THE ACTION OF GEOMETRIC AUTOMORPHISMS OF ASYMPTOTIC TEICHMÜLLER SPACES

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1. Introduction

The Teichmüller space $T(R)$ of a Riemann surface R is a deformation space of the complex structure of R, and the quasiconformal mapping class group $\mathrm{MCG}(R)$ of R is the set of all homotopy classes of quasiconformal automorphisms of R. The quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on the Teichmüller space $T(R)$ isometrically with respect to the Teichmüller distance, which induces the Teichmüller modular group $\mathrm{Mod}(R)$. If R is of analytically infinite type, then $T(R)$ is infinite dimensional and the action of $\mathrm{MCG}(R)$ on $T(R)$ is, in general, not discontinuous. This is equivalent to that the orbit of some point in $T(R)$ under the action of $\mathrm{MCG}(R)$ is not discrete. This phenomenon appears only when Teichmüller spaces are infinite dimensional, namely it does not occur for Riemann surfaces of analytically finite type. On the basis of this fact, in [8] and [9], we have introduced the notion of limit sets and regions of discontinuity of Teichmüller modular groups, analogous to the theory of Kleinian groups, and studied the dynamics of Teichmüller modular groups. In this paper, we apply the theory of dynamics to isometric automorphisms on a certain quotient space of $T(R)$ named the asymptotic Teichmüller space.

The asymptotic Teichmüller space $AT(R)$ of R was introduced in [14] when R is the upper half-plane and in [4] and [13] when R is an arbitrary hyperbolic Riemann surface. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is of analytically infinite type. Otherwise $AT(R)$ consists of just one point. It is similar to the action of $\mathrm{MCG}(R)$ on $T(R)$ that every element of $\mathrm{MCG}(R)$ induces an isometric automorphism of $AT(R)$. Namely, we have a homomorphism $\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Isom}(AT(R))$. We define the geometric automorphism group $\mathcal{G}(R)$ as the image $\iota_{AT}(\mathrm{MCG}(R))$.

We investigate the dynamical behavior of $\mathcal{G}(R)$ on $AT(R)$. However it is different from the action of $\mathrm{MCG}(R)$ on $T(R)$ that the homomorphism ι_{AT} is not injective. Furthermore, the action of $\mathrm{MCG}(R)$ on $AT(R)$ can be trivial. In fact, there exists such an example that the action of $\mathrm{MCG}(R)$ is trivial even if $AT(R)$ is non-trivial. Thus it is necessary to know when the action of $\mathrm{MCG}(R)$ on $AT(R)$ is non-trivial. We prove that if a Riemann surface R is of topologically infinite type and satisfies the upper bound condition, then the action of $\mathrm{MCG}(R)$ on $AT(R)$ is non-trivial. Furthermore, for a Riemann surface R that does not necessarily satisfy the upper bound condition, we give a condition for a quasiconformal automorphism of R to induce a non-trivial action on $AT(R)$.

In the last section, we define the limit set and the region of discontinuity of $\mathcal{G}(R)$, and observe a relationship between the limit set of $\mathrm{Mod}(R)$ on $T(R)$ and that of $\mathcal{G}(R)$ on $AT(R)$. We prove that if a Riemann surface R does not satisfy the lower bound condition, then both the limit set of $\mathrm{Mod}(R)$ on $T(R)$ and that of $\mathcal{G}(R)$ on $AT(R)$ coincide with the whole spaces respectively. On the other hand, we have an example of a Riemann surface R satisfying the lower and upper bound conditions for which the limit set of $\mathrm{Mod}(R)$ on $T(R)$ is empty but the limit set of $\mathcal{G}(R)$ on $AT(R)$ is not empty.

2. Preliminaries

2.1. **Teichmüller spaces and Teichmüller modular groups.** Throughout this paper, we assume that a Riemann surface R is hyperbolic. Namely, it is represented by a quotient space $\mathbb{H}/\Gamma$ of the upper half-plane $\mathbb{H}$ by a torsion free Fuchsian group Γ. We say that R is of *analytically finite type* if it is a compact surface with at most finitely many points removed, and R is of *topologically finite type* if it is a compact surface with at most finitely many points and disks removed. Furthermore, we say that R satisfies the *lower bound condition* if the injectivity radius at any point of R except cusp neighborhoods is uniformly bounded away from zero, and R satisfies the *upper bound condition* if there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R. The lower and upper bound conditions are quasiconformally invariant.

We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1 : R \to R$ is homotopic to the identity. Here the homotopy is considered to be relative to the ideal boundary at infinity. The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1], [f_2]) = \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$, which is called the Teichmüller distance. For fundamental facts on Teichmüller spaces, see [12] and [19].

The *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the set of all quasiconformal mapping classes on R. Here the homotopy is considered to be relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces an automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is an isometry with respect to d_T. Let $\mathrm{Isom}(T(R))$ be the group of all isometric automorphisms of $T(R)$. Then we have a homomorphism

$$\iota_T : \mathrm{MCG}(R) \to \mathrm{Isom}(T(R))$$

given by $[g] \mapsto [g]_*$ and we define the *Teichmüller modular group* by

$$\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R)).$$

It was proved in [3] and [7] that ι_T is injective (faithful) if R is of non-exceptional type. Here we say that a Riemann surface R is of exceptional type if R is of analytically finite type and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R.

2.2. **Asymptotic Teichmüller spaces.** We say that a quasiconformal homeomorphism f on R is *asymptotically conformal* if for every $\varepsilon > 0$, there exists a compact subset E of R such that the maximal dilatation f is less than $1 + \varepsilon$ on $R - E$. We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1 : R \to R$ is homotopic to the identity relative to the ideal boundary at infinity of R. The *asymptotic Teichmüller space* $AT(R)$ with the base Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. See [4] and [13] for details. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic. See also [5] and [6].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-E})$, where infimum is taken over all compact subsets E of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_{AT}([[f_1]], [[f_2]]) = \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_{AT} is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

3. Non-trivial actions of quasiconformal mapping classes on $AT(R)$

It is similar to the case of Teichmüller spaces that every element $[g] \in \mathrm{MCG}(R)$ induces an automorphism $[g]_*$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is an isometry with respect to d_{AT}. Let $\mathrm{Isom}(AT(R))$ be the group of all isometric automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Isom}(AT(R))$$

given by $[g] \mapsto [g]_*$. We define the *geometric automorphism group* by

$$\mathcal{G}(R) = \iota_{AT}(\mathrm{MCG}(R)),$$

and call an element of $\mathcal{G}(R)$ geometric automorphism.

