LIMIT SET OF QUASICONFORMAL MAPPING CLASS GROUP ON ASYMPTOTIC TEICHMÜLLER SPACE

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Abstract. In the infinite dimensional Teichmüller space for a Riemann surface of infinite type, the action of the quasiconformal mapping class group is not discontinuous, in general. In this paper, we first give a survey of our recent research for the discontinuous action. Then we investigate the dynamics of the action of the quasiconformal mapping class group on the asymptotic Teichmüller space by comparing that on the Teichmüller space.

1. Introduction

The Teichmüller space $T(R)$ of a Riemann surface R is the set of all Teichmüller equivalence classes of quasiconformal homeomorphisms of R, which has a complex Banach manifold structure. We consider a Riemann surface R of analytically infinite type. Then $T(R)$ is infinite dimensional. The quasiconformal mapping class group $\mathrm{MCG}(R)$ of R is the group of all homotopy equivalence classes of quasiconformal automorphisms of R, and every element of $\mathrm{MCG}(R)$ induces a biholomorphic automorphism of $T(R)$. Then we have the Teichmüller modular group $\mathrm{Mod}(R)$, which is coincident with the group of all biholomorphic automorphisms of $T(R)$. On the basis of the fact that the action of $\mathrm{MCG}(R)$ on $T(R)$ is not discontinuous in general, in a series of our papers [7], [8] and [15], we have considered the dynamics of the action of $\mathrm{MCG}(R)$ and investigated several phenomena which appear only when the Teichmüller space is infinite dimensional. In Section 2, we give a survey of our recent research on the discontinuous action of $\mathrm{MCG}(R)$. In particular, we define a stationary subgroup of $\mathrm{MCG}(R)$ as a generalization of the mapping class group of a topologically finite Riemann surface. Then a subgroup of $\mathrm{MCG}(R)$ with stationary property acts on $T(R)$ discontinuously under the bounded geometry condition on a Riemann surface. Moreover, the pure mapping class group and the essentially trivial mapping class group have the stationary property.

The asymptotic Teichmüller space $AT(R)$ is the set of all asymptotic equivalence classes of quasiconformal homeomorphisms of R and this equivalence is defined

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similarly to Teichmüller equivalence by using asymptotically conformal homeomorphisms. This is a quotient space of $T(R)$ and is also endowed with the complex structure. The quasiconformal mapping class group $\mathrm{MCG}(R)$ also acts biholomorphically on $AT(R)$ and we have the asymptotic Teichmüller modular group $\mathrm{Mod}_A(R)$, which is a subgroup of the biholomorphic automorphism group of $AT(R)$. However, a non-trivial element of $\mathrm{MCG}(R)$ can induce a trivial element of $\mathrm{Mod}_A(R)$. Moreover $\mathrm{MCG}(R)$ can act trivially on $AT(R)$. In Section 3, we review several results on sufficient conditions for non-trivial actions of $\mathrm{MCG}(R)$ on $AT(R)$.

The region of discontinuity on $T(R)$ for a subgroup $G \subset \mathrm{MCG}(R)$ is the largest open subset where G acts discontinuously, and the limit set is the complement of the region of discontinuity. Similarly, we also define the region of discontinuity and the limit set on $AT(R)$. We would like to know a relationship between the limit set on $T(R)$ and that on $AT(R)$. In [9], we have obtained a Riemann surface R such that the limit set on $T(R)$ for $\mathrm{MCG}(R)$ is empty but the limit set on $AT(R)$ for $\mathrm{MCG}(R)$ is not empty. Namely the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are quite different. However, we explore the problem that all limit points on $T(R)$ are projected to the limit set on $AT(R)$. In Section 4, we give a partial solution of this problem by using the results in Sections 2 and 3. We already know that a point of the region of discontinuity on $T(R)$ can be projected to the limit set on $AT(R)$. However we also prove that, for a cyclic group generated by a conformal automorphism of infinite order, the region of discontinuity on $AT(R)$ is not empty.

2. DISCONTINUITY OF TEICHMÜLLER MODULAR GROUP

2.1. **Teichmüller modular group.**

Throughout this paper, we assume that a Riemann surface R admits a hyperbolic structure and has non-abelian fundamental group. The *Teichmüller space* $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f of R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 of R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$ which is called the Teichmüller distance.

