EXTENDING A VECTOR FIELD ON A SUBMANIFOLD TO A
REEB VECTOR FIELD ON THE WHOLE CONTACT MANIFOLD

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Abstract. Given a vector field on a submanifold of a contact manifold, when
does it extend to a Reeb vector field on the whole manifold? We give an answer
to this question.

1. Introduction and statement of results

The purpose of this short note is to give a basic, but (as far as the author
thinks) important, result for Reeb vector fields. Maybe, it is known to experts, but
the author could not find it in the literature. We work in the smooth category.
Let \((M, \xi)\) be a contact manifold with a cooriented (i.e. transversely oriented)
contact plane field \(\xi\). We want to consider the following extension problem for a
vector field: Let \(N\) be a compact submanifold of \(M\) transverse to \(\xi\), and let \(X\)
be a vector field along \(N\) which is tangent to \(N\) and positively transverse to \(\xi\)
everywhere on \(N\). (Note: In this note throughout, a compact submanifold means a
compact submanifold without boundary.) In this situation, can we find a function
\(f\) on \(N\) and a contact form \(\alpha\) on \(M\) defining \((\xi)\) such that \(fX\)
coincides on \(N\) with the Reeb vector field \(R\) of \(\xi\)? Some cases are known. If
\(N\) is 1 dimensional (i.e. \(N\) is a disjoint union of a finite number of transverse
loops to \(\xi\)) and \(X\) is an arbitrary nonsingular vector field along \(N\) tangent to \(\xi\),
then, the answer is yes (folklore?). (In this case we need not multiply \(X\) by a
nonconstant function). If \(N\) is a contact submanifold of arbitrary dimension and
\(X\) is the Reeb vector field \(R\) of some contact form \(\alpha\) on \(N\) with \(\ker \alpha = TN \cap \xi\),
then the answer is also yes (e.g. [3, Proof of Theorem 2.5.15]). (In this case we
need not multiply \(X\) by a nonconstant function, either). In [1] we have shown the
following. Let \((\mathbb{R}^{2n+1}, \xi_{\text{std}})\) be the standard contact space, where
\(\xi_{\text{std}} = \ker \alpha_{\text{std}}\) and \(\alpha_{\text{std}} = dz + \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)\)
\(\left((x_1, y_1, \ldots, x_n, y_n, z)\right)\) being the coordinates
of \(\mathbb{R}^{2n+1}\), \(T^n\) the torus defined by \(z = 0\) and \(x_j^2 + y_j^2 = 1\) \((j = 1, \ldots, n)\)
and \(X\) be an arbitrary vector field on \(T^n\) transverse to \(\xi_{\text{std}}\). Then, the answer is again yes
(In this case we may need to multiply \(X\) by a nonconstant function). Here, remark
that \(T^n\) is not a contact submanifold of \((\mathbb{R}^{2n+1}, \xi_{\text{std}})\).

In this note we unite the above mentioned results in one and formulate as follows.

Theorem 1.1. Let \((M, \xi)\) be a cooriented contact manifold, \(N\) a compact subman-
ifold of \(M\) transverse to \(\xi\) and \(X\) a vector field along \(N\) which is tangent to \(N\)
and transverse to \(\xi\) everywhere on \(N\). Then, the following are equivalent.

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There exists a contact form $\alpha$ on $M$ defining $\xi$ such that the restriction of the Reeb vector field $R_\alpha$ on $N$ coincides with $X$.

2. $X$ preserves $TN \cap \xi$.

We have some corollaries to this theorem.

**Theorem 1.2** (Rescaling version). Let $(M,\xi)$ be a cooriented contact manifold, $N$ a compact submanifold of $M$ transverse to $\xi$ and $X$ a vector field along $N$ which is tangent to $N$ and transverse to $\xi$ everywhere on $N$. Then, the following are equivalent.

1. There exist a contact form $\alpha$ on $M$ defining $\xi$ and a function $f$ on $N$ such that the restriction of the Reeb vector field $R_\alpha$ on $N$ coincides with $fX$.
2. There exists a contact form $\alpha$ on $M$ defining $\xi$ such that $X \in \ker (j^*d\beta)_p$ for any $p \in N$, where $j : N \to M$ is the inclusion map.