It is different from the case of $\iota_T : \mathrm{MCG}(R) \to \mathrm{Isom}(T(R))$ that the homomorphism ι_{AT} is not injective, namely $\mathrm{Ker}\, \iota_{AT} \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. Furthermore, it is not always true that $\mathrm{MCG}(R)$ acts on $AT(R)$ non-trivially. Actually, if $AT(R)$ is trivial, then the action of $\mathrm{MCG}(R)$ is trivial. Even if $AT(R)$ is non-trivial, we have a following example.

Example 3.1. In [17], a Riemann surface R of analytically infinite type is constructed so that $\mathrm{Ker}\, \iota_{AT} = \mathrm{MCG}(R)$, namely the action of $\mathrm{MCG}(R)$ on $AT(R)$ is trivial. Note that R does not satisfy the upper bound condition.

In this section, we give a sufficient condition of a Riemann surface for the quasiconformal mapping class group to act on the asymptotic Teichmüller space non-trivially. By definition, the action of $\mathrm{MCG}(R)$ on $AT(R)$ is non-trivial if and only if there exist an element $[g] \in \mathrm{MCG}(R)$ and a point $\hat{\tau} = [[f]] \in AT(R)$ such that $[g]_*(\hat{\tau}) \neq \hat{\tau}$, namely the homotopy equivalence class of $f \circ g^{-1} \circ f^{-1}$ contains no asymptotically conformal automorphisms of $f(R)$.

For a non-trivial simple closed curve c on R, let $\ell(c)$ be the geodesic length for the free homotopy class of c, and let d be the hyperbolic distance on R.

Theorem 3.2. *Let $R = \mathbb{H}/\Gamma$ be a Riemann surface on which there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of infinitely many simple closed geodesics such that the hyperbolic lengths $\ell_n := \ell(c_n)$ are uniformly bounded from above. Then the action of $\mathrm{MCG}(R)$ on $AT(R)$ is non-trivial.*

For a proof of Theorem 3.2, the following lemma on extremal length is crucial. Let $F = \{\beta\}$ be a set of rectifiable curves β on a domain D in $\mathbb{C}$. Then the extremal length of the curve family F is defined by

$$\lambda(F) = \sup_{\rho} \frac{\{\inf_{\beta \in F} \int_\beta \rho(z)|dz|\}^2}{\iint_D \rho(z)^2 dxdy},$$

where the supremum is taken over all Borel measurable non-negative functions $\rho(z)$ on D. If the supremum is attained by $\rho_0(z)$, then we call $\rho_0(z)|dz|$ the extremal metric for the curve family F. For details, see [1] and [20].

Lemma 3.3. *Let Q be a quadrilateral in $\mathbb{C}$ and Q_0 be a subdomain of Q. Let F be a family of all curves on Q connecting a pair of the opposite sides of the boundary of Q. Let g be a K-quasiconformal homeomorphism on Q such that the restriction of g to Q_0 is $(1+\varepsilon)$-quasiconformal for some $\varepsilon > 0$. Then the extremal length $\lambda(g(F))$ of the curve family $g(F)$ on $g(Q)$ satisfies*

$$(1/C) \cdot \lambda(F) \leq \lambda(g(F)) \leq C \cdot \lambda(F),$$

where $C = C(K, \varepsilon, r) = K + (1 + \varepsilon - K)r$ and $r < 1$ is the ratio of the area of Q_0 to Q with respect to the extremal metric for the curve family F.

Proof. Let F' be a family of all curves on Q connecting another pair of the opposite sides of the boundary of Q. Then $\lambda(F)\lambda(F') = 1$ and $\lambda(g(F))\lambda(g(F')) = 1$. We will obtain a lower bound of $\lambda(F')$. Let $\lambda_0(z)|dz|$ be the extremal metric for the curve family F' that is the same as that for the curve family F up to a constant, and we define a metric $\rho(\zeta)|d\zeta|$ on $g(Q)$ by

$$\rho(\zeta) := \frac{\lambda_0(g^{-1}(\zeta))}{|g_z(g^{-1}(\zeta))| - |g_{\bar{z}}(g^{-1}(\zeta))|}.$$

Since $|d\zeta| \geq (|g_z(z)| - |g_{\bar{z}}(z)|)|dz|$, we have

$$\int_{g(\beta)} \rho(\zeta)|d\zeta| \geq \int_\beta \lambda_0(z)|dz|$$

for an arbitrary curve $\beta \in F'$. Furthermore

$$\begin{aligned}\iint_{g(Q)} \rho(\zeta)^2 d\xi d\eta &= \iint_Q \lambda_0(z)^2 \frac{|g_z|+|g_{\bar z}|}{|g_z|-|g_{\bar z}|} dxdy \\ &\leq K \iint_{Q-Q_0} \lambda_0(z)^2 dxdy + (1+\varepsilon) \iint_{Q_0} \lambda_0(z)^2 dxdy \\ &= (1-r)K \iint_Q \lambda_0(z)^2 dxdy + r(1+\varepsilon) \iint_Q \lambda_0(z)^2 dxdy \\ &= \{K + (1+\varepsilon-K)r\} \iint_Q \lambda_0(z)^2 dxdy.\end{aligned}$$

Thus we have

$$\lambda(g(F')) \geq \frac{1}{K+(1+\varepsilon-K)r} \lambda(F'), \tag{1}$$

which implies that $\lambda(g(F)) \leq C \cdot \lambda(F)$. Since the inequality (1) holds also for the curve family F, we have the other inequality. □

Lemma 3.3 holds also for annuli as follows, which was originally proved in [18].

