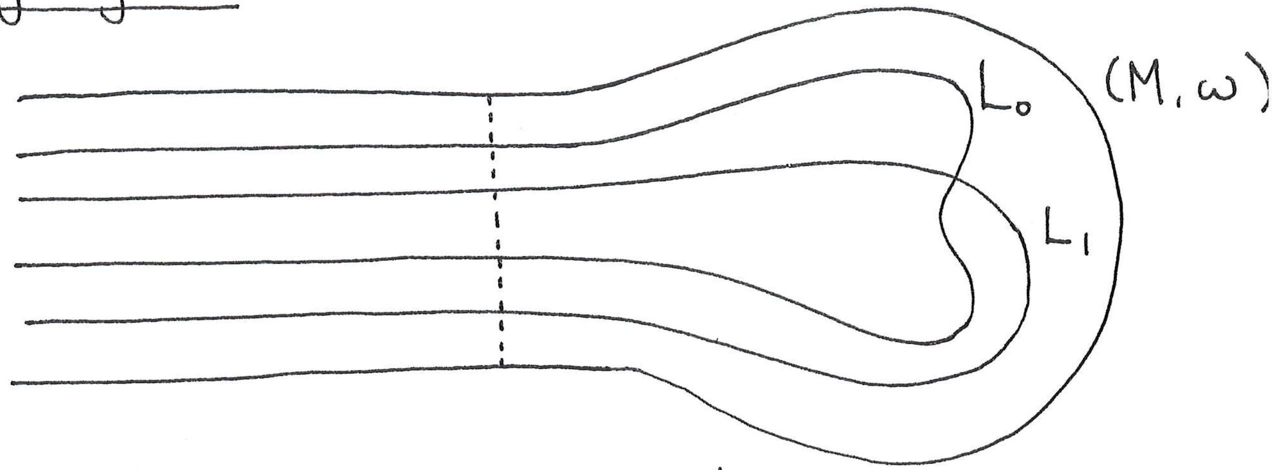


$(-\infty, 0] \times N$
$\;\;\;\;\;\;\;\;\;\;\;\;\;\; \Psi$
$\;\;\;\;\;\;\;\;\;\;\;\;\;\; t$

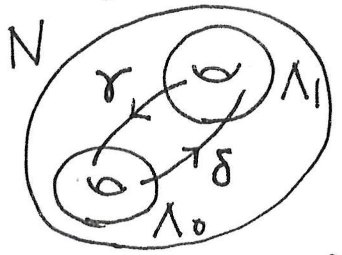
- N : compact, contact mfd with contact form λ
- $\omega|_{(-\infty,0]\times N} = d(e^t \lambda)$
- J' : almost cpx str on $\mathbb{R} \times N$ s.t. $\begin{cases} \mathbb{R}\text{-trans inv} \\ J' \frac{\partial}{\partial t} = \text{Reeb}, \; J'(\ker \lambda) = \text{ke} \end{cases}$
- J : almost cpx str on M s.t. $J|_{(-\infty,0]\times N} = J'$

Our Lagrangians :

(7)



- L_0, L_1 : Lagrangians in M s.t. $L_0 \pitchfork L_1$
- Λ_0, Λ_1 : Legendrians in N s.t.

