EXTENDING A VECTOR FIELD ON A SUBMANIFOLD TO A REEB VECTOR FIELD ON THE WHOLE CONTACT MANIFOLD

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Abstract. Given a vector field on a submanifold of a contact manifold, when does it extend to a Reeb vector field on the whole manifold? We give an answer to this question.

1. Introduction and statement of results

The purpose of this short note is to give a basic, but (as far as the author thinks) important, result for Reeb vector fields. It may be known to the experts, but the author could not find it in the literature. Manifolds, functions, forms, vector fields and plane fields appearing in this note are all smooth, and submanifolds are always assumed to be without boundary. Let (M, ξ) be a manifold with a cooriented (i.e. transversely oriented) contact plane field. First, we will consider the following extension problem for a vector field: Let N be a compact submanifold of M transverse to ξ, and let X be a vector field on N positively transverse to ξ. In this situation, can we find a positive function f on N and a contact form α on M defining ξ (i.e. $\ker\alpha = \xi$ and the coorientation of ξ is compatible with α) such that fX coincides on N with the Reeb vector field R_α of α? Some cases are known. If N is 1 dimensional (i.e. N is a finite union of loops), then, the answer is yes. (In this case we need not multiply X by a function). If N is a contact submanifold of arbitrary dimension and X is the Reeb vector field R_β of some contact form β on N defining $TN \cap \xi$, then the answer is also yes (e.g. [3, Proof of Theorem 2.5.15]). (In this case we need not multiply X by a function, either). In [1] we have shown the following. Let $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$ be the standard contact space, where $\xi_{\text{std}} = \ker\alpha_{\text{std}}$ and $\alpha_{\text{std}} = dz + \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$ ($(x_1, y_1, \cdots, x_n, y_n, z)$ being the coordinates of $\mathbb{R}^{2n+1}$), T^n the torus defined by $z = 0$ and ${x_j}^2 + {y_j}^2 = 1$ ($j = 1, \cdots, n$), and let X be an arbitrary vector field on T^n transverse to ξ_{std}. Then, the answer is again yes (In this case we may need to multiply X by a nonconstant function). Here, remark that T^n is not a contact submanifold of $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$.

In this note we unite the above mentioned results in one and formulate as follows.

Theorem 1.1. *Let (M, ξ) be a cooriented contact manifold, N a compact submanifold of M transverse to ξ and X a vector field on N positively transverse to ξ everywhere on N. Then, the following are equivalent.*

(1) *There exists a contact form α on M defining ξ such that the restriction of the Reeb vector field R_α on N coincides with X.*

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(2) *X preserves $TN \cap \xi$.*

We have some corollaries to this theorem.

Theorem 1.2 (Rescaling version)**.** *Let (M, ξ) be a cooriented contact manifold, N a compact submanifold of M transverse to ξ and X a vector field on N positively transverse to ξ. Then, the following are equivalent.*

(1) *There exist a contact form α on M defining ξ and a function f on N such that the restriction of the Reeb vector field R_α on N coincides with fX.*
(2) *There exists a contact form α on M defining ξ such that $i_X d\alpha = 0$ on TN.*

Remark. Under the assumption of Theorem 1.2, there exists a vector field which never extends to a Reeb vector field by rescaling. For example, if N is a contact submanifold and X has a hyperbolic contracting periodic orbit, then, no rescaling of X extends to a Reeb vector field.

Corollary 1.3 ([3, Proof of Theorem 2.5.15])**.** *Let (M, ξ) be a cooriented contact manifold and N a compact contact submanifold of M. Then, the Reeb vector field of any contact form on N defining $TN \cap \xi$ extends to the Reeb vector field of some contact form on M defining ξ.*

Before stating the next corollary, we introduce one notion. Let (M, ξ) be a cooriented contact manifold and N a submanifold of M. Throughout this note we denote the inclusion map of N into M by j. Let α be a contact form on M which defines ξ. We say that N is *isotropic* with respect to α if $j^*d\alpha$ vanishes. We simply say that N is *isotropic* if N is isotropic with respect to some α defining ξ.

Notice. This definition of isotropy is weaker than the one given in Geiges' book ([3, Definition 1.5.11]), where N is called isotropic if $j^*\alpha$ vanishes. Our definition coincides with the one of Foulon-Hasselblatt[2], although, there, it is defined not for a submanifold but for a tangent subspace and for a foliation.

