# ON THE ACTION OF THE MAPPING CLASS GROUP FOR RIEMANN SURFACES OF INFINITE TYPE

EGE FUJIKAWA, HIROSHIGE SHIGA AND MASAHIKO TANIGUCHI

*Dedicated to Professor Hiroki Sato on his 60th birthday*

ABSTRACT. We consider Riemann surfaces of infinite type and their reduced Teichmüller spaces. The reduced Teichmüller space admits the action of the reduced mapping class group. Generally, the action is not discrete while it is faithful. We give sufficient conditions for the discreteness of the action in terms of the geometry of Riemann surfaces.

## 1. INTRODUCTION

The mapping class group (or Teichmüller modular group) $\mathrm{Mod}(R)$ for a Riemann surface $R$ is the set of equivalence classes of quasiconformal automorphisms of $R$ (see [10]). Two quasiconformal automorphisms $h_1$ and $h_2$ of $R$ are equivalent if $h_2^{-1} \circ h_1$ is homotopic to the identity by a homotopy that keeps every points of ideal boundary $\partial R$ fixed throughout. In the theory of Teichmüller spaces of Riemann surfaces of analytically finite type, the mapping class group plays an important role in various fields. This is a group of the biholomorphic automorphisms of the Teichmüller space and it acts faithfully and properly discontinuously. On the other hand, it seems that there are few studies on $\mathrm{Mod}(R)$ for a Riemann surface $R$ of infinite type. Recently, Earle-Gardiner-Lakic showed in [3] that it acts faithfully on $T(R)$. In this paper, we consider the discreteness of the action of the mapping class group. We say that a subgroup $G$ of $\mathrm{Mod}(R)$ is discrete if the orbit of any point of $T(R)$ under the $G$ action is discrete.

For a Riemann surface of analytically finite type, $\mathrm{Mod}(R)$ is discrete, while in the case of infinite type, $\mathrm{Mod}(R)$ is not necessarily discrete. In particular, if $R$ has a boundary curve (border), $\mathrm{Mod}(R)$ is not discrete since a slight change of the boundary value of a quasiconformal map produces a different mapping class in Mod$(R)$. Thus, it is natural that we consider another group, the reduced mapping class group. The reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the set of homotopy classes of quasiconformal automorphisms of $R$ whose homotopy maps does not necessarily keep points of $\partial R$ fixed. The reduced mapping class group is also important because it naturally acts on the reduced Teichmüller space.

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We explore the problem of discreteness of the reduced mapping class group for Riemann surfaces of infinite type. Actually, if $R$ is a Riemann surface of topologically finite type, then $\mathrm{Mod}^{\#}(R)$ is discrete. However, $\mathrm{Mod}^{\#}(R)$ is not discrete in general. For example, if $R$ has a sequence of disjoint simple closed geodesics which are not freely homotopic to a boundary component and whose lengths tend to $0$, then we see that $\mathrm{Mod}^{\#}(R)$ is not discrete (See §3 and §6). The purpose of this paper is to give a sufficient condition for discreteness.

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## 2. The mapping class group for the reduced Teichmüller space

Throughout this paper, we assume that a Riemann surface $R$ is *hyperbolic*, that is, it is represented by $\mathbb{H}/\Gamma$ for some Fuchsian group $\Gamma$ acting on the upper half-plane $\mathbb{H}$ with the hyperbolic metric $|dz|/y$ $(z = x + y\sqrt{-1})$. We also assume that the Fuchsian group $\Gamma$ is always non-elementary. In other words, we assume that the group $\Gamma$ is non-abelian. A Riemann surface is called *of analytically finite type* if the hyperbolic area is finite, and is called *of analytically infinite type* if the area is not finite.

For an open Riemann surface $R$, a relatively non-compact connected component of the complement of a compact subset of $R$ is called an *end*. An end $V$ of $R$ is called a *hole* if it is doubly connected and the hyperbolic area of $V$ is infinite. A doubly connected end of $R$ is called a *cusp* if the hyperbolic ares of $V$ is finite. A cusp $V$ with smooth relative boundary is conformally equivalent to the punctured disk $\{0 < |z| < 1\}$. An ideal boundary of $R$ corresponding to the origin $z = 0$ is called a *puncture*.

**Notation.** The hyperbolic distance on $\mathbb{H}$ and on a Riemann surface $R$ is denoted by $d_{\mathbb{H}}(\cdot,\cdot)$ and $d_R(\cdot,\cdot)$ respectively. Further the hyperbolic length of a curve $c$ in $\mathbb{H}$ or in $R$ is denoted by $\ell(c)$.

We review the theory of Teichmüller spaces and mapping class groups. See [4], [6] and [10] for the details.

**Definition 1.** Fix a Riemann surface $R$. For pairs $(S_j, f_j)$ of Riemann surfaces $S_j$ and quasiconformal maps $f_j$ of $R$ onto $S_j$ $(j = 1, 2)$, we say that $(S_1, f_1)$ and $(S_2, f_2)$ are ***RT (reduced Teichmüller) equivalent*** if there exists a conformal map $h$ of $S_1$ onto $S_2$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity on $R$.

The *reduced Teichmüller space* $T^{\#}(R)$ with the base Riemann surface $R$ is the set of all the RT equivalence classes $[S, f]$ of such pairs $(S, f)$ as above.

**Definition 2.** We say that two quasiconformal automorphisms $h_1$ and $h_2$ of $R$ are *RT equivalent* if $h_2^{-1} \circ h_1$ is homotopic to the identity on $R$.

The *reduced mapping class group* $\mathrm{Mod}^{\#}(R)$ is the set of all the RT equivalence classes $[h]$ of quasiconformal automorphisms $h$ of $R$. Furthermore, for a simple closed geodesic $c$ on $R$, we set

$$\mathrm{Mod}_c^{\#}(R) = \{[f] \in \mathrm{Mod}^{\#}(R) \mid f(c) \text{ is freely homotopic to } c\}.$$

Every quasiconformal map of $R = \mathbb{H}/\Gamma$ induces an isomorphism of $\Gamma$ into $\mathrm{PSL}(2, \mathbb{R})$. We see that if two automorphisms $h_1$ and $h_2$ are RT equivalent then they induce the same isomorphism modulo $\mathrm{PSL}(2, \mathbb{R})$ conjugacy.

If $R$ is a compact Riemann surface, then the reduced Teichmüller space $T^{\#}(R)$ is nothing but the ordinary Teichmüller space $T(R)$ of $R$ and the reduced mapping class group $\mathrm{Mod}^{\#}(R)$ is the ordinary mapping class group $\mathrm{Mod}(R)$.

Similar to the case of $T(R)$, the reduced Teichmüller space $T^{\#}(R)$ is equipped with the reduced Teichmüller distance $d_T(\cdot, \cdot)$ defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \frac{1}{2} \inf_{f_1, f_2} \log K(f_1 \circ f_2^{-1}),$$

where $K(\cdot)$ is the maximal dilatation of a quasiconformal map and the infimum is taken over all quasiconformal maps $f_1$ and $f_2$ determining $[S_1, f_1]$ and $[S_2, f_2]$, respectively. It is known that $T^{\#}(R)$ is a complete metric space with respect to this $d_T$. An element $\omega = [h] \in \mathrm{Mod}^{\#}(R)$ induces an automorphism of $T^{\#}(R)$ by

$$[S, f] \mapsto [S, f \circ h^{-1}].$$

This is an isometric automorphism with respect to $d_T$ and denoted by $\omega_*$. Namely, we have a homomorphism $\mathrm{Mod}^{\#}(R) \to \mathrm{Aut}\,(T^{\#}(R))$.

**Remark 1.** In [3], it is proved that for any Riemann surface $R$ of infinite analytic type (and if $2g + n > 4$ when $R$ is of finite $(g, n)$-type), the homomorphism $\mathrm{Mod}^{\#}(R) \to \mathrm{Aut}\,(T^{\#}(R))$ as above is faithful. Therefore we can identify $\omega_*$ with $\omega$ and omit the asterisk hereafter.

**Definition 3.** We say that a subgroup $G$ of $\mathrm{Mod}^{\#}(R)$ is *discrete* if every sequence $\{\omega_n\} \subset G$ satisfying $\lim_{n\to\infty} \omega_n(p) = q$ for some pair of points $p, q$ in $T^{\#}(R)$ is eventually a constant sequence, that is, there exists an $N \in \mathbb{N}$ such that $\omega_n = \omega_N$ for every $n \geq N$.

## 3. EXAMPLES

As we noted in the introduction, if $R$ is a compact Riemann surface, then the action of $\mathrm{Mod}(R)$ on $T(R)$ is discrete. Contrary to this case, there are various kinds of examples which show non-discreteness of $\mathrm{Mod}^{\#}(R)$ for a Riemann surface $R$ of infinite type.