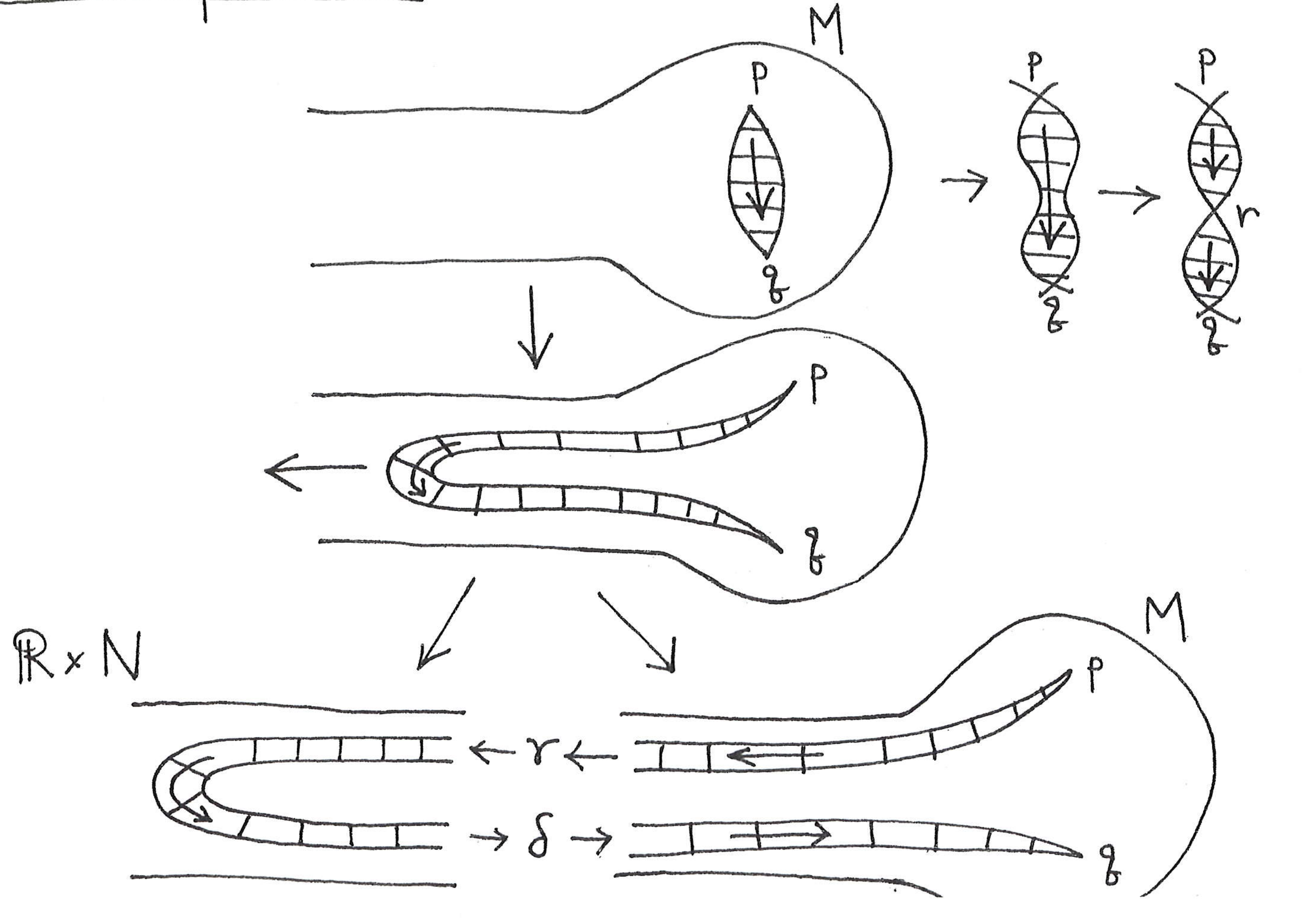


$\left.\begin{array}{l}\gamma : \text{"positive" Reeb chord} \\ \delta : \text{"negative" Reeb chord}\end{array}\right\}$ isolated

- $L_0|_{(-\infty,0]\times N} = (-\infty,0]\times\Lambda_0$, $L_1|_{(-\infty,0]\times N} = (-\infty,0]\times\Lambda_1$