The following is related to (and generalizes in some sense) [1, Theorem 1.1] since T^n is an isotropic submanifold of $(\mathbb{R}^{2n+1}, \xi_{\text{std}})$.

Corollary 1.4. *Let (M, ξ) be a cooriented contact manifold and N a compact isotropic submanifold of M transverse to ξ. Then, any vector field on N positively transverse to ξ extends, after multiplying a suitable function, to the Reeb vector field of some contact form on M defining ξ.*

Corollary 1.5 (folklore?)**.** *Let (M, ξ) be a cooriented contact manifold and N is a finite union of loops transverse to ξ. Then, any vector field on N positively transverse to ξ extends to the Reeb vector field of some contact form on M defining ξ.*

Next, we consider an extension problem of another type: the case where N is a compact Legendrian submanifold of (M, ξ) and $X = \{X_p \in T_p(M) \mid p \in N\}$ is a vector field *along* N which is positively transverse to ξ. Note that, unlike the previous case, in this case $X_p \notin T_p(N)$ for any $p \in N$.

Theorem 1.6. *Let (M, ξ) be a cooriented contact manifold, N a compact Legendrian submanifold of M and X a vector field along N positively transverse to ξ. Then, the following are equivalent.*

(1) *There exists a contact form α on M defining ξ such that the restriction of the Reeb vector field R_α on N coincides with X.*
(2) *For any contact form α on M defining ξ and satisfying $\alpha(X) = 1$ on N, it holds that $i_X d\alpha = 0$ on TN.*
(3) *There exists a contact form α on M defining ξ such that $\alpha(X) = 1$ on N and that $i_X d\alpha = 0$ on TN.*

We pose here the following

Problem. Find a statement equivalent to (1) which is expressed only in terms of X, ξ and N.

As an immediate consequence of the above theorem we have

Theorem 1.7 (Rescaling version)**.** *Let (M, ξ) be a cooriented contact manifold, N a compact Legendrian submanifold of M and X a vector field along N positively transverse to ξ. Then, the following are equivalent.*

(1) *There exist a contact form α on M defining ξ and a function f on N such that the restriction of the Reeb vector field R_α on N coincides with fX.*
(2) *There exists a contact form α on M defining ξ such that $i_X d\alpha = 0$ on TN.*

Remark. Note that, under the assumption of Theorem 1.7, there exists X which never extends to a Reeb vector field by any rescaling. In §4 we give an example.

2. Non-isotropy

Given a submanifold of a contact manifold, how can one know its non-isotropy? Here we give one simple criterion. Let (M, ξ) be a cooriented contact manifold and N a compact submanifold of M.

Proposition 2.1. *If there exists a contact form α on M defining ξ such that $j^* d\alpha$ never vanishes on N, then N is non-isotropic.*

Proof of Proposition 2.1. Suppose there exists an α as in the proposition. Let β be any contact form on M defining ξ. Then, $\beta = f\alpha$ for some non-vanishing function f on M. Since N is compact, there exists a point $p \in N$ such that $(j^* df)_p = 0$. Thus we have $(j^* d\beta)_p = (j^* d(f\alpha))_p = (j^*(df \wedge \alpha + f d\alpha))_p = (j^* df)_p \wedge (j^*\alpha)_p + f(p)(j^* d\alpha)_p = f(p)(j^* d\alpha)_p$. Hence $(j^* d\beta)_p$ does not vanish, implying that N is not isotropic with respect to β. Since β is arbitrary, we get the conclusion. □