**Example 1.** Suppose that $R$ has a sequence $\{c_n\}$ of distinct simple closed geodesics that are not freely homotopic to a boundary component and that these hyperbolic lengths tend to 0. Then the Dehn twist along each $c_n$ gives an element $\omega_n$ of $\mathrm{Mod}^{\#}(R)$ such that the sequence $\{\omega_n(p_0)\}$ converges to $p_0$ as $n \to \infty$, where $p_0 = [R, id]$ is the base point of $T^{\#}(R)$. Hence $\mathrm{Mod}^{\#}(R)$ is not discrete.

There exists a Riemann surface $R$ such that it has no short geodesics but that $\mathrm{Mod}^{\#}(R)$ is not discrete.

**Example 2.** We construct a Riemann surface $R$ such that it has no short geodesics and but contains a point with arbitrarily large injectivity radius with respect to the hyperbolic metric (in fact, $R$ does not satisfy the second condition in Theorem 1, which is stated in Section 4), and that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Set

$$R = \mathbb{C} - \bigcup_{n=1}^{\infty} \bigcup_{m \in \mathbb{Z}} \left\{ \frac{m}{n} + (2n+1)\sqrt{-1} \right\}.$$

To show that $\mathrm{Mod}^{\#}(R)$ is not discrete, set

$$f_n(z) = \begin{cases} x - (y - 2n - 2)/n + y\sqrt{-1} & (2n+1 \leq y < 2n+2) \\ x + (y - 2n)/n + y\sqrt{-1} & (2n \leq y < 2n+1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

Then $f_n$ are quasiconformal automorphisms of $R$ and the maximal dilatations of $\{f_n\}$ tend to 1. Thus $\mathrm{Mod}^{\#}(R)$ is not discrete.

Now, we see that $R$ does not satisfy the second condition in Theorem 1. We put $A_n = R \cap \{z \mid \mathrm{Im} z = 2n+1\}$ and $a_{m,n} = m/n + (2n+1)\sqrt{-1}$ $(n \in \mathbb{N}, m \in \mathbb{Z})$. Then we shall prove that

$$(1) \qquad d_R(A_n, A_{n+1}) \to \infty \qquad (n \to \infty).$$

To prove this, we show that the injectivity radii at $b_n = 2n\sqrt{-1}$ tend to $\infty$ as $n \to \infty$. The length of any non-trivial closed curve passing through $b_n$ is greater than $d_n = \inf_m d_R(b_n, I_{m,n} \cup I_{m,n+1})$, where $I_{m,n}$ is the segment connecting $a_{m,n}$ and $a_{m+1,n}$. Set

$$\varphi_{m,n}(z) = n(z - a_{m,n}).$$

Then, $\varphi_{m,n}$ is a conformal mapping from $\mathbb{C} - \{a_{m,n}, a_{m+1,n}\}$ onto the Riemann surface $S = \mathbb{C} \setminus \{0, 1\}$. From the decreasing property of the hyperbolic distance, we have

$$\begin{aligned} d_R(b_n, I_{m,n}) &\geq d_S(\varphi_{m,n}(b_n), \varphi_{m,n}(I_{m,n})) \\ &= d_S(-m - n\sqrt{-1}, (0,1)). \end{aligned}$$

Obviously, $d_S(-m - n\sqrt{-1}, (0,1)) \to \infty$ as $|m| + |n| \to \infty$. Therefore, $\lim_{n\to\infty} d_n = \infty$ and the injectivity radii at $b_n$ tend to $\infty$ as $n \to \infty$. Hence we see that, for any $M > 0$, there exists $n \in \mathbb{Z}$ such that $A_n$ and $A_{n+1}$ belong distinct component of $R_M$ each other. This implies that $R$ does not satisfy the second condition in Theorem 1.

Next, we show that $R$ has no short geodesics. Suppose that there exists a sequence $\{c_k\}$ of simple closed geodesics on $R$ such that $\lim_{k\to\infty} \ell(c_k) = 0$. From (1) we may assume that $c_k$ contains two distinct points $a_{m,n}$ and $a_{m',n}$. By translation, we may also assume that $c_k$ contains $a_{0,n}, a_{m,n}$ but does not contain $a_{-m,n}$. From the decreasing property of the hyperbolic metric as above, we see that the hyperbolic length of $c_k$ in $R$ is greater than the length in $R'$, where $R' = \hat{\mathbb{C}} \setminus \{\infty, a_{-m,n}, a_{0,n}, a_{m,n}\}$ which is conformally equivalent to $R_0 = \hat{\mathbb{C}} \setminus \{\infty, -1, 0, 1\}$. It is well known that the length of any closed geodesic in $R_0$ is greater than some positive constant $L$. Thus, we have $\ell(c_k) > L > 0$ and it is a contradiction.

We exhibit an example of a planar Riemann surface $R$ without cusps but containing a point with arbitrarily large injectivity radius with respect to the hyperbolic metric such that $\mathrm{Mod}^{\#}(R)$ is not discrete.

**Example 3.** We construct a Riemann surface $R$ without cusps such that it satisfies the first condition in Theorem 1 but does not satisfy the second condition and that $\mathrm{Mod}^{\#}(R)$ is not discrete.

For each $n \geq 2$, we set

$$I_n = [-1, 1] \cup \bigcup_{k=1}^{\infty} I_{n,k},$$

where

$$\begin{aligned} I_{n,k} &= \{x + (1 - 1/n)^k \sqrt{-1} \mid -1 \leq x \leq 1\} \\ &\bigcup \{x + (1 + (k-1)/n)\sqrt{-1} \mid -1 \leq x \leq 1\}. \end{aligned}$$

We take infinitely many copies $\{R_n\}$ of $\mathbb{C} - \{y\sqrt{-1} \mid y \leq -1\}$ and set $R_n{}' = R_n - I_n$ for each $n \geq 2$. We make a Riemann surface $R$ by gluing the right hand side of $\{y\sqrt{-1} \mid y < -1\}$ on $R_n{}'$ with the left hand side of $\{y\sqrt{-1} \mid y < -1\}$ on $R_{n-1}{}'$ $(n = 3, 4, \ldots)$ along the imaginary axis. By using the same argument as that of Example 2, we can show that the Riemann surface $R$ satisfies the first condition in Theorem 1 but does not satisfy the second condition in Theorem 1.

Consider a quasiconformal map $f_n$ of $R_n{}'$ defined by

$$f_n(z) = \begin{cases} x + (1 - 1/n)y\sqrt{-1} & (0 < y \leq 1) \\ x + (y - 1/n)\sqrt{-1} & (y > 1) \\ x + y\sqrt{-1} & \text{elsewhere.} \end{cases}$$

It is easily seen that the maximal dilatations of $f_n$ converge to 1 as $n \to \infty$. Obviously, $f_n$ is extended to a quasiconformal automorphism of $R$ by setting it the identity on $R - R_n{}'$ and we will write it by the same letter $f_n$. Thus the quasiconformal map $f_n$ gives an element $[f_n]$ of $\mathrm{Mod}^{\#}(R)$ such that $\{[f_n](p_0)\}$ converges to $p_0$ as $n \to \infty$, where $p_0 = [R, id]$ is the base point of $T^{\#}(R)$. Hence we conclude that $\mathrm{Mod}^{\#}(R)$ is not discrete.

Even if a Riemann surface $R$ has no short geodesics and no points with arbitrarily large injectivity radius, $\mathrm{Mod}^{\#}(R)$ may not be discrete.

**Example 4.** We construct a Riemann surface $R$ such that it has no short geodesics and no points with arbitrarily large injectivity radius but that $\mathrm{Mod}^{\#}(R)$ is not discrete. Consider a torus $S$ with two geodesic borders with the same length to each other. We take infinity many copies $\{S_n\}_{n=-\infty}^{\infty}$ of $S$. We denote the two geodesic borders of $S_n$ by $\ell_{n,1}$ and $\ell_{n,2}$. Construct a Riemann surface $R$ by gluing the $\ell_{n-1,2}$ with $\ell_{n,1}$ and gluing $\ell_{n,2}$ with $\ell_{n+1,1}$ for each $n$. Let $f$ be a conformal automorphism of $R$ which sends $S_n$ to $S_{n+1}$, and we set $f_n := f^n$. Then we see that $[f_n] \neq id$ as an element of $\mathrm{Mod}^{\#}(R)$. However, $[f_n](p_0) = p_0$ for all $n$, where $p_0 = [R, id] \in T^{\#}(R)$ because $f_n : R \to R$ is a conformal mapping. Hence, $\mathrm{Mod}^{\#}(R)$ is not discrete.

## 4. Main Results

As Example 1 shows, for the discreteness of the mapping class group, it is necessary that there exists no sequence of geodesics on the Riemann surface whose lengths converge to zero. Examples 2, 3 show that some conditions for the injectivity radius are required for the discreteness.

**Definition 4.** For a given $M > 0$, we define $R_M$ to be the subset of points $p \in R$ such that there exists a non-trivial simple closed curve passing through $p$ whose hyperbolic length is less than $M$. The set $R_\epsilon$ is called the *$\epsilon$-thin part* of $R$ if $\epsilon > 0$ is smaller than the Margulis constant. Further, a connected component of the $\epsilon$-thin part that corresponds to a puncture is called the *cusp neighborhood*.