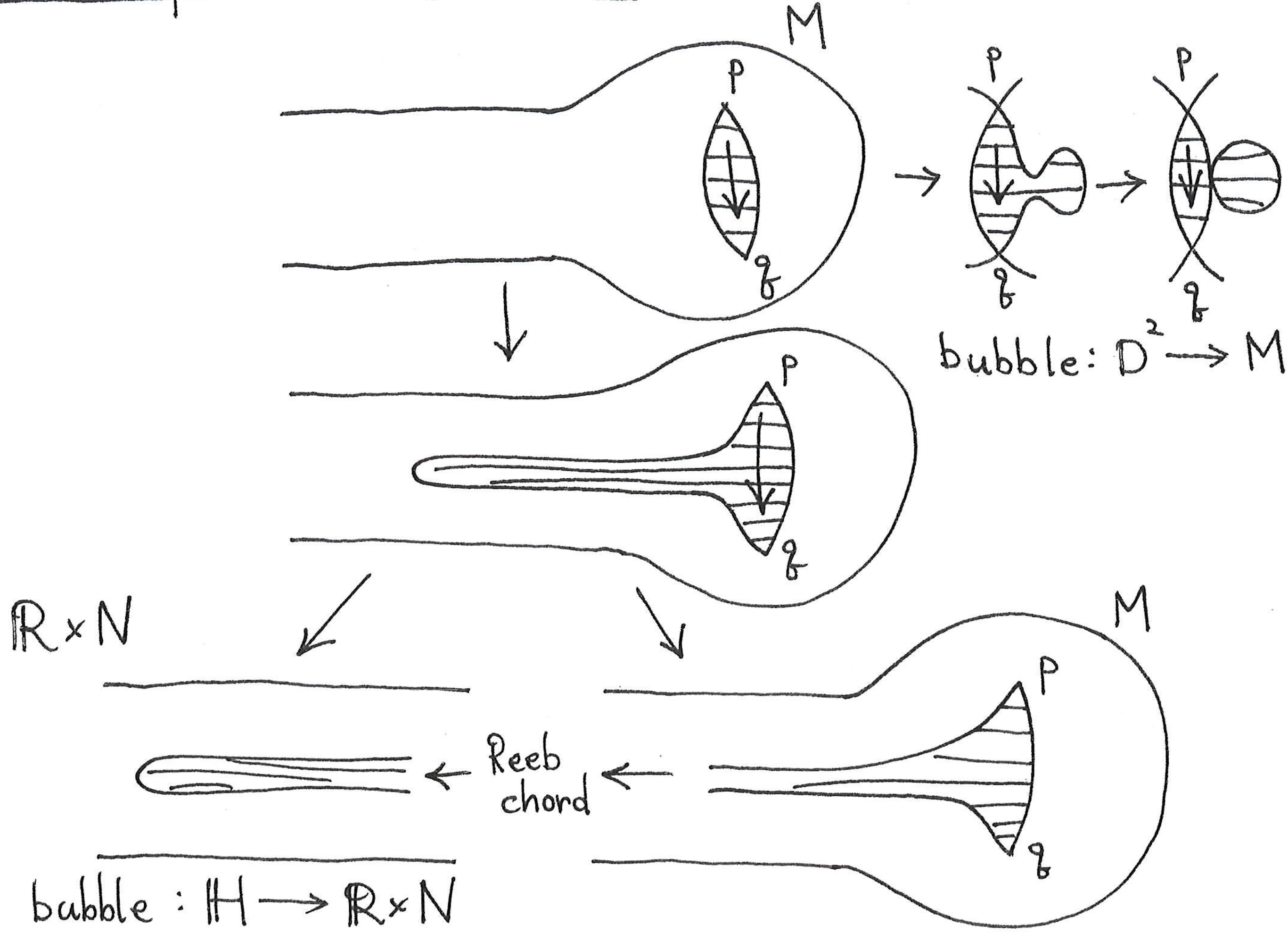
Codim = 1 phenomena :

(8)



Codim = 1 phenomena (bubble):

9



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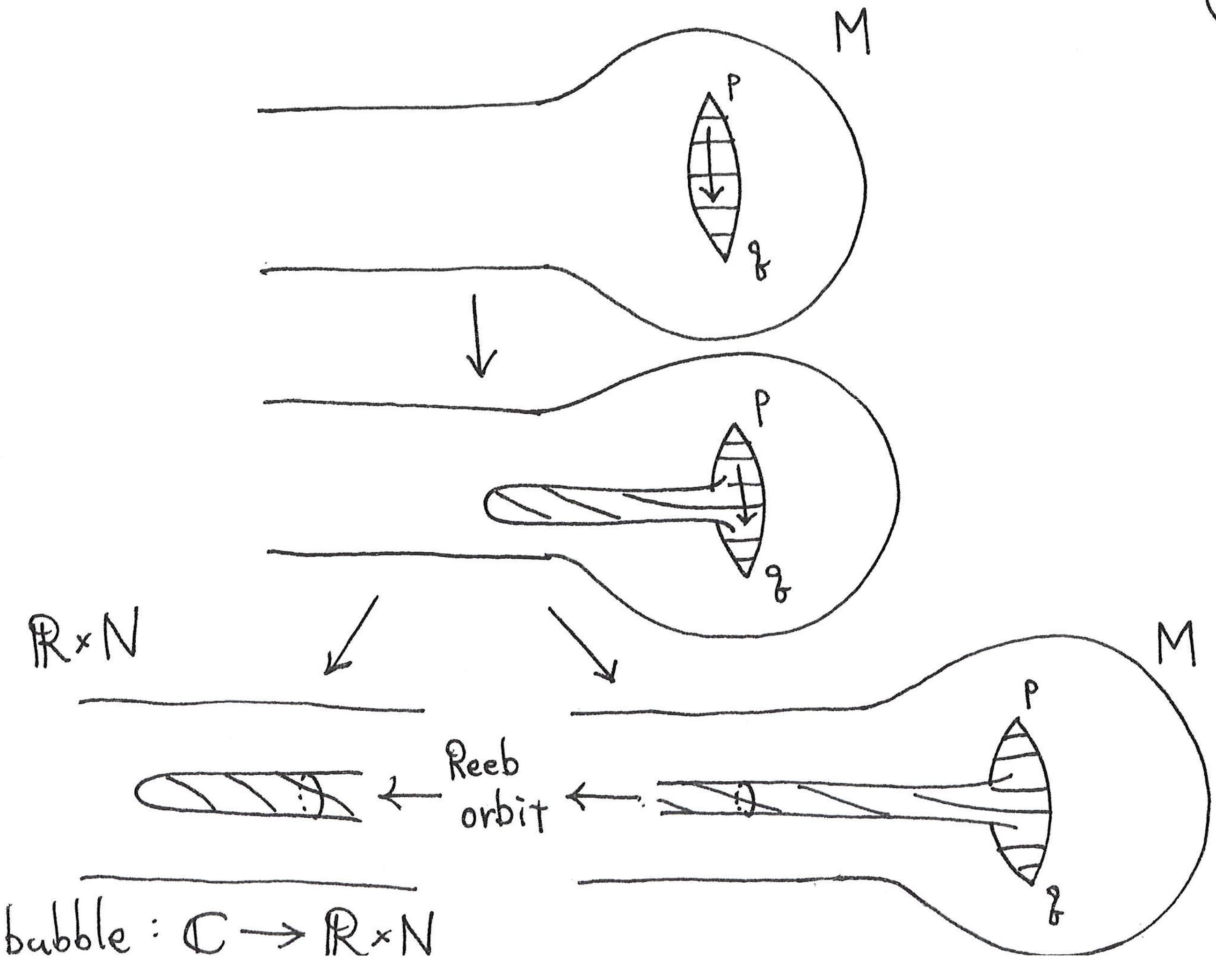