Lemma 3.4. *Let A be an annulus and A_0 be a subdomain of A. Let F be a family of all curves on A connecting distinct boundary components of A. If g is a K-quasiconformal homeomorphism on A such that the restriction of g to A_0 is $(1+\varepsilon)$-quasiconformal for some $\varepsilon > 0$, then the inequality*

$$(1/C) \cdot \lambda(F) \leq \lambda(g(F)) \leq C \cdot \lambda(F)$$

holds. The constant C is the same as in Lemma 3.3.

Proof. Let F' be a family of all curves on A separating the boundary components of A. Then $\lambda(F)\lambda(F') = 1$ and $\lambda(g(F))\lambda(g(F')) = 1$. Thus the proof of Lemma 3.3 can be applied. □

For arguments in this section, we recall some facts on the extremal length on annuli. Let c be a simple closed geodesic on a Riemann surface $R = \mathbb{H}/\Gamma$ with the hyperbolic length $\ell > 0$, and let γ be a hyperbolic element of Γ corresponding to c. We may assume that $\gamma(z) = e^{\ell} z$ and consider $A = \mathbb{H}/\langle\gamma\rangle$, which is an annular cover of R. Then A is conformally equivalent to $\{z \in \mathbb{C} \mid 1 < |z| < s\}$, where $\log s = 2\pi^2/\ell$. Let F be a family of all curves on A connecting distinct boundary components of A. Then the extremal metric for the curve family F is given by

$$\rho_s(z)|dz| := \frac{|dz|}{|z|\log s},$$

and $\lambda(F) = \log s/(2\pi)$. By this metric, the length of any radial segment is 1, the length of any concentric circle is $2\pi/\log s$ and the area of A is $2\pi/\log s$.

Set

$$\tilde{A}(\theta) = \left\{ z \in \mathbb{H} \mid \frac{\pi}{2} - \frac{\theta}{2} < \arg z < \frac{\pi}{2} + \frac{\theta}{2} \right\}$$

and $A(\theta) = \tilde{A}(\theta)/\langle\gamma\rangle$ which is a subdomain of A. Then with respect to the extremal metric $\rho_s(z)|dz|$ on A, the length of any radial segment in $A(\theta)$ is θ/π. Thus the ratio of the area of $A(\theta)$ to A is θ/π.

Proof of Theorem 3.2. By assumption, there exists a positive constant M such that $\ell_n \leq M$ for all n. Then c_n exit R, namely $d(p, c_n) \to \infty$ $(n \to \infty)$ for any point $p \in R$. By taking a subsequence, we may assume that c_n are all disjoint. See the proof of [16, Proposition 1].

For each n, we take an integer k_n so that $3M \leq k_n\ell_n \leq 4M$, and consider a mapping class caused by infinitely many Dehn twists with respect to each c_n in k_n times. Then there exists a quasiconformal automorphism g_0 in this mapping class such that the maximal dilatation $K(g_0)$ satisfies

$$K(g_0) \leq \sup_n \left(\left\{ \left(\frac{k_n\ell_n}{2\theta_n} \right)^2 + 1 \right\}^{\frac{1}{2}} + \frac{k_n\ell_n}{2\theta_n} \right)^2 \leq \left(\left(\frac{4M^2}{\theta^2} + 1 \right)^{\frac{1}{2}} + \frac{2M}{\theta} \right)^2,$$

where $\theta_n = 2\arctan\left\{(\sinh(\ell_n/2))^{-1}\right\}$ and $\theta = 2\arctan\left\{(\sinh(M/2))^{-1}\right\}$ (see [15]). Namely, $[g_0]$ is a quasiconformal mapping class. We will prove that $[g_0]$ contains no asymptotically conformal automorphisms of R. Then the element $[g_0]_* \in \mathcal{G}(R)$ does not fix the base point $[[id]]$ of $AT(R)$, and we have the assertion.

For each n, let γ_n be a hyperbolic element of Γ corresponding to c_n, and we take an annular cover $A_n = \mathbb{H}/\langle\gamma_n\rangle = \{z \in \mathbb{C} \mid 1 < |z| < s_n\}$ of R, where $\log s_n = 2\pi^2/\ell_n$. Let $Q_n = \{re^{i\theta} \in A_n \mid \ 0 < \theta < 2\pi\}$, which is a quadrilateral obtained by removing a geodesic from A_n with respect to the extremal metric $\rho_{s_n}(z)|dz|$, and let F_n be a family of all curves on Q_n connecting the distinct boundary components of A_n. For any element $g \in [g_0]$, let $\tilde{g}_n$ be a lift of g to A_n. Then $\tilde{g}_n$ fixes some boundary points a_1 and a_2 that belong to different components and, for a curve α connecting a_1 and a_2, the image $\tilde{g}_n(\alpha)$ wraps k_n times around A_n. In this circumstance, the extremal length $\lambda(\tilde{g}_n(F_n))$ of the curve family $\tilde{g}_n(F_n)$ on $\tilde{g}_n(Q_n)$ satisfies

$$\frac{\lambda(\tilde{g}_n(F_n))}{\lambda(F_n)} \geq \left((k_n - 1)\frac{\ell_n}{\pi} \right)^2 + 1 \geq \frac{4M^2}{\pi^2} + 1 =: M_0$$

for all n. See the proof of [15, Theorem 1].

We take a positive constant ε_0 so that $1 + \varepsilon_0 < M_0$. Suppose to the contrary that $[g_0]$ contains an asymptotically conformal automorphism h with the maximal dilatation $K_0 > 1$. Then there exists a compact subset E of R such that the restriction of h to $R - E$ is $(1 + \varepsilon_0)$-quasiconformal.

Since c_n exit R, we may assume that $c_n \subset R - E$ and set $d_n = d(E, c_n)$. For a fixed integer n, we may assume that $\gamma_n(z) = e^{\ell_n}z$, and consider $\tilde{A}(\theta_n)$ for $\theta_n = 2\arctan(\sinh d_n)$ and $A(\theta_n) = \tilde{A}(\theta_n)/\langle\gamma_n\rangle$. Then there is no lift of E in $A(\theta_n)$. The ratio of the area of $A(\theta_n)$ to A_n with respect to the extremal metric $\rho_{s_n}(z)|dz|$ is $\theta_n/\pi =: r_n$.