3. The case where N is transverse to ξ

Proof of Theorem 1.1. Obviously (1) implies (2). The implication from (2) to (1) is done just by adjusting the argument in [3, Proof of Theorem 2.5.15] to our situation. For the self-containedness of this note, we give the detail below. Let (M, ξ), N and X be as in the hypothesis of Theorem 1.1. Suppose further that X satisfies the condition (2), namely, X preserves $TN \cap \xi$. First, take any contact form α on M defining ξ. Then, in terms of α the condition (2) can be restated as follows: there exists a function λ on N such that $L_X j^*\alpha = \lambda j^*\alpha$. Now, we choose a function f on M so that $f = 1/\alpha(X)$ on N (Such f certainly exists.) and define $\beta = f\alpha$. Then, β is also a contact form on M defining ξ and $\beta(X) = 1$ on N. Moreover, by a usual computation, we have $L_X j^*\beta = 0$ on N. In fact, $L_X j^*\beta = L_X\{(1/\alpha(X))j^*\alpha\} = X(1/\alpha(X))j^*\alpha + (1/\alpha(X))L_X j^*\alpha = (1/(\alpha(X))^2\{-X(\alpha(X)) + \alpha(X)\lambda\}j^*\alpha = 0$, because $X(\alpha(X)) = i_X d i_X j^*\alpha = i_X L_X j^*\alpha = i_X(\lambda j^*\alpha) = \alpha(X)\lambda$. Since $\beta(X) = 1$ on N, $L_X j^*\beta = 0$ implies $i_X j^* d\beta = 0$ on N. Now, we will extend X to a Reeb vector field (for some contact form) on M. To this end, we look for a smooth function g on M satisfying (a) $g > 0$ on M, (b) $g = 1$ on N and (c) $i_X d(g\beta) = 0$ on $TM|_N$. Assuming (b), we have $i_X d(g\beta) = i_X(dg\wedge\beta + g d\beta) = (i_X dg)\beta - (i_X\beta)dg + g i_X d\beta = (Xg)\beta - \beta(X)dg + g i_X d\beta = -dg + i_X d\beta$. Thus, the condition (c) may be replaced with (c)': $dg = i_X d\beta$ on $TM|_N$. Now, to begin the construction of g, first set $g = 1$ on N. Then, dg and $i_X d\beta$ are both defined and vanish on TN, hence, the equality (c)' is consistent on the subbundle TN of $TM|_N$. Next, we will extend g over a neighborhood of N. Take a small tubular neighborhood $U = \exp(\nu)$ of N, where ν is a normal disk bundle of N in M and $\exp : \nu \to M$ is the exponential map along N with respect to some Riemannian metric on M. Define g on U by $g(\exp(v)) = 1 + d\beta(X, v)$ for $v \in \nu$. Then, g satisfies (c)' and is positive if U is small. Now, extending g over whole M preserving positivity is a routine work. The existence of g is proven. Using this g, we consider the contact form $\gamma = g\beta$. Then, its Reeb vector field R_γ is the desired one. In fact, by definition, R_γ satisfies $\gamma(R_\gamma) = 1$ and $\iota(R_\gamma)d\gamma = 0$. If we restrict these properties to $TM|_N$, these two properties of R_γ are the same with those of X. Since γ is a contact form on M, the vector field along N which satisfies these two properties is unique. Therefore, we conclude that $R_\gamma = X$. The proof is complete. □

Proof of Theorem 1.2. Let (M, ξ), N and X be as in the hypothesis of Theorem 1.2. That (1) implies (2) is obvious. To prove the converse implication, suppose that X satisfies $X_p \in \ker(j^*d\beta)_p$ for any $p \in N$. Then, the rescaled vector field $(1/\beta(X))X$ satisfies $L_{(1/\beta(X))X} j^*\beta = 0$, hence, preserves $TN \cap \xi$. Therefore, by Theorem 1.1, $(1/\beta(X))X$ is the restriction of the Reeb vector field of some contact form on M defining ξ, as desired.

Proof of Corollary 1.3. Let N be a compact contact submanifold of (M, ξ) and X the Reeb vector field of some contact form on N defining $TN \cap \xi$. Then, since X preserves $TN \cap \xi$, we can apply Theorem 1.1 and obtain the conclusion. □

Proof of Corollary 1.4. Let N be a compact isotropic submanifold of (M, ξ). Then, by the definition of an isotropic submanifold, for any vector field X on N transverse to ξ there exists a contact form β on M defining ξ such that X and β satisfy the condition (2) of Theorem 1.2. So, the conclusion of the rescaling version follows. □

Proof of Corollary 1.5. Let N, X be as in Corollary 1.5. Then, since $\dim N = 1$, X obviously preserves $TN \cap \xi$ (the zero dimensional plane field!). Hence, the result follows. □

We give a simple example of the case of a submanifold which is neither contact nor isotropic.