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Q. Floer homology for our settings ?

cf. Floer theory for closed case.

$\Leftarrow$ Morse homology of closed mfds.

A. Morse homology of mfds with boundary !

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Morse homology of closed mfds:

M : closed mfd
g : Riemannian metric on M
$f : M \to \mathbb{R}$ Morse fun

$\Big\}$ X_f : gradient vect field on M

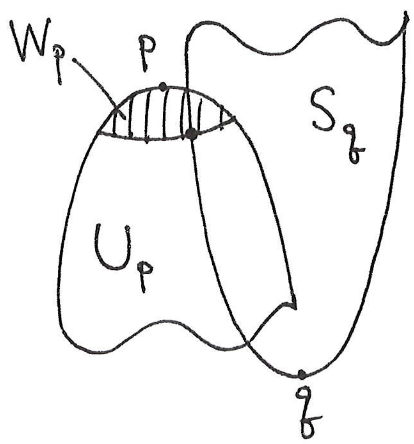
$M \ni p$: critical point

unstable mfd

$$U_p := \{ x \in M : x \xrightarrow{X_f} p \}$$

stable mfd

$$S_p := \{ x \in M : x \xrightarrow{-X_f} p \}$$



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$$M^k := \bigcup_{\mu(p)\leq k} U_p \quad \Leftarrow \quad f : \text{generic}$$

μ ← Morse index

CW complex

cellular homology | Morse homology

$$H_k(M^k, M^{k-1}) \cong \bigoplus_{\mu(p)=k} \mathbb{Z} U_p \iff C_k(f) := \bigoplus_{\mu(p)=k} \mathbb{Z} p$$

$$\delta_k : H_k(M^k, M^{k-1}) \to H_{k-1}(M^{k-1}, M^{k-2}) \iff \partial_k : C_k(f) \to C_{k-1}(f)$$

$$\delta_k U_p = \sum_{\mu(q)=k-1} \#(\partial W_p \cap S_q) U_q \iff \partial_k p := \sum_{\mu(q)=k-1} \#\left\{ p \xrightarrow{-X_f} q \right\} \cdot q$$

$$\frac{\operatorname{Ker} \delta_k}{\operatorname{Im} \delta_{k+1}} \cong H_k(M) \iff \frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}} \cong H_k(M)$$

singular homology | Morse homology

<u>Morse homology of mfds with boundary</u>: 14

M : mfd with ∂M

g : Riemannian metric on $M \setminus \partial M$ s.t.