Let $C = C(K, \varepsilon, r)$ be the constant obtained in Lemma 3.3, which tends to $1 + \varepsilon$ as $r \to 1$. We take an integer n_0 so that r_{n_0} satisfies $C(K_0, \varepsilon_0, r_{n_0}) < M_0$. A lift $\tilde{h}_{n_0}$ of h to A_{n_0} is K_0-quasiconformal and the restriction of $\tilde{h}_{n_0}$ to $A(\theta_{n_0})$ is $(1 + \varepsilon_0)$-quasiconformal. By Lemma 3.3, we have

$$\frac{\lambda(\tilde{h}_{n_0}(F_{n_0}))}{\lambda(F_{n_0})} \leq C(K_0, \varepsilon_0, r_0) < M_0.$$

This contradicts the above argument. Hence we conclude that $[g_0]$ contains no asymptotically conformal automorphisms of R. □

As an corollary to Theorem 3.2, we have the following.

Corollary 3.5. *Let R be a Riemann surface of topologically infinite type. Suppose that R satisfies the upper bound condition. Then the action of* $\mathrm{MCG}(R)$ *on* $AT(R)$ *is non-trivial.*

Proof. Since R satisfies the upper bound condition, there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R. Since R is of topologically infinite type, the subdomain R^* is of topologically infinite type. Then we can take a sequence of infinitely many simple closed geodesics in R^* whose hyperbolic lengths are uniformly bounded from above. Thus the statement follows from Theorem 3.2. $\square$

On the other hand, if R does not satisfy the upper bound condition, then the action of $\mathrm{MCG}(R)$ on $AT(R)$ can be trivial as in Example 3.1. For a Riemann surface R which does not necessarily satisfy the upper bound condition, we give a condition for a quasiconformal automorphism of R to induce a non-trivial action on $AT(R)$.

Theorem 3.6. *Let g be a quasiconformal automorphism of a Riemann surface R. Suppose there exists a constant $\delta > 1$ such that, for every compact subset E of R, there is a simple closed geodesic c on R outside of E satisfying either*

$$\frac{\ell(g(c))}{\ell(c)} \leq \frac{1}{\delta} \qquad \text{or} \qquad \frac{\ell(g(c))}{\ell(c)} \geq \delta.$$

Then g is not homotopic to any asymptotically conformal automorphism of R. In particular, the action of $[g] \in \mathrm{MCG}(R)$ on $AT(R)$ is non-trivial.

This theorem easily follows from [6, Lemma 13.1]. Here we give another elementary proof. For the proof, we need the following lemma.

Lemma 3.7. *Let $R = \mathbb{H}/\Gamma$ be a Riemann surface and c be a simple closed geodesic on R. Let E be a subset on R and $d = d(c, E)$ be the hyperbolic distance between c and E. If g is a K-quasiconformal homeomorphism of R onto another Riemann surface such that the restriction of g to $R - E$ is $(1+\varepsilon)$-quasiconformal for some $\varepsilon > 0$, then the inequality*

$$(1/\alpha) \cdot \ell(c) \leq \ell(g(c)) \leq \alpha \cdot \ell(c)$$

holds for a constant

$$\alpha = \alpha(K, \varepsilon, d) = K + (1 + \varepsilon - K)\frac{2\arctan(\sinh d)}{\pi}.$$

Remark. When $E = R$ (thus $d = 0$), this lemma is nothing but the result [21, Lemma 3.1]. For $\varepsilon = 0$, the lemma is proved in [18].

Proof of Lemma 3.7. We take an annular cover $A = A(c) = \mathbb{H}/\langle\gamma\rangle$ of R with respect to a hyperbolic element $\gamma \in \Gamma$ corresponding to c. We may assume that $\gamma(z) = kz$, where $\log k = \ell(c)$. Consider $\tilde{A}(\theta)$ for $\theta = 2\arctan(\sinh d)$ and $A(\theta) = \tilde{A}(\theta)/\langle\gamma\rangle$. Then there is no lift of E in $A(\theta)$.

Let F be a family of all curves on A connecting the distinct boundary components of A. Then the extremal length $\lambda(F)$ of F is $\pi/\ell(c)$.

We take an annular cover $A' = A'(g(c)) = \mathbb{H}/\langle\gamma'\rangle$ of $f(R)$ with respect to the hyperbolic element $\gamma' = \tilde{g} \circ \gamma \circ \tilde{g}^{-1}$ corresponding to $g(c)$. Here $\tilde{g}$ is a lift of g to $\mathbb{H}$.

Then the projection $\hat{g} : A \to A'$ of $\tilde{g}$ is a K-quasiconformal homeomorphism and the restriction of $\hat{g}$ to $A(\theta)$ is $(1+\varepsilon)$-quasiconformal. By Lemma 3.4, we have

$$\frac{1}{K+(1+\varepsilon-K)(\theta/\pi)} \cdot \frac{\pi}{\ell(c)} \leq \lambda(\hat{g}(F)) \leq \{K+(1+\varepsilon-K)(\theta/\pi)\} \cdot \frac{\pi}{\ell(c)}.$$

Since $\lambda(\hat{g}(F)) = \pi/\ell(g(c))$, this inequality is equivalent to the assertion. □

Proof of Theorem 3.6. We take a positive constant ε_0 so that $1+\varepsilon_0 < \delta$. Suppose to the contrary that g is homotopic to an asymptotically conformal automorphism h. Then there exists a compact subset E of R such that the restriction of h to $R-E$ is $(1+\varepsilon_0)$-quasiconformal. Let $\alpha = \alpha(K, \varepsilon, d)$ be the constant obtained in Lemma 3.7, which tends to $1+\varepsilon$ as $d \to \infty$, and we take a positive constant d_0 so that $\alpha_0 = \alpha(K(h), \varepsilon_0, d_0) < \delta$. By assumption, we can take a simple closed geodesic c on R so that $d(E, c) \geq d_0$ and that either $\ell(h(c))/\ell(c) \leq 1/\delta$ or $\ell(h(c))/\ell(c) \geq \delta$. On the other hand, by Lemma 3.7, we have $(1/\alpha_0) \cdot \ell(c) \leq \ell(h(c)) \leq \alpha_0 \cdot \ell(c)$. We get a contradiction. □