Now, we exhibit our main results.

**Theorem 1.** *Let $R$ be a Riemann surface with the non-abelian fundamental group. Suppose that $R$ satisfies the following two conditions:*

(1) *There exists a constant $\epsilon > 0$ such that the $\epsilon$-thin part of $R$ consists only of cusp neighborhoods.*
(2) *There exist a constant $M > 0$ and a connected component $R_M^*$ of $R_M$ such that the homomorphism of $\pi_1(R_M^*)$ to $\pi_1(R)$ which is induced by the inclusion map of $R_M^*$ to $R$ is surjective.*

*Then $\mathrm{Mod}_c^{\#}(R)$ is discrete for any simple closed geodesic $c$ on $R$.*

**Remark 2.** Example 1 shows that the first condition in Theorem 1 is necessary for the discreteness. On the other hand, the Riemann surfaces in Examples 2 and 3 satisfy the first condition but does not satisfy the second condition. Example 4 shows that there exits a Riemann surface such that it satisfies both the conditions but $\mathrm{Mod}^{\#}(R)$ is not discrete.

**Remark 3.** If $R$ satisfies the second condition in Theorem 1 for a constant $M$, then it satisfies the condition for all $M' \geq M$.

**Remark 4.** The region $R_M$ is not necessarily connected for large $M$ even if the homomorphism : $\pi_1(R_M) \to \pi_1(R)$ is surjective. Moreover, in Example 7 of §6, we give a Riemann surface $R$ and divergent sequences $\{M_n\}$, $\{M'_n\}$ such that

- $M_n < M'_n < M_{n+1} < M'_{n+1}$ $(n = 1, 2, \ldots)$.
- $R_{M_n}$ is connected for all $n$ and the homomorphism : $\pi_1(R_{M_n}) \to \pi_1(R)$ is surjective.
- $R_{M'_n}$ is not connected for all $n$ but the homomorphism : $\pi_1(R^*_{M'_n}) \to \pi_1(R)$ is surjective for some component $R^*_{M'_n}$ of $R_{M'_n}$.

For a hyperbolic Riemann surface $R = \mathbb{H}/\Gamma$, we consider the *convex core* $C(\Gamma)$ of the limit set of $\Gamma$, that is, the hyperbolic convex envelope of $\Lambda(\Gamma) \subset \mathbb{R} \cup \{\infty\}$ in $\mathbb{H}$. Since the convex core $C(\Gamma)$ is $\Gamma$-invariant, it determines a region $C(R)$ in $R$ and we call the region the *convex core* of $R$.

**Definition 5.** We say that a Riemann surface $R$ has *$\epsilon$-uniform geometry* if the following two conditions are satisfied for some $\epsilon > 0$:

(1) The $\epsilon$-thin part of $R$ consists of cusp neighborhoods.
(2) The injectivity radius on the convex core $C(R)$ of $R$ is less than $\epsilon^{-1}$.

Since $C(R)$ is connected and it contains any closed geodesic on $R$, from Theorem 1 we have the following immediately .

**Corollary 1.** *Let $R$ be a Riemann surface with $\epsilon$-uniform geometry for some $\epsilon > 0$. Then Mod$_c^{\#}(R)$ is discrete for any simple closed geodesic c on $R$.*

**Remark 5.** The conditions in Theorem 1 do not imply the uniform geometry. For example, set $R = \mathbb{C} - \mathbb{Z}$. Then $R$ has a Fuchsian model of the first kind, and hence the convex core $C(R)$ coincides with $R$. By considering a sequence $\{z_n\}$ in $R$ with $|\text{Im}\, z_n| \to \infty$ as $n \to \infty$, we see that $R$ has points with arbitrarily large injectivity radius. Hence, $R$ does not have $\epsilon$-uniform geometry for any $\epsilon > 0$. On the other hand, it is easily seen that $R$ satisfies the conditions in Theorem 1.

**Remark 6.** The conditions having uniform geometry were first stated as no short geodesics and no large disk condition. Nakanishi and Yamamoto[11] shows that under these conditions the out radius of the Teichmüller space is strictly less than 6. Ohtake[12] uses these conditions to show that the norm of the Poincaré series is strictly less than one which generalizes a result in McMullen[9].

It is important to give conditions for the mapping class group to be discrete. By using the above results, we have the following.

**Theorem 2.** *Let $R$ be a Riemann surface satisfying the conditions in Theorem 1 or Corollary 1. Suppose that either the genus, the number of cusps or the number of holes of $R$ is positive finite. Then* $\text{Mod}^{\#}(R)$ *is discrete.*

## 5. Proofs of main results

First of all, we note the geometry of a component of $R_M$.

**Proposition 1.** *For $M > 0$, let $R_M^*$ be a connected component of $R_M$ defined in Definition 4 and $R_\epsilon$ the $\epsilon$-thin part of $R$ for some small $\epsilon < M$. We assume that $R_M^* - R_\epsilon$ is not of type $(0, 3)$. Then there exists a constant $M_1 > 0$ depending only on $M$ and $\epsilon$ such that for any point $p \in R_M^* - R_\epsilon$ there exists a simple closed curve $c_p$ passing through $p$ with $\ell(c_p) < M_1$ which does not surround a puncture of $R$.*

*Proof.* Let $\Gamma$ be a Fuchsian group representing $R$. Take an arbitrary point $p$ in $R_M^* - R_\epsilon$. From the definition, we may find a simple closed curve $c_p \ni p$ whose length is less than $M$. If $c_p$ is not homotopic to a simple closed curve which surrounds a puncture of $R$, then there is nothing to prove.

Thus, we suppose that $c_p$ surrounds a puncture of $R$. Then, a parabolic transformation $\gamma \in \Gamma$ represents $c_p$. We may assume that $\gamma(z) = z + 1$. For $r > 0$, we take $\delta(r)$ so that

$$d_{\mathbb{H}}(\delta(r)\sqrt{-1}, \delta(r)\sqrt{-1} + 1) = r,$$

It is easily seen that $\delta(r) = (2\sinh r/2)^{-1}$ for $r > 0$. We put

$$S(M, \epsilon) = \{z \in \mathbb{H} \mid \delta(M) \leq \text{Im } z \leq \delta(\epsilon),\ 0 \leq \text{Re } z \leq 1\}.$$

Since $\ell(c_p) < M$ and $p \notin R_\epsilon$, a lift $C_p$ of $c_p$ contains a point in $S(M, \epsilon)$.

Let $L_z$ ($z \in \mathbb{H}$) denote the geodesic arc from $z$ to $z + 1$. Suppose that there exists a point $z \in S(M, \epsilon)$ such that the projection $l_z$ in $R$ of $L_z$ via the canonical projection $\pi : \mathbb{H} \to R = \mathbb{H}/\Gamma$ is not simple. Then $l_z$ contains a non-trivial simple closed curve $c'_z$ with $\ell(c'_z) < \ell(l_z) < M$.

If $c'_z$ does not surround a puncture of $R$, then connect $p$ and $c'_z$ by a simple arc on $R$. Then, we see that there exists a simple closed curve passing through $p$ with length less than $M_1 = 2(M + d_{\mathbb{H}}(\delta(\epsilon)\sqrt{-1}, \delta(M)\sqrt{-1}))$ which does not surround a puncture of $R$.

Next, suppose that $c'_z$ surrounds a puncture of $R$. Noting that $l_z$ is the projection of the geodesic arc $L_z$, we verify that $c'_z$ is not homotopic to $c_p$. In other words, the curve $c'_z$ surrounds another puncture of $R$. Connecting $c'_z$ and $c_p$, we have a simple closed curve passing through $p$ with length less than $M_1$. Since $R^*_M - R_\epsilon$ is not of type $(0, 3)$, the curve does not surround a puncture of $R$.

Finally, we suppose that $l_z$ is simple for any $z \in S(M, \epsilon)$. Let us consider a geodesic $L_z$ for $z \in \widetilde{R^*_M} \cap \{z \in \mathbb{H} \mid \text{Im } z = \delta(M)\}$, where $\widetilde{R^*_M}$ is a lift of $R^*_M$ with $\widetilde{R^*_M} \cap S(M, \epsilon) \neq \emptyset$. From the definition, $\ell(L_z) = M$. Therefore, there exists a simple closed curve $c_z$ in $R^*_M$ passing through $\pi(z)$ with $\ell(c_z) < M = \ell(L_z) = \ell(l_z)$. Obviously, the curve $c_z$ is not homotopic to $l_z = \pi(L_z)$ because $l_z$ is the shortest simple closed curve which passes through $\pi(z)$ and surrounds the puncture. Therefore, by using the same argument as above, we have a non-trivial simple closed curve passing through $p$ with length less than $M_1$ which does not surrounding a puncture of $R$. □

To prove the main results, the following proposition on the hyperbolic geometry is crucial.