$$g|_{(0,1]\times\partial M} = \frac{1}{r} dr \otimes dr + r g_{\partial M}$$

$(0,1] \ni r$

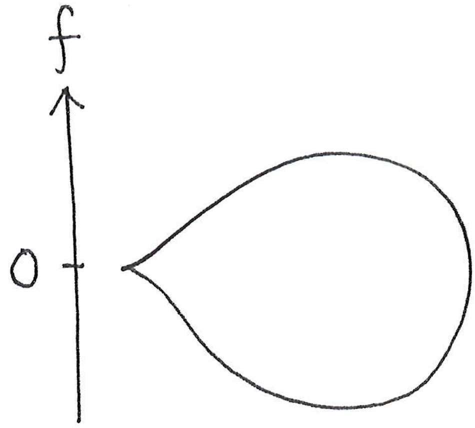
$g_{\partial M}$ ← Riem metric on ∂M

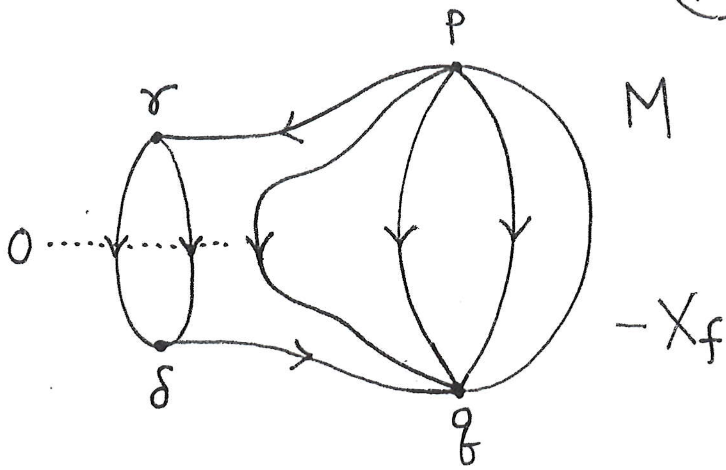
$f : M\setminus\partial M \to \mathbb{R}$ Morse fun s.t.

- $f|_{(0,1]\times\partial M} = r f_{\partial M}$

  $f_{\partial M}$ ← Morse fun on ∂M

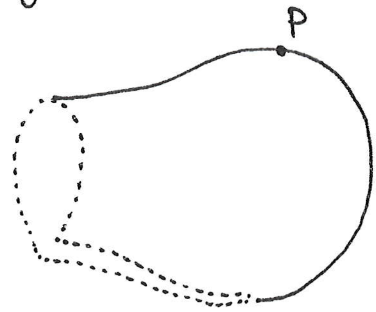
- $\partial M \ni x \quad df_{\partial M}(x) = 0 \Rightarrow f_{\partial M}(x) \neq 0$





γ : "positive" boundary crit pt p, q : interior crit pt

U_γ : 

U_p : 

δ : "negative" boundary crit pt

U_δ : 

"half cell"

U_q : 

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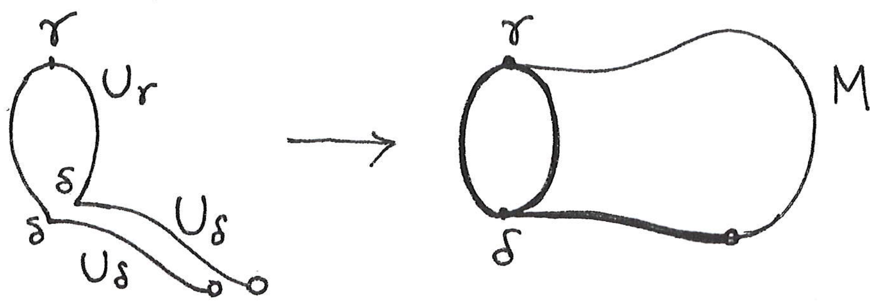
$$M^k := \bigcup_{\mu(p)\leqq k} U_p \cup \bigcup_{\mu(\gamma)\leqq k} U_\gamma \cup \underbrace{\bigcup_{\mu(\delta)\leqq k} U_\delta}_{\text{not CW complex}}$$

$$I_\gamma : U_\gamma \cup \underbrace{(U_{\delta_1} \cup \cdots \cup U_{\delta_1})}_{\#\{\gamma \searrow \delta_1\}} \cup \cdots \cup \underbrace{(U_{\delta_{N_\gamma}} \cup \cdots \cup U_{\delta_{N_\gamma}})}_{\#\{\gamma \searrow \delta_{N_\gamma}\}} \longrightarrow M$$

I_γ : 

$$M^k \underset{\text{homotopic}}{\sim} \bigcup_{\mu(p)\leqq k} U_p \cup \bigcup_{\mu(\gamma)\leqq k} \mathrm{Im}\, I_\gamma$$