A *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes on R. If R is of topologically infinite type, then $\mathrm{MCG}(R)$ is, in general, not a countable group. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$

of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$, and we define the *Teichmüller modular group* of R by

$$\mathrm{Mod}(R) = \iota(\mathrm{MCG}(R)).$$

It is proved in [2] that the homomorphism ι is injective (faithful) for all Riemann surfaces R of non-exceptional type. See also [6] and [21] for other proofs. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι is also surjective for every Riemann surface R of non-exceptional type, namely $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. The proof is a combination of the results of [1] and [20]. See [10] for a survey of the proof.

2.2. **Pure mapping class group.**

Definition 2.1. We say that a subgroup $G \subset \mathrm{MCG}(R)$ acts at a point $p \in T(R)$ *discontinuously* if the following equivalent conditions are satisfied:

(a) there exists a neighborhood U of p such that the number of elements $[g]_* \in \iota(G)$ satisfying $[g]_*(U) \cap U \neq \emptyset$ is finite.
(b) there exist no distinct elements $[g_n]_* \in \iota(G)$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$.
(c) the orbit $\iota(G)(p)$ is discrete and the stabilizer subgroup $\mathrm{Stab}_{\iota(G)}(p)$ is finite.

Set

$$\Omega(G) = \{p \in T(R) \mid G \text{ acts at } p \text{ discontinuously}\}.$$

We call $\Omega(G)$ the *region of discontinuity* of G. By definition, the region of discontinuity is an open subset on $T(R)$.

For a Riemann surface R of analytically finite type, the quasiconformal mapping class group $\mathrm{MCG}(R)$ acts on $T(R)$ discontinuously, namely $\Omega(\mathrm{MCG}(R)) = T(R)$ (see [16, Section 8]). However, for a Riemann surface R of analytically infinite type, the action of $\mathrm{MCG}(R)$ is, in general, not discontinuous.

On the basis of this fact, Gardiner and Lakic [18] considered a special planar Riemann surface, the complement $\hat{\mathbb{C}} - \mathrm{C}$ of the standard middle-thirds Cantor set C in the unit interval as a subset of the complex sphere $\hat{\mathbb{C}}$. They defined the pure mapping class group $P(\hat{\mathbb{C}} - \mathrm{C})$ as the group of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(\hat{\mathbb{C}} - \mathrm{C})$ such that g fixes all points of C, and proved the following.

Proposition 2.2. *The pure mapping class group* $P(\hat{\mathbb{C}} - \mathrm{C})$ *acts on the Teichmüller space* $T(\hat{\mathbb{C}} - \mathrm{C})$ *discontinuously.*

We extend Proposition 2.2 for general Riemann surfaces. First we define the pure mapping class group for all Riemann surfaces.

Definition 2.3. The *pure mapping class group* $P(R)$ of a Riemann surface R is the group of all quasiconformal mapping classes $[g] \in \mathrm{MCG}(R)$ such that g fixes all non-cuspidal ends of R.

Then we have the following, which is proved in [11].

Theorem 2.4. *Let R be a Riemann surface satisfying the bounded geometry condition and having more than two non-cuspidal ends. Then the pure mapping class group $P(R)$ acts on the Teichmüller space $T(R)$ discontinuously.*

For a definition of the bounded geometry condition, see the next section. An outline of a proof of Theorem 2.4 is given in Section 2.4.

2.3. **Bounded geometry.**

Definition 2.5. We say that a Riemann surface R satisfies the *bounded geometry condition* if it satisfies the following three conditions:

(i) *lower bound condition*: the injectivity radius at any point of R except cusp neighborhoods are uniformly bounded away from zero.
(ii) *upper bound condition*: there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R.
(iii) R has no ideal boundary at infinity, namely the Fuchsian model of R is of the first kind.

Moreover we say that a Riemann surface R satisfies the *strong bounded geometry condition* if R satisfies conditions (i), (iii) and the following condition (ii′).

(ii′) *strong upper bound condition*: the injectivity radius at any point of R is uniformly bounded from above.

The (strong) bounded geometry condition is invariant under quasiconformal deformations, and every non-universal normal cover of a Riemann surface of analytically finite type satisfies the bounded geometry condition (see [7, Proposition 3]). More generally, if a Riemann surface R has a uniform pants decomposition, then R satisfies the bounded geometry condition. Here we say that a Riemann surface R has a *uniform pants decomposition* if it is decomposed into pairs of pants with geodesic boundary possibly degenerating to a puncture such that the hyperbolic lengths of boundary geodesics of all pairs of pants in the decomposition are uniformly bounded from above and below. In particular, the complement of the Cantor set satisfies the bounded geometry condition. However we see that having a uniform pants decomposition does not imply the bounded geometry condition as the following proposition says.