**Proposition 2.** *Let $\Gamma$ be a Fuchsian model on the upper half-plane $\mathbb{H}$ of a Riemann surface $R$. Assume that $\Gamma$ is non-elementary. Let $M$ and $D$ be positive constants. Then there is a constant $A > 1$ depending only on $M$ and $D$ that satisfies the following property: for a quasiconformal automorphism $f$ of $\mathbb{H}$ such that $f \circ \Gamma \circ f^{-1} = \Gamma$, suppose that there exist distinct hyperbolic elements $g_1, g_2$ and $g_3$ in $\Gamma$ such that*

1. *translation lengths of $g_j$ ($j = 1, 2, 3$) are less than $M$,*
2. *the projections of the axes $\ell_j$ of $g_j$ to $R$ are simple closed geodesics,*
3. *the distances between a point $z_1$ on $\ell_1$ and $\ell_j$ ($j = 2, 3$) are less than $D$, and*
4. *an isomorphism $\chi$ of $\Gamma$ induced by $f$ satisfies*

$$\chi(g_1) = g_1, \quad \chi(g_2) = g_2, \quad \chi(g_3) \neq g_3.$$

*Then, $K(f) \geq A$.*

To prove this proposition, we prepare some known results.

**Lemma 1** ([7] Theorem 1)**.** *Let $f$ be a quasiconformal automorphism of $\mathbb{C}$ fixing $0$ and $1$, and suppose that there is a point $z_0$ in $\mathbb{C} - \{0, 1\}$ such that*

$$\log M = d_1(z_0, f(z_0)) > 0.$$

*Then $K(f) \geq M^2$, where $d_1(\ ,\ )$ is the hyperbolic distance on $\mathbb{C} - \{0, 1\}$.*

**Lemma 2** ([13] Lemma 3.1)**.** *Let $f$ be a quasiconformal mapping of a Riemann surface $R$ onto another Riemann surface $S$, and c be a simple closed geodesic on $R$ with hyperbolic length $L$. Then the hyperbolic length of a closed geodesic on $S$ homotopic to $f(c)$ is not greater than $K(f)L$.*

**Lemma 3** ([5])**.** *For a given $M > 0$, let $g$ and $g'$ be arbitrary two distinct hyperbolic elements of $\Gamma$ with translation lengths less than $M$. Suppose that the projections of the axes of $g$ and $g'$ to $R$ are simple closed geodesics which coincide or disjoint. Then the axes have a distance greater than $C > 0$ depending only on $M$.*

We also needs a variant of the above lemma.

**Lemma 4.** *For a given $M > 0$, let $g$ and $g'$ be arbitrary two hyperbolic elements of $\Gamma$ with translation lengths less than $M$. Suppose that the projections of the axes of $g$ and $g'$ to $R$ are simple closed geodesics which intersect to each other. Then the axes make an angle greater than $C > 0$ depending only on $M$.*

*Proof.* Assume that the translation length of $g'$ is not less than that of $g$. Then, on $R$, the closed geodesic $\ell$ induced by $g$ can not round more than once in the collar of the geodesic $\ell'$ induced by $g'$. Hence we have a desired lower bound for the intersection angle of $\ell$ and $\ell'$. ☐

*Proof of Proposition 2.* We may assume that fixed points of $g_1$ are $0$ and $\infty$, and that $z_1 = \sqrt{-1} \in \mathbb{H}$, hence $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$ for $j = 2, 3$. We may also assume that the maximal dilatation of $f$ is less than $2$. Then at least one of the fixed points of $g_j$ $(j = 2, 3)$ is not in $U = \{x \in \mathbb{R} \mid |x| < \delta \text{ or } |x| > 1/\delta\}$ for sufficiently small $\delta > 0$ which depends only on $M$ and $D$. Indeed, if both fixed points of $g_j$ are in $U_1 = \{x \in \mathbb{R} \mid |x| < \delta\}$ for small $\delta > 0$, then it contradicts $d_{\mathbb{H}}(\sqrt{-1}, \ell_j) \leq D$. The same argument works when both fixed points are in $U_2 = \{x \in \mathbb{R} \mid |x| > 1/\delta\}$. If one fixed point of $g_j$ is in $U_1$ and the other is in $U_2$, then it contradicts Lemma 3 if $\ell_1 \cap \ell_j = \emptyset$ and it contradicts Lemma 4 if $\ell_1 \cap \ell_j \neq \emptyset$ $(j = 2, 3)$ . Therefore, we verify that at least one of the fixed points of $g_j$ $(j = 2, 3)$ is not in $U$. By using the same argument, we see that there exists a constant $\delta' > 0$ depending only on $M$ and $D$ such that all fixed points of $g_2$ and $g_3$ are in $\{x \in \mathbb{R} \mid \delta' < |x| < 1/\delta'\}$.

Then, since $d_{\mathbb{H}}(\sqrt{-1}, \ell_3) \leq D$, the Euclidean diameter $\mathrm{diam}(\ell_3)$ of $\ell_3$ is greater than some $r = r(M, D) > 0$ which depends only on $M$ and $D$. Set $g_4 = f \circ g_3 \circ f^{-1}$. By the assumption we have $g_4 \neq g_3$. Then, we see that there exists a constant $C = C(M, D) > 0$ depending only on $M$ and $D$ such that an inequality

$$|b - f(b)| > C \tag{2}$$

holds for at least one fixed point $b$ of $g_3$. Indeed, since $K(f) < 2$, the translation length of $g_4$ is less than $2M$ by Lemma 2. Noting that diam$(\ell_3) > r$, we see that if $\ell_3 \cap \ell_4 \neq \emptyset$, then we have the assertion from Lemma 4 and that Lemma 3 yields the assertion if $\ell_3 \cap \ell_4 = \emptyset$.

Take a fixed point $a$ of $g_2$ with

$$\delta < |a| < 1/\delta. \tag{3}$$

Let $\phi$ be a Möbius transformation with $\phi(0) = 0, \phi(a) = 1$ and $\phi(\infty) = \infty$. As we noted, $\delta' < |b| < 1/\delta'$. Hence, (2) and (3) implies that

$$d_a(b, f(b)) > \log L$$

holds for some constant $L > 1$ depending only on $M$ and $D$, where $d_a(,)$ is the hyperbolic distance on $\mathbb{C} - \{0, a\}$. Considering $\{0, a, b\}$ instead of $\{0, 1, z_0\}$ for $z_0 = \phi(b)$ in Lemma 1, we verify that the assertion follows for $A = L^2$. □

Next, we show a fundamental property of Mod$_c^{\#}(R)$.

**Proposition 3.** *Let $R$ be a Riemann surface. For an arbitrary simple closed geodesic $c$ on $R$, let $\{[f_n]\}$ be a sequence of transformations of* $\mathrm{Mod}_c^{\#}(R)$ *that satisfies* $\lim_{n\to\infty} K(f_n) = 1$. *Then there exists a subsequence $\{[f_{n_j}]\}$ of $\{[f_n]\}$ such that $\{f_{n_j}\}$ locally uniformly converges to a conformal automorphism $f$ of $R$ which determines a transformation $[f] \in \mathrm{Mod}_c^{\#}(R)$.*

*Proof.* First we suppose that $c$ is not homotopic to a boundary component of $R$. Then there exists a simple closed geodesic $c'$ on $R$ with $c \cap c' \neq \emptyset$. Hence Lemma 5 (with $C = c$ and $K = c'$) below shows the desired result.

Next suppose that $c$ is homotopic to a boundary component of $R$. We may assume that the Riemann surface $R$ is not topologically finite. Consider the double $\hat{R}$ of $R$. Then, $\hat{R}$ is still hyperbolic and the curve $c$ is not homotopic to a boundary component of $\hat{R}$. And it is easily seen that quasiconformal mappings $f_n : R \to R$ $(n = 1, 2, \ldots)$ are extended to quasiconformal mappings $\hat{f}_n : \hat{R} \to \hat{R}$ with the same maximal dilatations. Therefore, by the same argument as above, we have the desired result. □

**Lemma 5.** *Let $\{f_n\}$ be a sequence of quasiconformal automorphisms of a hyperbolic Riemann surface $R$ that satisfies $\lim_{n\to\infty} K(f_n) = 1$. Suppose that there exist compact subsets $C$ and $K$ of $R$ such that $f_n(C) \cap K \neq \emptyset$ for all $n$. Then there exist a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ and a conformal automorphism $f$ of $R$ such that $\{f_{n_j}\}$ converge to $f$ locally uniformly on $R$.*

*Proof.* From the assumption, there exists a sequence $\{p_n\}$ on $C$ such that $f_n(p_n) \in K$. Since $C$ and $K$ are compact, there exist $p \in C$ and $q \in K$ such that $p_n \to p$ and $f_n(p_n) \to q$ as $n \to \infty$ Take lifts of $p_n, p$ and $q$ in $\mathbb{H}$, say $\tilde{p}_n$, $\tilde{p}$ and $\tilde{q}$, respectively, so that $\tilde{p}_n \to \tilde{p}$ as $n \to \infty$. We can take lifts $\tilde{f}_n : \mathbb{H} \to \mathbb{H}$ of $f_n$ satisfying $\tilde{f}_n(\tilde{p}_n) \to \tilde{q}$. Since $\{\tilde{f}_n\}$ is a normal family, a subsequence $\{\tilde{f}_{n_j}\}$ of $\tilde{f}_n$ converges locally uniformly, and the limit function $\tilde{f}$ is either a quasiconformal automorphism of $\mathbb{H}$ or a constant in $\mathbb{R} \cup \{\infty\}$ (see [8] Theorem 5. 3). Since $\tilde{f}(\tilde{p}) =$

$\tilde{q}$ is in $\mathbb{H}$, $\tilde{f}$ is not a constant. Thus, it follows from $\lim_{n\to\infty} K(f_n) = 1$ that $\tilde{f}$ is a conformal automorphism of $\mathbb{H}$. Hence, $\{f_{n_j}\}$ converges locally uniformly to a conformal automorphism $f$ of $R$ which is the projection of $\tilde{f}$. □

Before proving our main theorems, we shall give a sufficient condition for discreteness of a sequence of $\mathrm{Mod}^{\#}(R)$ under the conditions in Theorem 1.