4. Dynamics of geometric automorphisms of $AT(R)$

4.1. **Limit sets for groups of isometries.** In general, let X be a complete metric space with a distance d, and let $\mathrm{Isom}(X)$ be the group of all isometric automorphisms of X. For a subgroup $G \subset \mathrm{Isom}(X)$ and a point $x \in X$, it is said that $y \in X$ is a *limit point* of x for G if there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of distinct elements of G such that $d(g_n(x), y) \to 0$ $(n \to \infty)$. The set of all limit points of x for G is denoted by $\Lambda(G, x)$, and the *limit set* for G is defined by $\Lambda(G) = \bigcup_{x \in X} \Lambda(G, x)$. It is said that $x \in X$ is a *recurrent point* for G if $x \in \Lambda(G, x)$, and the set of all recurrent points for G is called the *recurrent set* for G and is denoted by $\mathrm{Rec}(G)$. It is evident from the definition that $\mathrm{Rec}(G) \subset \Lambda(G)$ and these sets are G-invariant. Moreover, it was proved in [10, Proposition 2.2] that $\Lambda(G) = \mathrm{Rec}(G)$, which is a closed set.

The complement $X - \Lambda(G)$ of the limit set is denoted by $\Omega(G)$. Then $\Omega(G)$ is the largest open subset in X where G acts discontinuously. This means that for every point $x \in \Omega(G)$, there exists a neighborhood U of x such that the number of elements $g \in G$ satisfying $g(U) \cap U \neq \emptyset$ is finite. Hence we call $\Omega(G)$ the *region of discontinuity* for G.

4.2. **Limit sets for geometric automorphism groups on $AT(R)$.** We apply the above mentioned notation to the Teichmüller space $T(R)$ with the Teichmüller distance d_T. Since the Teichmüller modular group $\mathrm{Mod}(R)$ is a subgroup of $\mathrm{Isom}(T(R))$, the limit set $\Lambda_T(G)$ and the region of discontinuity $\Omega_T(G)$ on $T(R)$ for a subgroup G of $\mathrm{Mod}(R)$ can be defined. See also [8]. This notion is of interest only when R is of analytically infinite type and has no ideal boundary at infinity. Indeed, we always have $T(R) = \Omega_T(\mathrm{Mod}(R))$ if R is of analytically finite type, and have $T(R) = \Lambda_T(\mathrm{Mod}(R))$ if R has ideal boundary at infinity. Hereafter, we assume that R is of analytically infinite type and has no ideal boundary at infinity, namely the Fuchsian model of R is infinitely generated of the first kind.

Similarly, for the asymptotic Teichmüller space $AT(R)$ with the asymptotic Teichmüller distance d_{AT}, we can consider the limit set $\Lambda_{AT}(G)$ on $AT(R)$ for a subgroup G of the geometric automorphism group $\mathcal{G}(R)$. In [8, Theorem 2], we proved that if R does not satisfy the lower bound condition, then $\Lambda_T(\mathrm{Mod}(R)) = T(R)$. The same statement holds also for the limit set of $\mathcal{G}(R)$ on $AT(R)$.

Theorem 4.1. *Let R be a Riemann surface that does not satisfy the lower bound condition. Then $\Lambda_{AT}(\mathcal{G}(R)) = AT(R)$.*

Proof. First we will construct quasiconformal automorphisms h_i $(i = 1, 2, \cdots)$ of R such that they are not homotopic to any asymptotically conformal automorphisms of R and that $H(h_i) \to 1$ $(i \to \infty)$. Then the base point $[[id]]$ of $AT(R)$ belongs to the limit set $\Lambda_{AT}(\mathcal{G}(R))$. By assumption, there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of simple closed geodesics on R such that $\ell_n := \ell(c_n) \to 0$ as $n \to \infty$. We may assume that ℓ_n are strictly decreasing. For a given integer $i \in \mathbb{N}$ and for all $n \geq i$, we take a positive integer $k(n, i)$ satisfying $3\ell_i \leq k(n, i)\ell_n \leq 4\ell_i$, and consider a mapping class caused by infinitely many Dehn twists with respect to each c_n $(n \geq i)$ in $k(n, i)$ times. Then there exists a quasiconformal automorphism h_i in this mapping class such that the maximal dilatation $K(h_i)$ satisfies

$$\begin{aligned} K(h_i) &\leq \sup_{n\geq i} \left(\left\{ \left(\frac{k(n,i)\ell_n}{2\,\theta_n} \right)^2 + 1 \right\}^{\frac{1}{2}} + \frac{k(n,i)\ell_n}{2\,\theta_n} \right)^2 \\ &\leq \left(\left(\frac{4\ell_i^2}{\theta_i^2} + 1 \right)^{\frac{1}{2}} + \frac{2\ell_i}{\theta_i} \right)^2, \end{aligned}$$

where $\theta_n = 2 \arctan\{(\sinh(\ell_n/2))^{-1}\}$. Namely, $[h_i]$ is a quasiconformal mapping class and $K(h_i) \to 1$ $(i \to \infty)$. Hence $H(h_i) \to 1$. By repeating the same argument as in the proof of Theorem 3.2 for each h_i, we see that $[h_i]$ contains no asymptotically conformal automorphisms of R.