Proposition 2.6. *There exists a Riemann surface R such that it satisfies the bounded geometry condition but it does not have a uniform pants decomposition.*

Proof. Let $\check{R}$ be a compact Riemann surface of genus 2, and $\dot{R} = \check{R} - \{\check{x}\}$ for a point $\check{x} \in \check{R}$. Let $\check{H}$ be a Fuchsian model of $\check{R}$ acting on the unit disc Δ, $F \subset \Delta$ a fundamental region of $\check{H}$, and $\pi : \Delta \to \check{R}$ the universal covering projection. We take a point $x \in F$ such that $\pi(x) = \check{x}$, and set $\dot{F} = \overline{F} - \{x\}$. Let $R = \bigcup_{h \in \check{H}} h(\dot{F})$, which is a normal covering surface of $\dot{R}$ embedded in Δ. Since $\dot{R}$ is of analytically finite type, we see that R satisfies the bounded geometry condition. On the other hand, R does not have a uniform pants decomposition. Indeed, for every pants decomposition of R, a sequence of boundary geodesics of pairs of pants converges to the boundary $\partial\Delta = \{z \in \mathbf{C} \mid |z| = 1\}$ in the Hausdorff topology. Since the hyperbolic length on R is greater than that on Δ, we conclude that the hyperbolic lengths of the boundary geodesics are not bounded from above. $\square$

2.4. **Stationary subgroup.**

In this section, we explain our approach to a proof of Theorem 2.4. More generally, we consider the action of a stationary subgroup of the quasiconformal mapping class group, which is a generalization of the mapping class group of a topologically finite Riemann surface.

Definition 2.7. A subgroup G of $\mathrm{MCG}(R)$ is said to be *stationary* if there exists a compact subsurface W of R such that $g(W) \cap W \neq \emptyset$ for every representative g of every element of G.

It is known that a sequence of normalized quasiconformal homeomorphisms whose maximal dilatations are uniformly bounded is sequentially compact in compact open topology. The stationary property of mapping classes corresponds to the normalization in this context and hence such a sequence of mapping classes also has the compactness property if they are uniformly bounded. By using this observation, we have the following.

Theorem 2.8. *Let R be a Riemann surface satisfying the bounded geometry condition. Then* (i) $\Omega(\mathrm{MCG}(R)) \neq \emptyset$, (ii) $\Omega(G) = T(R)$ *for every stationary subgroup G of* $\mathrm{MCG}(R)$.

See [7] and [8] for a proof of Theorem 2.8.

Remark. There exist Riemann surfaces R such that $\emptyset \neq \Omega(\mathrm{MCG}(R)) \subsetneqq T(R)$. A typical example is a non-universal normal covering surface of a Riemann surface of analytically finite type.

Remark. There exist a Riemann surface R and a subgroup G of $\mathrm{MCG}(R)$ such that G is non-stationary but $\Omega(G) = T(R)$. See [13, Proposition 3.1]. In the paper, we further constructed a Riemann surface R satisfying the bounded geometry condition such that $\mathrm{MCG}(R)$ is non-stationary but $\Omega(\mathrm{MCG}(R)) = T(R)$.

By Theorem 2.8 (ii), the following theorem proved in [11] completes a proof of Theorem 2.4.

Theorem 2.9. *If a Riemann surface R has more than two non-cuspidal ends, then the pure mapping class group $P(R)$ is stationary.*

We have another example of stationary subgroup.

Example 2.10. For a simple closed geodesic c on R, let $G_c(R)$ be the set of all elements $[g] \in \mathrm{MCG}(R)$ such that $g(c)$ is freely homotopic to c. Then $G_c(R)$ is stationary, and thus it acts on $T(R)$ discontinuously if R satisfies the bounded geometry condition. See [8] and [15].

2.5. Essentially trivial mapping class group.

In this section, we define a subgroup of the pure mapping class group, which is called the essentially trivial mapping class group.

Definition 2.11. A quasiconformal mapping class $[g] \in \mathrm{MCG}(R)$ is said to be *essentially trivial* if there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. The *essentially trivial mapping class group* $E(R)$ is the group of all essentially trivial mapping classes.