**Proposition 4.** *Let $R$ be a Riemann surface satisfying the two conditions in Theorem 1, and $\{f_n\}$ be a sequence of quasiconformal automorphisms of $R$ satisfying the following conditions:*

- *$\{(f_n)_*\}$ converges to the identity, where $(f_n)_* : \pi_1(R) \to \pi_1(R)$ is an isomorphism induced by $f_n$.*
- $\lim_{n\to\infty} K(f_n) = 1$.

*Then, $f_n$ is homotopic to the identity for sufficiently large $n$.*

*Proof.* Let $\Gamma$ be a Fuchsian model of $R$, and $\tilde{f}_n$ a lift of $f_n$ for each $n$. We may take $\tilde{f}_n$ so that the isomorphisms $\chi_n : \Gamma \to \Gamma$ induced by $\tilde{f}_n$ converge to the identity. Suppose that $\chi_n$ are not eventually the identity. Then the following lemma gives us three hyperbolic elements $g_{1,n}, g_{2,n}$ and $g_{3,n}$ in $\Gamma$ for each $n$ which satisfy the conditions in Proposition 2 for some constants $M'$ and $D$. Hence, we have

$$K(f_n) \geq A = A(M', D) > 1.$$

Since constants $M'$ and $D$ are independent of $n$, this contradicts $\lim_{n\to\infty} K(f_n) = 1$. Hence we have proved this proposition. □

**Lemma 6.** *Let $R$ be a Riemann surface satisfying the two conditions in Theorem 1, and $\chi_n$ are isomorphisms of the Fuchsian model $\Gamma$ of $R$ such that $\chi_n \to id$ and that they are not eventually the identity. Then, for each $n$, there exist hyperbolic elements $g_{j,n}$ $(j = 1, 2, 3)$ of $\Gamma$ with axes $\ell_{j,n}$ such that they satisfy the following four conditions.*

(1) *the projections $L_{j,n}$ of $\ell_{j,n}$ to $R$ are simple closed geodesics.*
(2) *there is a constant $M'$ independent of $n$ such that the lengths of $L_{j,n}$ are less than $M'$.*
(3) *there is a constant $D$ independent of $n$ such that the distances between a point on $\ell_{1,n}$ and $\ell_{j,n}$ $(j = 2, 3)$ are less than $D$, and*
(4) *$\chi_n(g_{j,n}) = g_{j,n}$ for $j = 1, 2$, and $\chi_n(g_{3,n}) \neq g_{3,n}$.*

*Proof.* First, we observe a fundamental property of $R_M$. For an arbitrary point $p_0$ in $R^*_M - R_\epsilon$, there exists a non-trivial simple closed curve $C_{p_0}$ passing through $p_0$ such that it is not homotopic to a puncture and $\ell(C_{p_0}) < M_1$, where $M_1 = M_1(M, \epsilon)$ is a constant in Proposition 1 depending only on $M$ and $\epsilon$. Then there exists a simple closed geodesic $L_{p_0}$ which is homotopic to $C_{p_0}$. The length of $L_{p_0}$ is greater than $\epsilon$ and we have

$$0 < \epsilon/M_1 \leq \ell(L_{p_0})/\ell(C_{p_0}).$$

Hence there exists a constant $B = B(M, \epsilon)$ depending only on $\epsilon$ and $M$ such that the hyperbolic distance between $p_0$ and $L_{p_0}$ on $R$ is less than $B$ (FIGURE 1). This

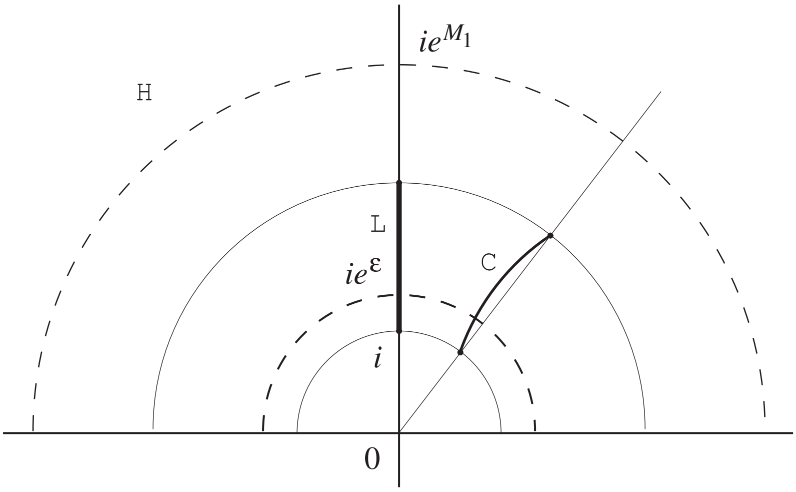


FIGURE 1. $L$: a lift of $L_{p_0}$, $C$: a lift of $C_{p_0}$

implies that for every point $z_0$ in a lift of $R^*_M$, say $\widetilde{R^*_M}$, if it is not projected to $R_\epsilon$, then there is an axis $\ell_0$ of a hyperbolic element of $\Gamma$ such that $d_{\mathbb{H}}(z_0, \ell_0) \leq B$ and that the projection to $R$ is a simple closed geodesic with length less than $M_1$.

By Remark 3, the homomorphism $\pi_1(R^*_{M''}) \to \pi_1(R)$ is surjective for all $M'' \geq M$. Thus, we take a constant $M$ sufficiently large so that there exist two disjoint simple closed geodesics $L^0_1$ and $L^0_2$ on $R^*_M$ whose lengths are less than $M$. Let $\gamma_j$ $(j = 1, 2)$ be hyperbolic elements of $\Gamma$ which represent $L^0_j$. Since $\chi_n \to id$ $(n \to \infty)$, $\chi_n(\gamma_1) = \gamma_1$ and $\chi_n(\gamma_2) = \gamma_2$ for sufficiently large $n$. Since $\chi_n$ is not eventually the identity, we may find a $\gamma_n \in \Gamma$ so that $\chi_n(\gamma_n) \neq \gamma_n$. The following lemma shows more, that is, we may take better one as $\gamma_n$.

**Lemma 7.** *Let $\gamma_1, \gamma_2$ and $\chi_n$ be the same ones as above. For sufficiently large $n$, there exists a hyperbolic element $\gamma_n$ of $\Gamma$ that satisfies the following two conditions:*

(1) $\chi_n(\gamma_n) \neq \gamma_n$.
(2) *the projection of the axis of $\gamma_n$ on $R$ is a simple closed geodesic with length less than $M$.*

*Proof.* Since $\chi_n \neq id$, there exists an element $\alpha_n$ of $\Gamma$ such that $\chi_n(\alpha_n) \neq \alpha_n$. We will show that either $\alpha_n \circ \gamma_1 \circ \alpha_n^{-1}$ or $\alpha_n \circ \gamma_2 \circ \alpha_n^{-1}$ is a desired element. It is obvious that both of them satisfy the second condition of the lemma. Hence, it suffices to show that one of them satisfies the first condition.

Suppose that $\chi_n$ fixes $\alpha_n \circ \gamma_j \circ \alpha_n^{-1}$ $(j = 1, 2)$. Then $\beta_n \circ \gamma_j \circ \beta_n^{-1} = \gamma_j$ $(j = 1, 2)$, where $\beta_n = \alpha_n^{-1} \circ \chi_n(\alpha_n)$. Thus, $\beta_n$ fixes all fixed points of $\gamma_1$ and $\gamma_2$. Since $\gamma_1$ and $\gamma_2$ are non-commutative, the Möbius transformation $\beta_n$ fixes four points and it must be the identity map. This contradicts $\chi_n(\alpha_n) \neq \alpha_n$. □

Let $\gamma_n$ be an element in Lemma 7. By the proof of Lemma 7, we may assume that $\gamma_n = \alpha_n \circ \gamma_1 \circ \alpha_n^{-1}$ for some $\alpha_n \in \Gamma$. We denote by $\ell^0_1$, $\ell^0_2$ and $\ell_n$ the axes of $\gamma_1$, $\gamma_2$ and $\gamma_n$, respectively. The projection of $\ell_n$ to $R$ is the same as that of $\ell^0_1$.