"cellular" homology :

$$H_k(M^k, M^{k-1}) \cong \bigoplus_{\mu(p)=k} \mathbb{Z}\, U_p \oplus \bigoplus_{\mu(\gamma)=k} \mathbb{Z}\, I_\gamma$$

$$\delta_k : H_k(M^k, M^{k-1}) \longrightarrow H_{k-1}(M^{k-1}, M^{k-2})$$

$$\delta_k U_p = \sum_{\mu(p')=k-1} \#(\partial W_p \cap S_{p'})\, U_{p'} + \sum_{\mu(\gamma')=k-1} \#(\partial W_p \cap S_{\gamma'})\, I_{\gamma'}$$

$$\delta_k I_\gamma = \sum_{\mu(\gamma')=k-1} \#(\partial W_\gamma \cap S_{\gamma'})\, I_{\gamma'} + \sum_{\substack{\mu(\delta)=k-1 \\ \mu(p')=k-1}} \#(\partial W_\gamma \cap S_\delta)\#(\partial W_\delta \cap S_{p'})\, U_{p'}$$

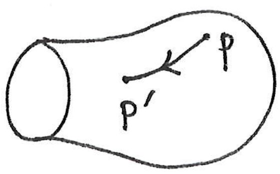
Thm (A.)

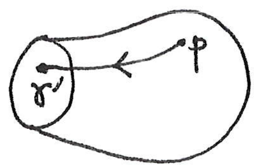
$$\frac{\operatorname{Ker} \delta_k}{\operatorname{Im} \delta_{k+1}} \cong H_k(M) \quad \text{absolute singular homology}$$

Morse homology :

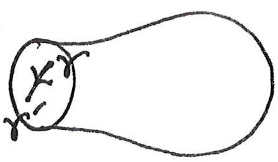
$$C_k(f) := \bigoplus_{\mu(p)=k} \mathbb{Z}\, p \oplus \bigoplus_{\mu(\gamma)=k} \mathbb{Z}\, \gamma$$

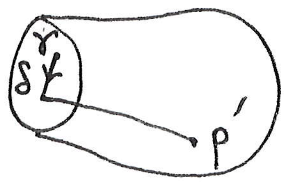
$$\partial_k : C_k(f) \longrightarrow C_{k-1}(f)$$

$\partial_k p := \sum_{\mu(p')=k} \# \{$ 

$\} p' + \sum_{\mu(\gamma')=k-1} \# \{$ 

$\} \gamma'$

$\partial_k \gamma := \sum_{\mu(\gamma')=k-1} \# \{$ 

$\} \gamma' + \sum_{\substack{\mu(\delta)=k-1 \\ \mu(p')=k-1}} \# \{$ 

$\} p'$

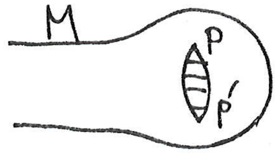
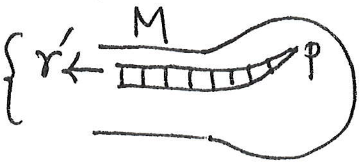
Thm (A.)

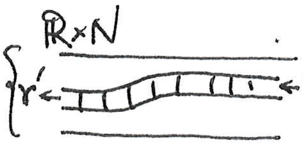
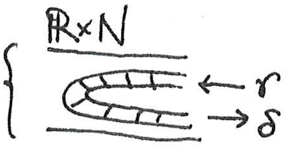
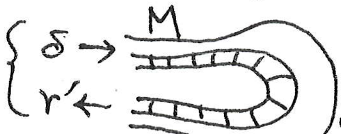
$$\frac{\operatorname{Ker} \partial_k}{\operatorname{Im} \partial_{k+1}} \cong H_k(M) \quad \text{absolute singular homology}$$

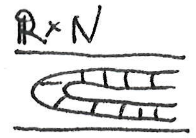
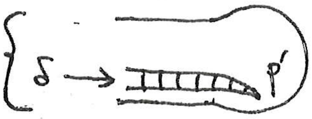
Floer homology :

$$C(L_0,L_1) := \bigoplus_{p\in L_0\cap L_1} \mathbb{Z}_2\, p \oplus \bigoplus_{\gamma:\Lambda_1\to\Lambda_0} \mathbb{Z}_2\, \gamma$$

$$\partial : C(L_0,L_1) \to C(L_0,L_1)$$

$\partial p := \sum_{p'\in L_0\cap L_1} \#\{$  $\}\, p' + \sum_{\gamma':\Lambda_1\to\Lambda_0} \#\{$  $\}\, \gamma'$

$\partial \gamma := \sum_{\gamma':\Lambda_1\to\Lambda_0} \#\{$  $\}\, \gamma' + \sum_{\substack{\delta:\Lambda_0\to\Lambda_1\\ \gamma':\Lambda_1\to\Lambda_0}} \#\{$  $\}\ \#\{$  $\}$