For an arbitrary point $[[f]] \in AT(R)$, the Riemann surface $f(R)$ does not satisfy the lower bound condition. Then, by a similar consideration as above, we see that there exist quasiconformal automorphisms of $f(R)$ such that they are not homotopic to any asymptotically conformal automorphisms of $f(R)$ and that their boundary dilatations tend to 1. Hence $[[f]] \in \Lambda_{AT}(\mathcal{G}(R))$. □

On the other hand, suppose that R satisfies the lower and upper bound conditions. Then we proved in [8, Theorem 3] that $\Omega_T(\mathrm{Mod}(R)) \neq \emptyset$. Furthermore, if R satisfies an extra condition, then $\Omega_T(\mathrm{Mod}(R)) = T(R)$.

Theorem 4.2. *There exists a Riemann surface R satisfying the lower and upper bound conditions such that $\Lambda_T(\mathrm{Mod}(R)) = \emptyset$ and $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$.*

Proof. Let R_0 be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group $\langle\phi\rangle$ generated by a conformal automorphism ϕ of R_0 of infinite order. Set $R = R_0 - \{p\}$ for a point $p \in R_0$. Then R satisfies the lower and upper bound conditions and $\Lambda_T(\mathrm{Mod}(R)) = \emptyset$ by [11, Theorem 2] or [9, Corollary 4.12].

By Proposition 4.3 below, we have $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$. This is equivalent to that $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$ by the following facts. The asymptotic Teichmüller spaces $AT(R_0)$ and $AT(R)$ are isometric. In fact, for a quasiconformal homeomorphism f on R_0, consider the restriction $f|_R$ on R. Then the correspondence $[[f]] \mapsto [[f|_R]]$ produces an isometry between $AT(R_0)$ and $AT(R)$. Furthermore, the subgroups $\mathcal{G}(R_0)$ of $\mathrm{Isom}(AT(R_0))$ and $\mathcal{G}(R)$ of $\mathrm{Isom}(AT(R))$ can be identified. Indeed, for every element $[g] \in \mathrm{MCG}(R_0)$, we take a simply connected closed subset V of R_0 containing p and $g(p)$, and take a quasiconformal automorphism h of R_0 such that $h(g(p)) = p$ and h is the identity outside of V. Then $[h \circ g|_R] \in \mathrm{MCG}(R)$. This mapping class depends on the choice of h. However, the correspondence $[g]_* \mapsto [h \circ g|_R]_*$ gives an isomorphism between $\mathcal{G}(R_0)$ and $\mathcal{G}(R)$. Hence we may identify the action of $\mathcal{G}(R_0)$ on $AT(R_0)$ with that of $\mathcal{G}(R)$ on $AT(R)$, and thus $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$ if and only if $\Lambda_{AT}(\mathcal{G}(R)) \neq \emptyset$. $\square$

The following proposition completes a proof of Theorem 4.2.

Proposition 4.3. *Let R_0 be a normal cover of a compact Riemann surface of genus 2 whose covering transformation group is a cyclic group $\langle \phi \rangle$ generated by a conformal automorphism ϕ of R_0 of infinite order. Then $\Lambda_{AT}(\mathcal{G}(R_0)) \neq \emptyset$.*

Proof. Using the method of dividing simple closed geodesics a_i $(i \in \mathbb{Z})$ on R_0 whose hyperbolic lengths are all the same, we divide R_0 into infinitely many tori A_i $(i \in \mathbb{Z})$ with two boundary components $a_{i,-}$ and $a_{i,+}$ so that $\phi(A_i) = A_{i+1}$. Furthermore, for each integer i, we divide A_i into two pairs of pants by two non-dividing simple closed geodesics b_i and c_i on R_0 belonging to A_i. Then we have pairs of pants $P_{i,-}$ with boundary components $a_{i,-}$, $b_{i,-}$ and $c_{i,-}$, and $P_{i,+}$ with boundary components $a_{i,+}$, $b_{i,+}$ and $c_{i,+}$.

The symmetry axes of a pair of pants are the fixed point loci of the canonical orientation-reversing isometric involution on it. Let $\alpha_{i,-}$ be the symmetry axis of $P_{i,-}$ connecting $b_{i,-}$ and $c_{i,-}$. Similarly, $\beta_{i,-}$ is the symmetry axis connecting $c_{i,-}$ and $a_{i,-}$, and $\gamma_{i,-}$ is the one connecting $a_{i,-}$ and $b_{i,-}$. We parametrize the boundary components of $P_{i,-}$ counterclockwise by a normalized arc length parameter θ $(0 \leq \theta \leq 1)$ with respect to the hyperbolic metric such that $a_{i,-}(0) = a_{i,-}(1) \in \gamma_{i,-}$, $b_{i,-}(0) = b_{i,-}(1) \in \alpha_{i,-}$ and $c_{i,-}(0) = c_{i,-}(1) \in \beta_{i,-}$. Similarly, we parametrize the boundary components of $P_{i,+}$ counterclockwise by a normalized arc length parameter θ.

Let $L^\infty(\mathbb{Z})$ be the Banach space of all bounded bilateral infinite sequence of real numbers, and let $(\xi_i)_{i \in \mathbb{Z}}$ $(0 < \xi_i \leq 1)$ be a point of $L^\infty(\mathbb{Z})$ defined in [10, Definition 4.3] as follows: set $\xi_0 = 1$ and $\xi_1 = \xi_{-1} = (1/2)\xi_0 = 1/2$. We proceed as $\xi_i = \xi_{i-6} = (2/3)\xi_{i-3}$ for $i = 2, 3, 4$ and $\xi_i = \xi_{i-18} = (3/4)\xi_{i-9}$ for $i = 5, \ldots, 13$. Inductively, set

$$\xi_i = \xi_{i-2\cdot 3^k} = \frac{k+1}{k+2} \cdot \xi_{i-3^k}$$

for $\sum_{j=0}^{k-1} 3^j + 1 \leq i \leq \sum_{j=0}^{k} 3^j$ stratified with the indices $k \in \mathbb{N}$. This is equivalent to the following direct definition by using 3-adic expansion. Every integer $i \in \mathbb{Z}$ is uniquely written as $i = \sum_{j=0}^{\infty} \varepsilon_j(i) \cdot 3^j$, where $\varepsilon_j(i)$ is either -1, 0 or 1. Then ξ_i is defined by $\xi_i = \prod_{\varepsilon_j(i) \neq 0} (j+1)/(j+2)$, where the product is taken over all $j \in \mathbb{N}$ satisfying $\varepsilon_j(i) \neq 0$.