Fix a point $z_1$ on $\ell_1^0$. There exists the nearest point $z_n$ on $\ell_n$ from $z_1$. Since $z_1$ and $z_n$ belong to $\widetilde{R_M^*}$ and since $\widetilde{R_M^*}$ is connected by the second condition in Theorem 1, there exists an oriented smooth curve $C_n$ in $\widetilde{R_M^*}$ from $z_n$ to $z_1$. Furthermore, we can take the curve $C_n$ so that the projection of $C_n$ is in $R_M^* - R_\epsilon$.

Now, we shall show the statement for $M' = M_1$ and $D = \max(4(B + M_1 + 1), d_{\mathbb{H}}(z_1, \ell_2^0))$ ; we consider the following two cases for $d_{\mathbb{H}}(z_1, \ell_n)$.

**1:** $d_{\mathbb{H}}(z_1, \ell_n) \leq 4(B + M_1 + 1)$.

In this case, we set $g_{1,n} = \gamma_1$, $g_{2,n} = \gamma_2$ and $g_{3,n} = \gamma_n$. Then the third condition of the lemma holds for $D$. Other three conditions are trivial from the choice of these transformations.

**2:** $d_{\mathbb{H}}(z_1, \ell_n) > 4(B + M_1 + 1)$.

In this case, there are points $z_2$ and $w_2$ on $C_n$ such that $z_n$, $z_2$ and $w_2$ are located in this order with respect to the orientation of $C_n$ and they satisfy

$$d_{\mathbb{H}}(z_n, z_2) = d_{\mathbb{H}}(z_2, w_2) = 2(B + M_1 + 1).$$

Since $z_2$ and $w_2$ are points on $\widetilde{R_M^*}$ which are not projected to $R_\epsilon$, it follows from the above observation that there exists an axis $\ell_2'$ (resp. $\ell_2''$) such that $d_{\mathbb{H}}(z_2, \ell_2') \leq B$ (resp. $d_{\mathbb{H}}(w_2, \ell_2'') \leq B$) and that the projections of $\ell_2'$ and $\ell_2''$ to $R$ are simple closed geodesics whose lengths are less than $M_1$. Since $d_{\mathbb{H}}(z_2, w_2) > 2(B + M_1)$, we see that $\ell_2'$ and $\ell_2''$ are distinct. Let $\gamma_2'$ and $\gamma_2''$ be hyperbolic elements of $\Gamma$ whose axes are $\ell_2'$ and $\ell_2''$ respectively. Take a point $\zeta_2 \in \ell_2'$ so that $d_{\mathbb{H}}(z_2, \zeta_2) \leq B$ (FIGURE 2).

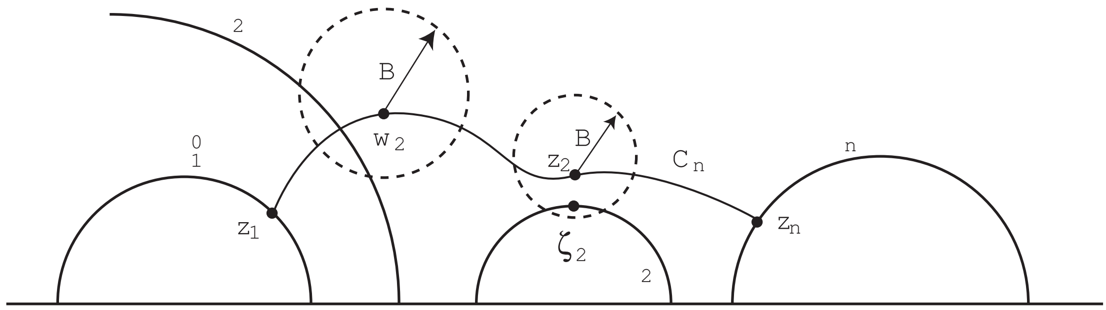


FIGURE 2

If $\chi_n(\gamma_2') = \gamma_2'$ and $\chi_n(\gamma_2'') = \gamma_2''$, set $g_{1,n} = \gamma_2'$, $g_{2,n} = \gamma_2''$ and $g_{3,n} = \gamma_n$. Noting that

$$d_{\mathbb{H}}(\zeta_2, \ell_2'') \leq 2(B + M_1 + 1) + 2B$$

and

$$d_{\mathbb{H}}(\zeta_2, \ell_n) \leq 2(B + M_1 + 1) + B,$$

we see that the third condition of the lemma holds for $D > 4B + 2(M_1 + 1)$. Thus, we obtain desired elements.

We consider the case when $\chi_n(\gamma_2') \neq \gamma_2'$ or $\chi_n(\gamma_2'') \neq \gamma_2''$. We may assume that $\chi_n(\gamma_2') \neq \gamma_2'$ because the argument below works for the case that $\chi_n(\gamma_2'') \neq \gamma_2''$.

If $\chi_n(\gamma_2') \neq \gamma_2'$ and $d_{\mathbb{H}}(z_1, \ell_2') \leq 4(B + M_1 + 1)$, then we see that $g_{1,n} = \gamma_1$, $g_{2,n} = \gamma_2$ and $g_{3,n} = \gamma_2'$ are desired ones as in the first case.

If $\chi_n(\gamma_2') \neq \gamma_2'$ and $d_{\mathbb{H}}(z_1, \ell_2') > 4(B + M_1 + 1)$, then we use the argument in the second case and we have $z_3, w_3$ on $C_n$ such that $z_2, z_3$ and $w_3$ are located in this order with respect to the orientation of $C_n$ and $d_{\mathbb{H}}(z_2, z_3) = d_{\mathbb{H}}(z_3, w_3) = 2(B + M_1 + 1)$. Also, we have axes $\ell_3', \ell_3''$ and $\gamma_3', \gamma_3'' \in \Gamma$ as above. Repeating this argument, we get desired elements since $d_{\mathbb{H}}(z_1, \ell_k') \leq 4(B + M_1 + 1) \leq D$ for some $k \in \mathbb{N}$. □

*Proof of Theorem 1.* Let $p_0 = [R, id]$ be the base point of $T^{\#}(R)$. We first suppose that there exists a sequence $\{g_n\}$ of quasiconformal automorphisms of $R$ which determine distinct elements of $\mathrm{Mod}_c^{\#}(R)$ such that $\lim_{n\to\infty} g_n(p_0) = p$ for some $p$ in $T^{\#}(R)$. Consider the sequence $\{f_n' = g_{n+1}^{-1} \circ g_n\}$. Then we see that $f_n'(p_0)$ converges to $p_0$. Thus there exist quasiconformal mappings $f_n : R \to R$ ($n = 1, 2, \ldots$) such that $f_n$ is RT-equivalent to $f_n'$ and that $\lim_{n\to\infty} K(f_n) = 1$. From Proposition 3, there exists a conformal automorphism $f$ of $R$ such that $[f_n \circ f] \in \mathrm{Mod}_c^{\#}(R)$ and $f_n \circ f$ converge to the identity on $R$ locally uniformly. Since $\lim_{n\to\infty} K(f_n \circ f) = \lim_{n\to\infty} K(f_n) = 1$, it follows from Proposition 4 that $[f_n \circ f] = [id]$ for sufficiently large $n$. Hence $[f_n] = [f^{-1}]$ for sufficiently large $n$. This contradicts the assumption that all $f_n$ are distinct.

Finally, we see that the same argument as above is valid for an arbitrary point $q = [S, f]$ in $T^{\#}(R)$. To see this, it suffices to show that the conditions of Theorem 1 are invariant under the quasiconformal deformation. Namely, the following lemma concludes the theorem. □

**Lemma 8.** *Let $R$ and $S$ be Riemann surfaces, and $f : R \to S$ be a $K$-quasiconformal map. If $R$ satisfies the conditions in Theorem 1, then $S$ also satisfies them.*

*Proof.* Let $\tilde{f} : \mathbb{H} \to \mathbb{H}$ be a lift of $K$-quasiconformal map $f$. The quasiconformal map $\tilde{f}$ can be extended to $\mathbb{H} \cup \hat{\mathbb{R}}$ with $\tilde{f}(\infty) = \infty$ and the restriction $\tilde{f}|\mathbb{R}$ of $\tilde{f}$ to $\mathbb{R}$ is a quasisymmetric function. The Douady-Earle extension $\Phi(\tilde{f})$ of $\tilde{f}|\mathbb{R}$ to $\mathbb{H}$ is a quasiconformal and bilipschitz map, and the bilipschitz constant $K'$ depends only on $K$ (cf. [2]). The projection $\phi_f : R \to S$ of $\Phi(\tilde{f})$ satisfies

$$(1/K')\ell(c) \leq \ell(\phi_f(c)) \leq K'\ell(c)$$

for an arbitrary curve $c$ on $R$, and $[S, f] = [S, \phi_f]$ in $T^{\#}(R)$. Then for an arbitrary point $a$ in $\phi_f(R_M^*)$, there exists a non-trivial simple closed curve $c_0$ containing $a$ such that $\ell(c_0) \leq K'M$. Thus, $\phi_f(R_M^*) \subset S_{K'M}$. Therefore, we see that the Riemann surface $S$ satisfies the second condition in Theorem 1 for a connected component of $S_{K'M}$ containing $\phi_f(R_M^*)$.