Example. Let N be a submanifold of $(\mathbb{R}^7, \xi_{\text{std}} = \ker \alpha_{\text{std}})$ defined by $z = 0$, ${x_1}^2 + {y_1}^2 + {x_2}^2 + {y_2}^2 = 1$ and ${x_3}^2 + {y_3}^2 = 1$. Then, N is diffeomorphic to $S^3 \times S^1$ and transverse to ξ. As a submanifold of $(\mathbb{R}^7, \xi_{\text{std}})$, N is neither contact nor isotropic. In fact, it is obviously non-contact by the dimension reason and non-isotropic because $j^* d\alpha_{\text{std}}$ never vanishes in the S^3-direction (see Proposition 2.1). Let (r_i, θ_i) be the polar coordinates for (x_i, y_i). Then, at each point of N, $\ker(j^* d\alpha_{\text{std}})$ is spanned by $\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}$ and $\frac{\partial}{\partial\theta_3}$, while $\ker(j^* d\alpha_{\text{std}}) \cap \xi$ is spanned by $\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2} - \frac{\partial}{\partial\theta_3}$. Thus, any vector field X on N of the form $g(\frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}) - h\frac{\partial}{\partial\theta_3}$ (g, h are functions on N such that $g > h$) satisfies the condition (2) of Theorem 1.2 for α_{std}, hence, extends to a Reeb vector field on $(\mathbb{R}^7, \xi_{\text{std}})$ after a suitable function multiplication.

In closing this section, we want to pose the following

Question. Let (M, ξ) be a cooriented contact manifold, $\mathcal{L}$ a compact oriented smooth lamination of M with 1-dimensional leaves positively transverse to ξ. Then, find a (general) condition on $\mathcal{L}$ under which there exists a contact form α on M defining ξ such that the Reeb vector field R_α is everywhere tangent to the leaves of $\mathcal{L}$.

Here, we say that a lamination $\mathcal{L}$ on M is *smooth* if there exist an open neighborhood U of $\mathcal{L}$ in M and a smooth foliation $\mathcal{F}$ on U such that $\mathcal{F}$ contains $\mathcal{L}$ as a saturated subset.

At present the author does not know how to approach this question from a general viewpoint. The only positive answer the author can say is the following

Corollary 3.1. *Let (M, ξ) be a cooriented contact manifold and N a compact isotropic submanifold of M transverse to ξ. Then, for any compact oriented smooth 1-dimensional lamination $\mathcal{L}$ on N positively transverse to ξ, there exists a contact form α on M defining ξ such that the Reeb vector field R_α is everywhere tangent to the leaves of $\mathcal{L}$.*

Proof. Suppose that N is isotropic with respect to a contact form α on M defining ξ. Then, the plane field $TN \cap \xi$ defines a foliation on N given by the closed 1-form $i^*\alpha$. By Tischler's theorem [4], N fibers over S^1 so that the tangent bundle to the fibers are arbitrary close to $TN \cap \xi$. This and the assumption that $\mathcal{L}$ is transverse to $TN \cap \xi$ imply that there exists an oriented smooth foliation $\mathcal{F}$ on N such that $\mathcal{F}$ contains $\mathcal{L}$ as a saturated subset and that $\mathcal{F}$ is everywhere transverse to $TN \cap \xi$. Now, since any vector field tangent to $\mathcal{F}$ transverse to $TN \cap \xi$ satisfies the hypothesis of Corollary 1.4, the result follows. □

4. The case where N is Legendrian

Proof of Theorem 1.6.

(1) $\Longrightarrow$ (2): Suppose that X extends to the Reeb vector field of some contact form β on M defining ξ. Let α be any contact form on M defining ξ and satisfying $\alpha(X) = 1$ on N. Then, since α and β are both contact forms defining ξ, there exists a nowhere vanishing function f on M such that $\beta = f\alpha$. On N, we have $\beta(X) = 1$ (because $X = R_\beta$ on N) and $\alpha(X) = 1$ (by the assumption), hence $f = 1$ on N. Since $X = R_\beta$ on N and $\beta = f\alpha$, we have $i_X d(f\alpha) = 0$ on TN. By computation, on TN we have $i_X(df \wedge \alpha + f d\alpha) = 0$, hence, $(Xf)\alpha - \alpha(X)df + f i_X d\alpha = 0$. Here

we have $\alpha = 0$ on $TN \subset \xi$, and $df = 0$ (because $f = 1$ on N) and $f = 1$ on N. Therefore, we conclude that $i_X d\alpha = 0$ on TN, as desired.