It is clear that $E(R)$ is a subgroup of $P(R)$. Thus Theorem 2.4 yields that $E(R)$ acts on $T(R)$ discontinuously if a Riemann surface R has the bounded geometry condition and has more than two non-cuspidal ends. Although Theorem 2.4 is not true for Riemann surfaces the number of whose non-cuspidal ends are at most two, the assumption on the number of ends can be removed for the discontinuous action of $E(R)$ as follows. For a proof, see [11].

Theorem 2.12. *Let R be an analytically infinite Riemann surface satisfying the bounded geometry condition. Then $E(R)$ acts on $T(R)$ discontinuously and freely.*

3. ASYMPTOTIC TEICHMÜLLER MODULAR GROUP ON ASYMPTOTIC TEICHMÜLLER SPACE

3.1. Asymptotic Teichmüller space.

In this section, we consider the asymptotic Teichmüller spaces of a Riemann surface R, which was introduced in [19] when R is the upper half-plane and in [2], [3] and [17] when R is an arbitrary hyperbolic Riemann surface. We say that a quasiconformal homeomorphism f on R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity by a homotopy that keeps every point of the ideal boundary at infinity fixed throughout. The *asymptotic Teichmüller space* $AT(R)$ of a Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic. See also [4] and [5].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-V})$, where the infimum is taken over all compact subsets V of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where the infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_A([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_A is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

3.2. **Non-trivial action of mapping class group on $AT(R)$.**

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_A. See [4]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_A : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$, and we define the *asymptotic Teichmüller modular group* of R (the geometric automorphism group of $AT(R)$) by

$$\mathrm{Mod}_A(R) = \iota_A(\mathrm{MCG}(R)).$$

It is different from the case of $\iota : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_A is not injective, namely $\mathrm{Ker}\, \iota_A \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. Moreover there exists a Riemann surface R such that $\mathrm{MCG}(R) = \mathrm{Ker}\, \iota_A$ as follows.

Example 3.1. In [23], a Riemann surface R of analytically infinite type is constructed so that $\mathrm{MCG}(R) = \mathrm{Ker}\,\iota_A$. Note that this Riemann surface R does not satisfy the bounded geometry condition and $\mathrm{MCG}(R)$ consists of only countable number of elements.

On the basis of this fact, we give a sufficient condition for non-trivial action of $\mathrm{MCG}(R)$ in [9].

Proposition 3.2. *Let R be a Riemann surface of topologically infinite type. Suppose that R satisfies the upper bound condition. Then* $\mathrm{Ker}\,\iota_A \subsetneqq \mathrm{MCG}(R)$.

For a proof of Proposition 3.2, we have shown that there exists a quasiconformal automorphism of R that is not homotopic to any asymptotically conformal automorphism of R if R satisfies the upper bound condition. Then the base point $[[id]]$ of $AT(R)$ is not a common fixed point of $\mathrm{Mod}_A(R)$ and we have the assertion. Since the upper bound condition is invariant under quasiconformal deformations, we can apply the same argument for all points $[[f]] \in AT(R)$ to prove that there exists a quasiconformal automorphism of $f(R)$ that is not homotopic to any asymptotically conformal automorphism of $f(R)$ and then $[[f]]$ is not a common fixed point of $\mathrm{Mod}_A(R)$. Thus we have the following extension of Proposition 3.2.

Theorem 3.3. *If a Riemann surface R has the upper bound condition, then* $\mathrm{Mod}_A(R)$ *does not have a common fixed point in $AT(R)$.*

3.3. **Elements of** $\mathrm{Ker}\,\iota_A$.

We characterize the subgroup $\mathrm{Ker}\,\iota_A$. In [11], we prove the following.

Theorem 3.4. *The inclusion relation $E(R) \subset \mathrm{Ker}\,\iota_A \subset P(R)$ holds.*

On the other hand, the following theorem gives a condition for a quasiconformal mapping class which does not belong to $\mathrm{Ker}\,\iota_A$.

Theorem 3.5. *Let g be a quasiconformal automorphism of a Riemann surface R. Suppose there exists a constant $\delta > 1$ such that, for every compact subset E of R, there is a simple closed geodesic c on R outside of E satisfying either*

$$\frac{\ell(g(c))}{\ell(c)} \leq \frac{1}{\delta} \qquad \textit{or} \qquad \frac{\ell(g(c))}{\ell(c)} \geq \delta.$$

Then g is not homotopic to any asymptotically conformal automorphism of R. In particular, $[g] \notin \mathrm{Ker}\,\iota_A$.