The same argument also shows that the first condition is satisfied by $S$. □

*Proof of Theorem 2.* We may assume that $R$ is a Riemann surface of infinite type. Suppose that $R$ is a Riemann surface of positive finite genus $g$ and satisfies the conditions in Theorem 1. Further suppose that $\mathrm{Mod}^{\#}(R)$ is not discrete. Then there exists a sequence $\{f_n\}$ of quasiconformal automorphisms of $R$ which determine distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} K(f_n) = 1$. Let $l$ be a dividing simple closed curve such that one of components of $R - l$ is a Riemann surface $S$ of genus $g$ with only one boundary component. Take a non-dividing simple closed geodesic $c$ on $S$. Then $f_n(c) \cap \bar{S} \neq \emptyset$ for all $n$. Indeed, if $f_n(c) \cap \bar{S} = \emptyset$, then $f_n(c)$ should be a dividing curve. Since $c$ is a non-dividing curve and $f_n$ is a homeomorphism, it can not occur. Then from Lemma 5, there exists a subsequence of $\{f_n\}$ which converges to a conformal automorphism $f$ of $R$ locally uniformly on $R$. Hence we can apply Proposition 4, and we conclude a contradiction.

Next suppose that $R$ has finite positive number of cusps and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as above. Let $V$ be a cusp neighborhood of a puncture of $R$. Since $R$ has only finitely many cusps, we may assume that $f_n(V) \cap V \neq \emptyset$ for all $n$ by taking a subsequence of $\{f_n\}$. Let $S$ be a pair of pants in $R$ such that it contains $V$ and that the boundaries of $S$ consist of the puncture and two dividing simple closed geodesics, say $c_1$ and $c_2$. We may assume that two geodesics $c_1$ and $c_2$ are not homotopic to a boundary component of $R$. If $f_n(c_1)$ is homotopic to $c_1$ for infinity many $n$, then they determine elements of $\mathrm{Mod}^{\#}_{c_1}(R)$. Hence, they must be discrete from Theorem 1. Assume that $f_n(c_1)$ is not homotopic to $c_1$ for all $n$. Since $f_n(V) \cap V \neq \emptyset$ and $f_n(S)$ is still a pair of pants for each $n$, we see that $f_n(c_1) \cap (\bar{S} \setminus V) \neq \emptyset$ or $f_n(c_2) \cap (\bar{S} \setminus V) \neq \emptyset$. We may assume that $f_n(c_1) \cap (\bar{S} \setminus V) \neq \emptyset$. Then from Lemma 5 and Proposition 4, we conclude a contradiction.

Finally, suppose that $R$ has finite positive number of borders and satisfies the conditions in Theorem 1. If $\mathrm{Mod}^{\#}(R)$ is not discrete, then there exists a sequence $\{f_n\}$ as before. Let $B$ be a one of borders of $R$. Since $R$ has only finite number of borders, we may assume that $f_n(B) = B$ for all $n$. Let $c$ be a simple closed geodesic which is homotopic to $B$. Then $f_n(c)$ is homotopic to $c$. Thus $f_n \in \mathrm{Mod}^{\#}_c(R)$, and $\{f_n\}$ is discrete by Theorem 1. This contradicts $\lim_{n\to\infty} K(f_n) = 1$. Hence $\mathrm{Mod}^{\#}(R)$ is discrete. □

## 6. Further examples

In Example 4, we showed that there exists a Riemann surface $R$ that satisfies the two conditions in Theorem 1, but that $\mathrm{Mod}^{\#}(R)$ is not discrete. In this case, there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\omega_n(p_0) = p_0$ for any $n$, where $p_0 = [R, id] \in T^{\#}(R)$ . By modifying this example, we exhibit another kind of examples of Riemann surfaces $R$ which also show that $\mathrm{Mod}^{\#}(R)$ are not discrete.

**Example 5.** We construct a Riemann surface $R$ such that there exists a sequence $\{\omega_n\}$ of distinct elements of $\mathrm{Mod}^{\#}(R)$ such that $\lim_{n\to\infty} d_T(\omega_n(p), p) = 0$ for some $p \in T^{\#}(R)$ and $\omega_n(p) \neq p$ for any $n$.

First, we consider a torus $A_0$ with two geodesic borders of the same length. Let $B_0$ be another torus obtained via the $(1 + \epsilon_0)$ quasiconformal deformation of $A_0$ for some $\epsilon_0 > 0$. Attach two copies of $B_0$ to $A_0$ along the borders suitably, and we obtain a Riemann surface $A_1$. Hence, it is a Riemann surface of genus 3 with two geodesic borders.

Next we take a Riemann surface $B_1$ which is the $(1+\epsilon_1)$ quasiconformal deformation of $A_1$ for some $\epsilon_1 > 0$. Attach two copies of $B_1$ to $A_1$ along the borders suitably, and we obtain a Riemann surface $A_2$ which is a Riemann surface of genus 9 with two geodesic borders. Repeating this process for some sequence $\{\epsilon_n\}$ of positive numbers, we have a sequence of Riemann surfaces $\{A_n\}$. More precisely, $A_{n+1}$ is a Riemann surface consisting of $A_n$ and two copies of $B_n$ which is $(1+\epsilon_n)$ quasiconformal deformation of $A_n$. Thus, $A_n$ is obtained by gluing $3^n$ surfaces homeomorphic to $A_0$, say $S_{-\alpha(n)}, S_{-\alpha(n)+1}, \ldots, S_{-1}, S_0, S_1, \ldots, S_{\alpha(n)-1}, S_{\alpha(n)}$ for $\alpha(n) = (3^n - 1)/2$. We construct a Riemann surface $R$ as the inductive limit of these $A_n$. Namely, $R$ is a Riemann surface obtained by gluing $S_k$ and $S_{k+1}$ $(k = 0, \pm 1, \pm 2, \ldots)$. If the sequence $\{\epsilon_n\}$ is bounded, then we see that $R$ satisfies

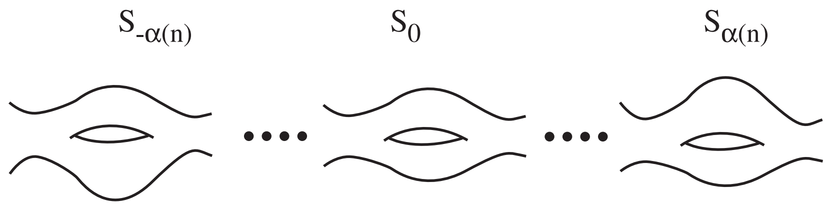


FIGURE 3

the above conditions on the injectivity radius.

Let $g_n$ be a quasiconformal automorphism of $R$ which sends a part corresponding to $S_k$ to a part corresponding to $S_{k+3^n}$ $(k = 0, \pm 1, \pm 2, \ldots)$. We shall show that there exists a quasiconformal automorphism $f_n$ homotopic to $g_n$ such that the maximal dilatations of $f_n$ $(n = 1, 2, \ldots)$ converge to one as $n \to \infty$.

We construct such maps inductively. If $0 \le |k| \le \alpha(n)$, then we set $f_n|_{S_k} = h_n$, where $h_n : A_n \to B_n$ is the $(1+\epsilon_n)$-quasiconformal mapping as above. If $\alpha(n) < |k| \le \alpha(n+1)$, then we may set $f_n|_{S_k} = h_{n+1} \circ h_n \circ h_{n+1}^{-1}$ and the maximal dilatation of $f_n|_{S_k}$ is less than $(1+\epsilon_n)(1+\epsilon_{n+1})^2$. Similarly, if $\alpha(m-1) < |k| \le \alpha(m)$ for $m(> n)$ and $h_m^{-1}(S_k) = S_\ell$ for some $\ell$ with $|\ell| < \alpha(p)$, then we may take $f_n|_{S_k} = h_m \circ h_\ell \circ h_m^{-1}$ on $S_k$. Therefore, we see that the maximal dilatation of $f_n$ is less than $(1+\epsilon_n) \prod_{k=n+1}^{\infty} (1+\epsilon_k)^2$.

If we take a sequence $\{\epsilon_n\}$ converges to zero rapidly so that $\sum_{n=1}^{\infty} \epsilon_n < \infty$, then we verify that the maximal dilatations of $f_n$ converge to 1 as $n \to \infty$. Thus, the quasiconformal automorphism $f_n$ induces an element of $\mathrm{Mod}^{\#}(R)$ whose orbits of $p_0 = [R, id]$ converge to $p_0$ in $T^{\#}(R)$.