(2) $\Longrightarrow$ (3): Take an arbitrary contact form β on M which defines ξ. Let f be a nowhere vanishing function on M which coincides with $1/\beta(X)$ on N and put $\alpha = f\beta$. Then, since α is still a contact form defining ξ and satisfies the hypothesis of (2), it follows from the conclusion of (2) that α satisfies (3), as desired.

(3) $\Longrightarrow$ (1): Let α be a contact form on M such that $\alpha(X) = 1$ on N and that $i_X d\alpha = 0$ on TN. First, choose a subbundle E of $\xi|_N$ so that $\xi|_N = TN \oplus E$, thus $TM|_N = \langle X \rangle \oplus TN \oplus E$, where $\langle X \rangle$ is the line bundle on N generated by X. Then, we can easily construct a nowhere vanishing function f on M so that $f = 1$ on N and that at each point of N, the differential df is 0 on $\langle X \rangle \oplus TN$ and coincides with $i_X d\alpha$ on E. Now, put $\beta = f\alpha$. Then, β is a contact form on M such that $\beta(X) = 1$ on N and that $i_X d\beta = 0$ on $TM|_N$. This means that $X = R_\beta$ on N. □

Proof of Theorem 1.7. Let (M, ξ), N and X be as in the hypothesis in Theorem 1.6. Assume (1). Then, by the condition of the Reeb vector field we have $i_X d\alpha = (1/f) i_{fX} d\alpha = (1/f) \cdot 0 = 0$, implying (2). Conversely, assume (2). Namely, assume that there exists a contact form α on M defining ξ such that $i_X d\alpha = 0$ on TN. Then, if we take a function f on M satisfying $f = 1/\alpha(X)$ on N, we see that $\alpha(fX) = f\alpha(X) = 1$ and $i_{fX} d\alpha = f i_X d\alpha = 0$ on TN. This means that α and fX satisfy the condition (3) in Theorem 1.6. Thus, by Theorem 1.6 the condition (1) in Theorem 1.7 follows. This completes the proof.

Finally, we give a simple example of X which, on the assumption of Theorem 1.7, cannot extend to a Reeb vector field by any rescaling.

Example. Let $M = T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with coordinates (x, y, z). Define a contact form α on M by $\alpha = (\sin 2\pi z)dx + (\cos 2\pi z)dy$ and put $\xi = \ker \alpha$. Then, the Reeb vector field of α is $R_\alpha = (\sin 2\pi z)\partial/\partial x + (\cos 2\pi z)\partial/\partial y$. Let N be a Legendrian circle in M defined by $x = y = 0$, and let X be a vector field along N defined by $X = R + W$, where $W = -(\cos 2\pi z)\partial/\partial x + (\sin 2\pi z)\partial/\partial y$. Then, $i_X d\alpha = 2\pi dz$. Now, we will show that, for this X, the statement (2) of Theorem 1.7 does not hold. Namely, we will prove that, for any contact form β of M defining ξ, there exists a point p of N such that $(i_X d\beta)|_{TN}$ does not vanish at p. In fact, suppose on the contrary that $(i_X d\beta)|_{TN}$ vanishes identically on N for some β. Since α and β commonly define ξ, there is a positive smooth function h on M such that $\beta = h\alpha$. Thus we have $0 = i_X d\beta = i_X d(h\alpha) = (Xh)\alpha - \alpha(X)dh + h i_X d\alpha$. Since $\alpha = 0$ on TN, we obtain on TN the equality $d \log h = (2\pi/\alpha(X))dz$. Here, the right hand side never vanishes on TN while the left hand side must vanish at a critical point of $\log h|_N$. This contradiction proves that (2) of Theorem 1.7 does not hold for X, hence, by Theorem 1.7, X does not extend to a Reeb vector field.

References

[1] T. Arai, T. Inaba and Y. Kano, *Reeb orbits trapped by Denjoy minimal sets*, Arch. Math. (Basel) 103 (2014), 381–388.

[2] P. Foulon and B. Hasselblatt, *Godbillon-Vey invariants for maximal isotropic C^2 foliations*, Adv. Stud. Pure Math. **72** (2017), Geometry, Dynamics and Foliations 2013, pp. 349–365.

[3] H. Geiges, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math., vol. 109, Cambridge Univ. Press, Cambridge, 2008.

[4] D. Tischler, *On fibering certain foliated manifolds over S^1*, Topology **9** (1970), 153–154.

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