For each integer i, we consider a pair of pants $P'_{i,-}$ with geodesic boundary components $a'_{i,-}$, $b'_{i,-}$ and $c'_{i,-}$ such that $\ell(a'_{i,-}) = \ell(a_{i,-})$, $\ell(b'_{i,-}) = \ell(b_{i,-})$ and $\ell(c'_{i,-}) = 1+\xi_i$. By the same way as we did for the boundary of $P_{i,-}$, we parametrize each of boundary components of $P'_{i,-}$ by a normalized arc length parameter θ. Similarly, we consider a pair of pants $P'_{i,+}$ with parametrized boundary components $a'_{i,+}$, $b'_{i,+}$ and $c'_{i,+}$ such that $\ell(a'_{i,+}) = \ell(a_{i,+})$, $\ell(b'_{i,+}) = \ell(b_{i,+})$ and $\ell(c'_{i,+}) = 1 + \xi_i$. We take a quasiconformal homeomorphism $f_{i,-} : P_{i,-} \to P'_{i,-}$ so that $f_{i,-}(a_{i,-}(\theta)) = a'_{i,-}(\theta)$, $f_{i,-}(b_{i,-}(\theta)) = b'_{i,-}(\theta)$ and $f_{i,-}(c_{i,-}(\theta)) = c'_{i,-}(\theta)$ for all

θ, and take a quasiconformal homeomorphism $f_{i,+} : P_{i,+} \to P'_{i,+}$ with similar properties as $f_{i,-}$.

We glue $P'_{i,-}$ and $P'_{i,+}$ by identifying $b'_{i,-}(\theta)$ with $b'_{i,+}(1-\theta)$ and identifying $c'_{i,-}(\theta)$ with $c'_{i,+}(1-\theta)$ for all θ. Then we obtain a torus A'_i with two boundary components $a'_{i,-}$ and $a'_{i,+}$. Furthermore, we glue A'_i and A'_{i+1} by identifying $a'_{i,+}(\theta)$ and $a'_{i+1,-}(1-\theta)$ for all θ. Then we obtain a Riemann surface R'_0 of infinite genus and a quasiconformal homeomorphism $f : R_0 \to R'_0$ such that the restriction of f to $P_{i,\pm}$ is $f_{i,\pm}$. The Riemann surface R'_0 has simple closed geodesics $c'_i(\theta)$ $(= c'_{i,-}(\theta) = c'_{i,+}(1-\theta))$ with hyperbolic length $1+\xi_i$.

Lemma 4.4. *For the point $\tau = [f] \in T(R_0)$ and for the element $[\phi]_* \in \mathrm{Mod}(R_0)$, we have $d_T([\phi^{3^k}]_*(\tau), \tau) \to 0$ $(k \to \infty)$.*

Proof. Set $\psi = f \circ \phi^{-1} \circ f^{-1}$. Then $d_T([\phi^n]_*(\tau), \tau) = \log K(\varphi_n)$, where φ_n is an extremal quasiconformal automorphism of R'_0 in the homotopy class of ψ^n. The quasiconformal automorphism ψ^n of R'_0 maps $P'_{i,\pm}$ to $P'_{i-n,\pm}$ for each i, and ψ^n satisfies $\ell(\psi^n(a'_{i,\pm})) = \ell(a'_{i,\pm})$, $\ell(\psi^n(b'_{i,\pm})) = \ell(b'_{i,\pm})$ and $\ell(\psi^n(c'_{i,\pm})) = 1 + \xi_{i-n}$. By applying [2, Theorem 1.1] to $\psi^n|_{P'_{i,\pm}}$, we see that there exists a quasiconformal automorphism $\tilde{\varphi}_n$ such that it is homotopic to ψ^n and satisfies $K(\tilde{\varphi}_n|_{A'_i}) \le 1 + C\varepsilon_{i,n}$ on each A'_i. Here $C > 0$ is a constant independent of i and n, and $\varepsilon_{i,n} = \log\{(1+\xi_i)/(1+\xi_{i-n})\}$. Since we proved in [10, Lemma 4.4] that $\xi_{i-3^k} \to \xi_i$ as $k \to \infty$ for all i, we see that $\varepsilon_{i,3^k} \to 0$ as $k \to \infty$. Thus we have $K(\tilde{\varphi}_{3^k}) \to 1$. Hence $d_T([\phi^{3^k}]_*(\tau), \tau) \to 0$ as $k \to \infty$. □

By Lemma 4.4, we have $d_{AT}([\phi^{3^k}]_*(\hat{\tau}), \hat{\tau}) \to 0$ $(k \to \infty)$ for $\hat{\tau} = [[f]] \in AT(R_0)$ which is a projection of τ and for $[\phi^{3^k}]_* \in \mathcal{G}(R_0)$. The following lemma, which is proved as an application of Theorem 3.6, concludes that $\hat{\tau} \in \Lambda_{AT}(\mathcal{G}(R_0))$ and completes the proof of Proposition 4.3.

Lemma 4.5. *$[\phi^{3^k}]_* \ne [\phi^{3^m}]_*$ in $\mathcal{G}(R_0)$ for every $k \ne m$.*

Proof. We will prove that $[\phi^{3^k}]_*(\hat{\tau}) \ne [\phi^{3^m}]_*(\hat{\tau})$, namely $\psi^{3^m-3^k}$ is not homotopic to any asymptotically conformal automorphism of R'_0. We may assume that $m > k \ge 0$. For an arbitrary integer $n > 0$, we see that

$$\begin{aligned}\frac{\ell(c'_{3^{m+n}})}{\ell(\psi^{3^m-3^k}(c'_{3^{m+n}}))} &= \frac{1+\xi_{3^{m+n}}}{1+\xi_{3^{m+n}-(3^m-3^k)}} = \frac{1+\xi_{3^{m+n}}}{1+\xi_{3^{m+n}}\xi_{3^m}\xi_{3^k}} \\ &> \frac{1+\xi_{3^m}}{1+(\xi_{3^m})^2\,\xi_{3^k}} \ (>1).\end{aligned}$$

In the last inequality, we used the fact that the function $(1+x)/(1+ax)$ $(a<1)$ is strictly increasing for $x > 0$ and that $\xi_{3^{m+n}} > \xi_{3^m}$. Since the last constant in the above inequality is independent of n, by applying Theorem 3.6 to the quasiconformal automorphism $\psi^{3^m-3^k}$, we conclude that it is not homotopic to any asymptotically conformal automorphism of R'_0. □

By the proof of Lemma 4.5, we have the following.