The following example shows that Theorem 2 does not necessarily hold for a planar Riemann surface.

**Example 6.** Set $R = \mathbb{C} - \mathbb{Z}$, which is a planar Riemann surface satisfying the conditions in Theorem 1, and set $f_n(z) = z + n$ $(n = 1, 2, 3, \cdots)$. Since $f_n(z)$ is a conformal automorphism of $R$, we see that $[f_n](p_0) = p_0$ for all $n$, where $p_0 = [R, id] \in T^{\#}(R)$. Hence $\mathrm{Mod}^{\#}(R)$ is not discrete (cf. Example 4).

Further we see that there exists a point $p$ in $T^{\#}(R)$ such that the set of the orbit of $p$ under the action of $\mathrm{Mod}^{\#}(R)$ is not discrete. To show this, consider a following Riemann surface $S$: Set

$$z_n = \begin{cases} n + \frac{\sqrt{-1}}{j(n)+1} & (n \neq 0), \\ 0 & (n = 0) \end{cases}$$

where $j(n)$ is the power of the factor 2 when we decompose $|n|$ to the product of primes, and set $S = \mathbb{C} - \cup_{n=-\infty}^{\infty}\{z_n\}$. Since there exists a quasiconformal automorphism $h$ of $\mathbb{C}$ such that $h(n) = z_n$ $(n \in \mathbb{Z})$, $S$ is a quasiconformal deformation of $R$.

For every positive $m$, we take a locally affine quasiconformal automorphism $g_m$ of $S$ such that $\mathrm{Re}\, g_m(z) = \mathrm{Re}\, z + 2^m$ (and hence $g_m(z_n) = z_{(n+2^m)}$). Then, since $j(n+2^m) = j(n)$ for $j(n) < m$ and $j(n+2^m) = m$ for $j(n) \geq m$, we may take the locally affine map $g_m$ so that the maximal dilatations of $g_m$ tend to 1. Hence we see that the set of the orbit of $p = [S, h] \in T^{\#}(R)$ under the action of $\mathrm{Mod}^{\#}(R)$ is not discrete.

We shall construct a Riemann surface $R$ and sequences $\{M_n\}$, $\{M'_n\}$ having the properties referred in Remark 3 in §4.

**Example 7.** We consider right-angled hexagons $H_n$ $(n = 1, 2, \ldots)$ in the hyperbolic plain $\mathbb{H}$. The sides of the hexagon $H_n$ are labelled $a_{j,n}$ $(j = 1, 2, \ldots, 6)$ counterclockwise. We construct the hexagon so that $\ell(a_{2,n}) = \ell(a_{6,n})$, $\ell(a_{3,n}) = \ell(a_{5,n}) = 1$ and $\ell(a_{1,n}) = (2n)^{-1}$. Then $\{H_n\}$ converges to a pentagon with one cusp as $n \to \infty$. Thus, we see that

(4) $$d_{\mathbb{H}}(P_n, a_{2,n}) = d_{\mathbb{H}}(P_n, a_{6,n}) \leq M < \infty$$

holds for some $M$ independent of $n$, where $P_n$ is the midpoint of $a_{4,n}$. Take the

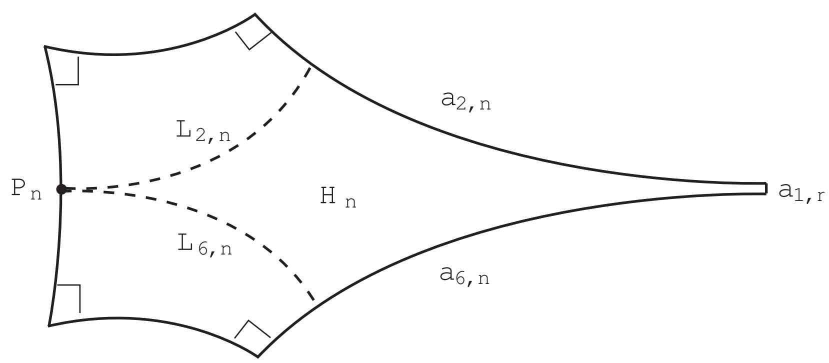


FIGURE 4. A hexagon close to a pentagon with one cusp

perpendicular line $L_{j,n}$ $(j = 2, 6)$ from $P_n$ to $a_{j,n}$. Since $d_{\mathbb{H}}(P_n, a_{1,n}) \to \infty$ as

$n \to \infty$, it follows from (4) that $d_{\mathbb{H}}(a_{1,n}, L_{2,n}) = d_{\mathbb{H}}(a_{1,n}, L_{6,n}) \to \infty$ (FIGURE 4).

Now, we take $k(n)$ copies of $H_n$, say $H_n^1, \ldots, H_n^{k(n)}$, so that

$$\frac{1}{3}d_{\mathbb{H}}(a_{1,n}, L_{2,n}) \leq 2k(n)\ell(a_{1,n}) = \frac{1}{n}k(n) \leq \frac{1}{2}d_{\mathbb{H}}(a_{1,n}, L_{2,n}). \tag{5}$$

Obviously, $k(n)/n \to \infty$ as $n \to \infty$. Let $a_{j,n}^i$ $(i = 1, 2, \ldots k(n); j = 1, 2, \ldots, 6)$ denote the sides of $H_n^i$ corresponding to $a_{j,n}$. Glue $H_n^i$ and $H_n^{i+1}$ along $a_{6,n}^i$ and $a_{2,n}^{i+1}$. Then, we have a right-angled $(2k(n) + 4)$-gon $D_n$ in $\mathbb{H}$. Label the side of $D_n$ formed by $a_{1,n}^1 \cup \ldots \cup a_{1,n}^{k(n)}$ as $b_{1,n}$ and the rest of sides as $b_{2,n}, \ldots, b_{2k(n)+4,n}$ counterclockwise.

We take a copy of $D'_n$ of $D_n$ with sides $b'_{j,n}$ $(j = 1, 2, \ldots, 2k(n) + 4)$ corresponding to $b_{j,n}$ of $D_n$. We glue $D_n$ and $D'_n$ along $b_{j,n}$ and $b'_{2k(n)+6-j,n}$ for $j = 2, 4, \ldots, 2k(n) + 2$ and $2k(n) + 4$. Then we have a hyperbolic bordered surface $S_n$ of type $(0, k(n) + 1)$. The boundary $\partial S_n$ consists of one long curve $c_{1,n}$ and $k(n)$ short curves $c_{2,n}, \ldots, c_{k(n),n}$. It follows from the construction that

$$\ell(c_{1,n}) = \frac{k(n)}{n},$$
$$\ell(c_{2,n}) = \ell(c_{k(n),n}) = 2,$$

and

$$\ell(c_{3,n}) = \ldots = \ell(c_{k(n)-1,n}) = 4.$$

From (4), we verify that $(S_n)_{4M}$ is connected and the naturel map of $\pi_1((S_n)_{4M})$ to $\pi_1(S_n)$ is surjective. On the other hand, it follows from (5) that $(S_n)_{k(n)/n}$ is not connected while both $(S_n)_{k(n)/2n}$ and $(S_n)_{2k(n)/n+4M}$ are connected.

We take a sequence $\{j_n\}$ so that

$$4M < \frac{k(j_n)}{j_n} < \frac{k(j_{n+1})}{10j_{n+1}}. \qquad (n = 1, 2, \ldots)$$

We glue $S_{j_n}$ and $S_{j_{n+1}}$ along $c_{k(j_n),j_n}$ of $\partial S_{j_n}$ and $c_{2,j_{n+1}}$ of $\partial S_{j_{n+1}}$. Then we have a bordered Riemann surface $S$, and a Riemann surface $R$ whose convex core is $S$. From the construction we verify that $R_{M_n}$ is connected for $M_n = k(j_n)/2j_n$ but $R_{M'_n}$ is not connected for $M'_n = k(j_n)/j_n$. Since $M_n, M'_n > 4M$, the natural maps of $\pi_1(R_{M_n})$ and $\pi_1(R^*_{M'_n})$ to $\pi_1(R)$ are surjective, where $R^*_{M'_n}$ is the "core component" of $R_{M'_n}$. Thus, $R$, $\{M_n\}$ and $\{M'_n\}$ are our desired ones.

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Ege Fujikawa
Department of Mathematics, Tokyo Institute of Technology
Oh-okayama Meguro-ku Tokyo 152-8551 Japan
fujikawa@math.titech.ac.jp

Hiroshige Shiga
Department of Mathematics, Tokyo Institute of Technology
Oh-okayama Meguro-ku Tokyo 152-8551 Japan
shiga@math.titech.ac.jp

Masahiko Taniguchi
Department of Mathematics, Graduate School of Science, Kyoto University
Kitashirakawa Sakyo-ku Kyoto 606-8502 Japan
tanig@kusm.kyoto-u.ac.jp