**Corollary 1.3** ([3, Proof of Theorem 2.5.15]). Let $(M,\xi)$ be a cooriented contact manifold and $N$ a compact contact submanifold of $M$ which is positively transverse to $\xi$. Then, the Reeb vector field of any contact form on $N$ defining $TN \cap \xi$ extends to the Reeb vector field of some contact form on $M$ defining $\xi$.

Before stating the next corollary, we introduce one notion. Let $(M,\xi)$ be a cooriented contact manifold and $N$ a submanifold of $M$. Let $\alpha$ be a contact form on $M$ which defines $\xi$. We say that $N$ is **isotropic** with respect to $\alpha$ if $j^*d\alpha$ vanishes. We simply say that $N$ is **isotropic** if $N$ is isotropic with respect to some $\alpha$ defining $\xi$.

**Notice.** This definition of isotropy is weaker than the one given in Geiges’ book ([3, Definition 1.5.11]), where $N$ is called isotropic if $j^*\alpha$ vanishes. Our definition coincides with the one of Foulon-Hasselblatt[2], although, there, it is defined not for a submanifold but for a tangent subspace and for a foliation.

The following is related to (and generalizes in some sense) [1, Theorem 1.1] since $T^n$ is an isotropic submanifold of $(\mathbb{R}^{2n+1},\text{std})$.

**Corollary 1.4.** Let $(M,\xi)$ be a cooriented contact manifold and $N$ a compact isotropic submanifold of $M$ transverse to $\xi$. Then, any vector field on $N$ positively transverse to $\xi$ extends, after multiplying a suitable positive function, to the Reeb vector field of some contact form on $M$ defining $\xi$.

**Corollary 1.5** (folklore?). Let $(M,\xi)$ be a cooriented contact manifold and $N$ is a disjoint union of finite number of loops positively transverse to $\xi$ Then, any vector field on $N$ extends to the Reeb vector field of some contact form on $M$ defining $\xi$.

### 2. Non-isotropy

Given a submanifold of a contact manifold, how can one know its non-isotropy? Here we give one simple criterion. Let $(M,\xi)$ be a contact manifold and $N$ a compact submanifold of $M$.

**Proposition 2.1.** If there exists a contact form $\alpha$ on $M$ defining $\xi$ such that $j^*d\alpha$ never vanishes on $N$, then $N$ is non-isotropic.
Proof of Proposition 2.1. Suppose there exists an $\alpha$ as in the proposition. Let $\beta$ be any contact form on $M$ defining $\xi$. Then, $\beta = f\alpha$ for some positive function $f$ on $M$. Since $N$ is compact, there exists a point $p \in N$ such that $(j^*df)_p = 0$. Thus we have $(j^*d\beta)_p = (j^*d(f\alpha))_p = (j^*(df \wedge \alpha + f\alpha))_p = (j^*df)_p \wedge (j^*\alpha)_p + f(p)(j^*f\alpha)_p = f(p)(j^*\alpha)_p$. (Because $(j^*df)_p = 0$.) Hence $(j^*d\beta)_p$ does not vanish, implying that $N$ is not isotropic with respect to $\beta$. Since $\beta$ is arbitrary, we get the conclusion. $\square$