Corollary 4.6. *Let a Riemann surface R_0 and a conformal automorphism ϕ be the same as in Proposition 4.3. The action of the element $[\phi] \in \mathrm{MCG}(R_0)$ on $AT(R_0)$ is non-trivial.*

Proof. In the proof of Lemma 4.5, we put $k = 0$ and $m = 1$. Then ψ^2 is not homotopic to any asymptotically conformal automorphism of R_0'. This yields that ψ is not homotopic to any asymptotically conformal automorphism. Thus $[\phi]_*(\hat{\tau}) \neq \hat{\tau}$, and then $[\phi] \in \mathrm{MCG}(R_0)$ acts on $AT(R_0)$ non-trivially. □

We now classify the limit points into two types. For a subgroup G of $\mathcal{G}(R)$, we define $\Lambda_{AT,0}(G)$ as the set of points $\hat{\tau} \in \Lambda_{AT}(G)$ such that there exists a sequence $\{\chi_n\}_{n=1}^{\infty}$ of distinct elements of G satisfying both $d_{AT}((\chi_n)_*(\hat{\tau}), \hat{\tau}) \to 0$ $(n \to \infty)$ and $(\chi_n)_*(\hat{\tau}) \neq \hat{\tau}$ for all n, and $\Lambda_{AT,\infty}(G)$ as the set of points $\hat{\tau} \in \Lambda(G)$ such that $\mathrm{Stab}_G(\hat{\tau})$ consists of infinitely many elements (cf. [8, Definition 2]).

Proposition 4.7. *Let R_0 be the Riemann surface in Proposition 4.3. Then $\Lambda_{AT,0}(\mathcal{G}(R_0)) \neq \emptyset$ and $\Lambda_{AT,\infty}(\mathcal{G}(R_0)) \neq \emptyset$.*

Proof. By Lemma 4.4 and the proof of Lemma 4.5, we see that $\hat{\tau} \in \Lambda_{AT,0}(\mathcal{G}(R_0))$. The base point $[[id]] \in AT(R_0)$ is fixed by all elements $[\phi^n]_* \in \mathcal{G}(R_0)$ $(n \in \mathbb{Z})$. By Lemma 4.5, we conclude that $[[id]] \in \Lambda_{AT,\infty}(\mathcal{G}(R_0))$. □

References

[1] L. Ahlfors *Conformal invariants*, McGraw-Hill 1973
[2] C. J. Bishop, *Quasiconformal mapping of Y-pieces*, Rev. Mat. Iberoamericana **18** (2002), 627–652.
[3] C. J. Earle, F. P. Gardiner and N. Lakic, *Teichmüller spaces with asymptotic conformal equivalence*, I.H.E.S., preprint (1995).
[4] C. J. Earle, F. P. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part I: The complex structure,* Contemporary Math. **256** (2000), 17–38.
[5] C. J. Earle, F. P. Gardiner and N. Lakic, *Asymptotic Teichmüller space, Part II: The metric structure,* Contemporary Math. **355** (2004), 187–219.
[6] C. J. Earle, V. Markovic and D. Saric, *Barycentric extension and the Bers embedding for asymptotic Teichmüller space,* Contemporary Math. **311** (2002), 87–105.
[7] A. Epstein, *Effectiveness of Teichmüller modular groups*, Contemporary Math. **256** (2000), 69–74
[8] E. Fujikawa, *Limit sets and regions of discontinuity of Teichmüller modular groups*, Proc. Amer. Math. Soc. **132** (2004), 117–126.
[9] E. Fujikawa, *Modular groups acting on infinite dimensional Teichmüller spaces*, Contemporary Math. **355** (2004), 239–253.
[10] E. Fujikawa and K. Matsuzaki, *Recurrent and periodic points for isometries of L^∞ spaces*, Indiana Univ. Math. J., to appear.
[11] E. Fujikawa, H. Shiga and M. Taniguchi, *On the action of the mapping class group for Riemann surfaces of infinite type*, J. Math. Soc. Japan **56** (2004), 1069–1086.
[12] F. P. Gardiner, *Teichmüller theory and quadratic differentials*, Wiley-Interscience, New York 1987.
[13] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller Theory,* Mathematical Surveys and Monographs **76**, Amer. Math. Soc., 2000.
[14] F. P. Gardiner and D. P. Sullivan, *Symmetric structure on a closed curve,* Amer. J. Math. **114** (1992), 683–736.
[15] K. Matsuzaki, *The infinite direct product of Dehn twists acting on infinite dimensional Teichmüller spaces*, Kodai Math. J. **26** (2003), 279–287.
[16] K. Matsuzaki *A countable Teichmüller modular group*, Trans. Amer. Math. Soc. **357** (2005), 3119–3131.
[17] K. Matsuzaki, *A quasiconformal mapping class group acting trivially on the asymptotic Teichmüller space*, Proc. Amer. Math. Soc., to appear.
[18] K. Matsuzaki, *Dynamics of Teichmüller modular groups and general topology of moduli spaces*, preprint.
[19] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, John Wiley & Sons, 1988.

[20] A. Vasil'ev, *Moduli of families of curves for conformal and quasiconformal mappings*, Lecture Notes in Math. **1788**, Springer 2002.

[21] S. A. Wolpert, *The length spectra as moduli for compact Riemann surfaces*, Ann. of Math. **109** (1979), 323–351.

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