$+ \sum_{\substack{\delta:\Lambda_0\to\Lambda_1\\ p'\in L_0\cap L_1}} \#\{$  $\}\ \#\{$  $\}\, p'$

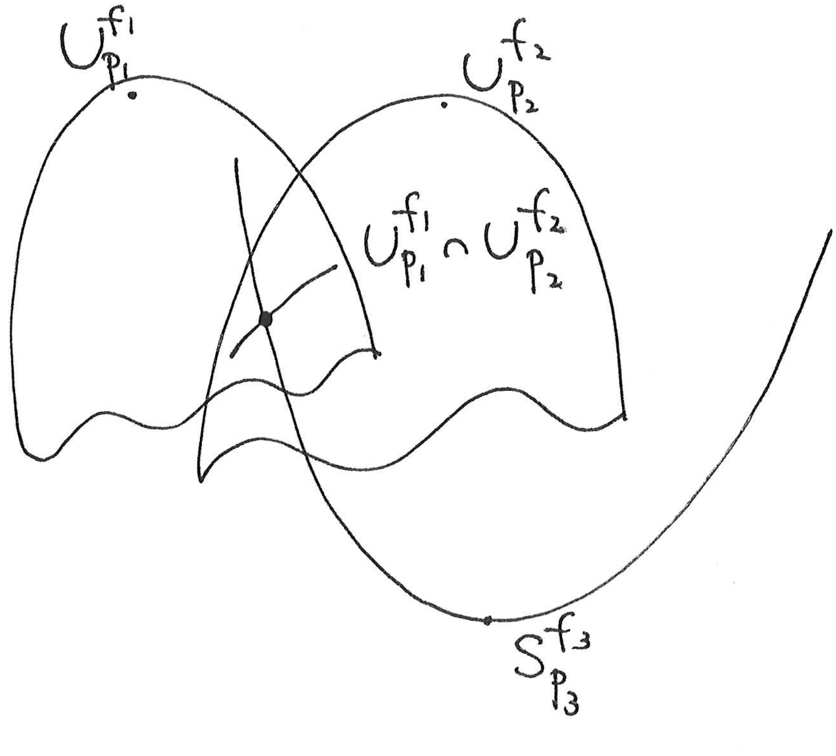
Thm (A.)

Suppose no bubble $\Rightarrow \partial\circ\partial = 0$

cup products on closed mfds:

$$f \rightsquigarrow \text{homology}$$

$$f_1, f_2, f_3 \rightsquigarrow \text{cup products (Fukaya)}$$



cup prod on cellular hom: (21)

$$M_i^{k_i} := \bigcup_{\mu(p_i)\leq k_i} U_{p_i}^{f_i}, \quad i=1,2,3$$

$$m_2 : H_{k_1}(M_1^{k_1}, M_1^{k_1-1}) \otimes H_{k_2}(M_2^{k_2}, M_2^{k_2-1}) \longrightarrow H_{k_3}(M_3^{k_3}, M_3^{k_3-1}), \quad k_1+k_2-n=k$$

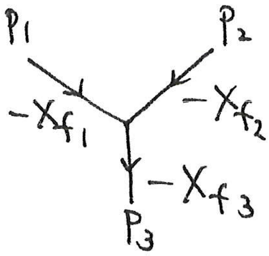
$$m_2(U_{p_1}^{f_1}, U_{p_2}^{f_2}) = \sum_{\mu(p_3)=k_3} \#(U_{p_1}^{f_1} \cap U_{p_3}^{f_2} \cap S_{p_3}^{f_3})\, U_{p_3}^{f_3}$$

Thm (Fukaya)

- m_2 gives cup products.
- m_2 satisfies the Leibnitz rules.

cup prod on Morse hom: (22)

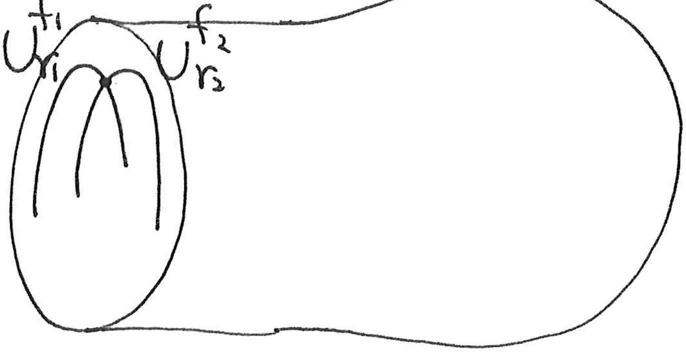
$$m_2 : C_{k_1}(f_1) \otimes C_{k_2}(f_2) \longrightarrow C_{k_3}(f_3), \quad k_1+k_2-n=k_3$$

$$m_2(p_1, p_2) := \sum_{p_3} \# \Big\{$$  $$\Big\}\, p_3$$

Thm (Fukaya)