3. Proof

Proof of Theorem 1.1. Obviously (1) implies (2). The implication from (2) to (1) is done just by adjusting the argument in [3, Proof of Theorem 2.5.15] to our situation. For the self-containedness of this note, we give the detail below: Let $(1)$ is done just by adjusting the argument in [3, Proof of Theorem 2.5.15] to our situation. Suppose further that $X$ satisfies the condition (2), namely, $X$ preserves $TN \cap \xi$. First, take any contact form $\alpha$ on $M$ defining $\xi$. Then, in terms of $\alpha$ the condition (2) can be restated as follows: there exists a function $\lambda$ on $M$ such that $L_X j^*\alpha = \lambda j^*\alpha$ holds on $N$, where $j: N \rightarrow M$ is the inclusion map. Now, we choose a positive function $f$ on $M$ so that $f = 1/\alpha(X)$ on $N$ (one can easily construct such a function) and define $\beta = f\alpha$. Then, $\beta$ is also a contact form on $M$ defining $\xi$ and $\beta(X) = 1$ on $N$. Moreover, by a usual calculation, we have $L_X j^*\beta = 0$ on $N$. In fact, $L_X j^*\beta = L_X ([1/(\alpha(X))+\lambda]^*\alpha) = X(1/(\alpha(X))+\lambda)^*\alpha + (1/(\alpha(X))L_X [1/(\alpha(X))+\lambda] = 1/(\alpha(X))^2 \{ -X(\alpha(X)) + \alpha(X) \})^*\alpha = 0$, because $X(\alpha(X)) = i_X d_X j^*\alpha = i_X L_X j^*\alpha = i_X (\lambda^*\alpha) = \alpha(X)$. Since $\beta(X) = 1$ on $N$, $L_X j^*\beta = 0$ implies $i_X j^*\beta = 0$ on $N$. Now, we will extend $X$ to a Reeb vector field (for some contact form) on $M$. To this end, we look for a smooth function $g$ on $M$ satisfying (a) $g > 0$ on $M$, (b) $g = 1$ on $N$ and (c) $i_X d(g\beta) = 0$ on $TM|N$. Assuming (b), we have $i_X d(g\beta) = i_X (dg \wedge \beta + gd\beta) = (i_X dg)\beta - (i_X \beta) dg + gi_X d\beta = (Xg)\beta - \beta(Xg) dg + gi_X d\beta = -dg + i_X d\beta$. Thus, the condition (c) may be replaced with (c'): $dg = i_X d\beta$ on $TM|N$. Now, to begin the construction of $g$, first set $g = 1$ on $N$. Then, $dg$ and $i_X d\beta$ are both defined and vanish on $TN$, hence, the equality (c') is consistent on the subbundle $TN$ of $TM|N$. Next, we will extend $g$ over a neighborhood of $N$. Take a small tubular neighborhood $U = \pi(\nu)$ of $N$, where $\nu$ is a normal disk bundle of $N$ in $M$. And define $g$ on $U$ by $g(\pi(v)) = 1 + d\beta(X, v)$ for $v \in \nu$. Then, $g$ satisfies (c') and is positive if $U$ is small. Now, extending $g$ over whole $M$ preserving positivity is a routine work. The existence of $g$ is proven. Using this $g$, we consider the contact form $\gamma = g^3$. Then, its Reeb vector field $R_\gamma$ is the desired one. In fact, by definition, $R_\gamma$ satisfies $\gamma(R_\gamma) = 1$ and $\nu(R_\gamma) d\gamma = 0$. If we restrict these properties to $TM|N$, these two properties of $R_\gamma$ are the same with those of $X$. Since $\gamma$ is a contact form on $M$, the vector field along $N$ which satisfies these two properties is unique. Therefore, we conclude that $R_\gamma = X$. The proof is complete. $\square$

Proof of Theorem 1.2. Let $(M, \xi)$ and $N$ be as in Theorem 1.2. Let $X$ be a vector field along $N$ which is tangent to $N$ and transverse to $\xi$ everywhere on $N$. That (1) implies (2) is obvious. So, to prove the converse implication, suppose that $X$ satisfies $X_p \in \ker(j^*d\beta)_p$ for any $p \in N$. Then, the $(1/\beta(X))X$ satisfies $L_{(1/\beta(X))X} j^*\beta = 0$, hence, preserves $TN \cap \xi$. Therefore, by Theorem 1.1, $(1/\beta(X))X$ is the restriction of the Reeb vector field of some contact form on $M$ defining $\xi$, as desired.
Proof of Corollary 1.3. Let $N$ be a contact submanifold of $(M, \xi)$ and $X$ the Reeb vector field of some contact form on $N$ defining $TN \cap \xi$. Then, since $X$ preserves $TN \cap \xi$, we can apply Theorem 1.1 and obtain the conclusion. \qed

Proof of Corollary 1.4. Let $N$ be an isotropic submanifold of $(M, \xi)$. Then, by the definition of an isotropic submanifold, for any vector field $X$ on $N$ transverse to $\xi$ there exists a contact form $\beta$ on $M$ defining $\xi$ such that $X$ and $\beta$ satisfy the condition (2) of Theorem 1.2. So, the conclusion of the rescaling version follows. \qed

Proof of Corollary 1.5. Let $N, X$ be as in Corollary 1.5. Then, since dim $N = 1$, $X$ obviously preserves $TN \cap \xi$ (the zero dimensional plane field!). Hence, the result follows. \qed

We give a simple example of the case of a submanifold which is neither contact nor isotropic.

Example. Let $N$ be a submanifold of $(\mathbb{R}^7, \xi_{\text{std}} = \ker(o_{\text{std}}))$ defined by $z = 0$, $x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$ and $x_3^2 + y_3^2 = 1$. Then, $N$ is diffeomorphic to $S^3 \times S^1$ and transverse to $\xi$. As a submanifold of $(\mathbb{R}^7, \xi_{\text{std}})$, $N$ is neither contact nor isotropic. In fact, it is non-contact because $j^*d\alpha_{\text{std}}$ is degenerate in the $S^1$-direction, non-isotropic because $j^*d\alpha_{\text{std}}$ is nondegenerate in the $S^3$-direction (see Proposition 2.1). Let $(r_i, \theta_i)$ be the polar coordinates for $(x_i, y_i)$. Then, for each $p \in N$, $T_pN$ is spanned by $\frac{\partial}{\partial r_1}$, $\frac{\partial}{\partial \theta_1}$, $\frac{\partial}{\partial r_2}$, and $r_2\frac{\partial}{\partial r_1} - r_1\frac{\partial}{\partial \theta_2}$. $T_pN \cap \xi$ is spanned by $\frac{\partial}{\partial r_1} + \frac{\partial}{\partial \theta_2}$, $\frac{\partial}{\partial r_2}$ and $r_2\frac{\partial}{\partial r_1} - r_1\frac{\partial}{\partial \theta_2}$. The kernel of $(j^*d\alpha_{\text{std}})_p$ is spanned by $\frac{\partial}{\partial r_1}$, $\frac{\partial}{\partial \theta_2}$ $\frac{\partial}{\partial r_2}$ and $r_2\frac{\partial}{\partial r_1} - r_1\frac{\partial}{\partial \theta_2}$. Hence, any nonsingular vector field $X$ on $N$ positively transverse to $\xi_{\text{std}}$ with $X \in \ker j^*d\alpha_{\text{std}}$ satisfies the condition (2) of Theorem 1.2.

Finally, we want to pose the following

Question. Let $(M, \xi)$ be a cooriented contact manifold, $\mathcal{L}$ a compact oriented smooth lamination of $M$ with 1-dimensional leaves positively transverse to $\xi$. Then, find a (general) condition on $\mathcal{L}$ under which there exists a contact form $\alpha$ on $M$ defining $\xi$ such that the Reeb vector field $R_\alpha$ is everywhere tangent to the leaves of $\mathcal{L}$.

Here, we say that a lamination $\mathcal{L}$ on $M$ is smooth if there exist an open neighborhood $U$ of $\mathcal{L}$ in $M$ and an oriented smooth foliation $\mathcal{F}$ on $U$ such that $\mathcal{F}$ contains $\mathcal{L}$ as a saturated subset.

At this point the author does not know how to approach this question from a general viewpoint. The only positive answer the author can say at present is the following

Corollary 3.1. Let $(M, \xi)$ be a cooriented contact manifold and $N$ a compact isotropic submanifold of $M$ transverse to $\xi$. Then, for any compact oriented smooth 1-dimensional lamination $\mathcal{L}$ on $N$ positively transverse to $\xi$, there exists a contact form $\alpha$ on $M$ defining $\xi$ such that the Reeb vector field $R_\alpha$ is everywhere tangent to the leaves of $\mathcal{L}$.

Proof. Suppose that $N$ is isotropic with respect to a contact form $\alpha$ on $M$ defining $\xi$. Then, the plane field $TN \cap \xi$ defines a foliation on $N$ given by the closed 1-form $i^*\alpha$. By Tischler’s theorem [4], $N$ fibers over $S^1$ so that the tangent bundle to the
fibers are arbitrary close to $TN \cap \xi$. This and the assumption that $\mathcal{L}$ is transverse to $TN \cap \xi$ imply that there exists an oriented smooth foliation $\mathcal{F}$ on $N$ such that $\mathcal{F}$ contains $\mathcal{L}$ as a saturated subset and that $\mathcal{F}$ is everywhere transverse to $TN \cap \xi$. Now, since any vector field tangent to $\mathcal{F}$ positively transverse to $TN \cap \xi$ satisfies the hypothesis of Corollary 1.4, the result follows.

\section*{References}


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