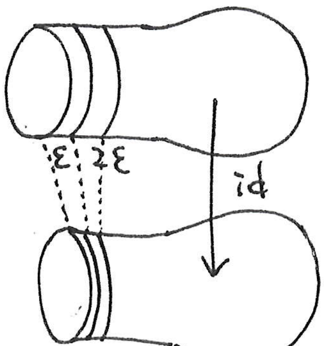
- m_2 gives cup products.
- m_2 satisfies the Leibnitz rules.

cup prod on mfds with boundary: (23)



$0 < \varepsilon \ll 1$

$\Psi_\varepsilon : M \xrightarrow{\text{diff}} M \setminus [0,\varepsilon) \times \partial M$



$U^{f_1}_{\gamma_1} \cap U^{f_2}_{\gamma_2} = x \in \partial M$

$U^{f_1}_{\gamma_1} \cap \Psi_\varepsilon(U^{f_2}_{\gamma_2}) = \phi$

$\Rightarrow x$: non-transverse intersection

cup prod on "cellular" hom: (24)

$$m_2 : H_{k_1}(M_1^{k_1}, M_1^{k_1-1}) \otimes H_{k_2}(M_2^{k_2}, M_2^{k_2-1}) \longrightarrow H_{k_3}(M_3^{k_3}, M_3^{k_3-1}), \quad k_1+k_2-n=k_3$$

$$m_2(U^{f_1}_{p_1}, U^{f_2}_{p_2}) := \sum_{p_3} \#(U^{f_1}_{p_1} \cap \Psi_\varepsilon(U^{f_2}_{p_2}) \cap S^{f_3}_{p_3})\, U^{f_3}_{p_3}$$
$$+ \sum_{\gamma_3} \#(U^{f_1}_{p_1} \cap \Psi_\varepsilon(U^{f_2}_{p_2}) \cap S^{f_1}_{\gamma_3})\, I^{f_3}_{\gamma_3}$$

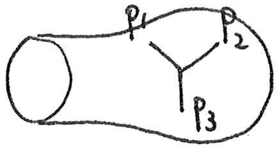
$m_2(U^{f_1}_{p_1}, I^{f_2}_{\gamma_3}) := \cdots$

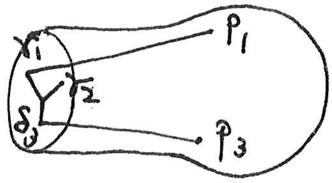
Thm (A.)

- m_2 satisfies the Leibnitz rules.

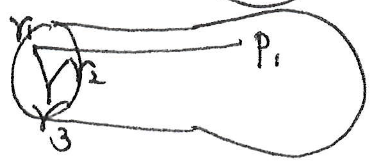
cup prod on Morse hom:

$m_2 : C_{k_1}(f_1) \otimes C_{k_2}(f_2) \longrightarrow C_{k_3}(f_3), \quad k_1 + k_2 - n = k_3$

$m_2(p_1, p_2) := \sum_{p_3} \# \{$  $\} p_3 + \sum_{\gamma_3} \# \{$  $\} \gamma_3$

$m_2(p_1, \gamma_2) := \sum_{\gamma_1, \delta_3, p_3} \# \{$  $\} p_3$

$+ \sum_{\delta_2, p_3} \# \{$  $\} p_3$

$+ \sum_{\gamma_1, \gamma_3} \# \{$  $\} \gamma_3$

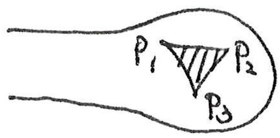
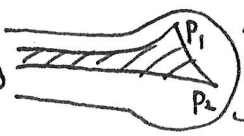
$m_2(\gamma_1, p_2) := \cdots$

Thm (A.)

- m_2 satisfies the Leibnitz rules.

product structures on Floer hom:

$m_2 : C(L_0, L_1) \otimes C(L_1, L_2) \longrightarrow C(L_0, L_1)$

$m_2(p_1, p_2) := \sum_{p_3 \in L_0 \cap L_1} \# \{$  $\} p_3 + \sum_{\gamma_3 : \Lambda_2 \to \Lambda_0} \# \{$  $\}$

$m_2(p_1, \gamma_2) := \cdots$

Thm (A.) Suppose no bubble

$\Rightarrow$ m_2 satisfies the Leibnitz rules.

Q. A_∞ structures ?