A proof was given in [9]. As an application of Theorem 3.5, we also proved the following.

Proposition 3.6. *Let R be a normal cover of a compact Riemann surface whose covering transformation group is a cyclic group $\langle g \rangle$ generated by a conformal automorphism g of R of infinite order. Then $[g] \notin \mathrm{Ker}\,\iota_A$.*

Furthermore, Proposition 3.6 is extended as follows in [14] and [24].

Theorem 3.7. *Let R be a Riemann surface and let g be a conformal automorphism of R of infinite order. Then $[g] \notin \operatorname{Ker} \iota_A$.*

4. LIMIT SET AND REGION OF DISCONTINUITY OF ASYMPTOTIC TEICHMÜLLER MODULAR GROUP

For a subgroup $G \subset \mathrm{MCG}(R)$ and a point $p \in T(R)$, it is said that $q \in T(R)$ is a *limit point* of p for G if there exists a sequence $\{[g_n]_*\}_{n=1}^{\infty}$ of distinct elements of $\iota(G) \subset \mathrm{Mod}(R)$ such that $d([g_n]_*(p), q) \to 0$ as $n \to \infty$. The set of all limit points of p for G is denoted by $\Lambda(G, p)$, and the *limit set* for G is defined by $\Lambda(G) = \bigcup_{p \in T(R)} \Lambda(G, p)$. It is said that $p \in T(R)$ is a *recurrent point* for G if $p \in \Lambda(G, p)$, and the set of all recurrent points for G is called the *recurrent set* for G and is denoted by $\mathrm{Rec}(G)$. It is evident from the definition that $\mathrm{Rec}(G) \subset \Lambda(G)$ and these sets are G-invariant. Moreover, we proved in [12, Proposition 2.2] that $\Lambda(G) = \mathrm{Rec}(G)$, which is a closed set. Then the complement $T(R) - \Omega(G)$ of the regions of discontinuity $\Omega(G)$ is coincident with $\Lambda(G)$.

For a subgroup $G \subset \mathrm{MCG}(R)$, the region of discontinuity $\Omega_A(G)$ on the asymptotic Teichmüller space $AT(R)$ is defined as the set of all points $\hat{p} \in AT(R)$ where G acts discontinuously (cf. Definition 2.1), and the limit set $\Lambda_A(G)$ on $AT(R)$ is defined in a similar way as the previous paragraph. Then the complement $AT(R) - \Omega_A(G)$ of the regions of discontinuity $\Omega_A(G)$ is coincident with $\Lambda_A(G)$.

In this section, we investigate a relationship between the limit set on $T(R)$ and that on $AT(R)$. In [9], we have proved the following proposition, which implies that a point in the region of discontinuity on $T(R)$ can be mapped into the limit set on $AT(R)$ by the projection $\pi : T(R) \to AT(R)$.

Proposition 4.1. *There exists a Riemann surface R satisfying the bounded geometry condition such that $\Omega(\mathrm{MCG}(R)) = T(R)$ and $\Omega_A(\mathrm{MCG}(R)) \subsetneqq AT(R)$.*

On the other hand, if a Riemann surface has a certain property, then the dynamics of the actions of the quasiconformal mapping class group on $T(R)$ and on $AT(R)$ are the same. The following theorem is also proved in [9].

Proposition 4.2. *Let R be a Riemann surface that does not satisfy the lower bound condition. Then $\Lambda(\mathrm{MCG}(R)) = T(R)$ and $\Lambda_A(\mathrm{MCG}(R)) = AT(R)$.*

We consider the problem whether $\pi(\Lambda(\mathrm{MCG}(R))) \subset \Lambda_A(\mathrm{MCG}(R))$ for all Riemann surfaces R. Proposition 4.2 is a special case of this problem. The following statement gives a partial solution of this problem.

Theorem 4.3. *Let g be a conformal automorphism of infinite order of a Riemann surface R. For the cyclic group $G = \langle [g] \rangle$ generated by $[g] \in \mathrm{MCG}(R)$, we have $\pi(\Lambda(G)) \subset \Lambda_A(G)$.*

Proof. We take a limit point $p \in \Lambda(G)$ arbitrary. Then there exists a sequence $\{n_i\}$ of integers such that $d([g^{n_i}]_*(p), p) \to 0$ as $i \to \infty$. Then $d_A([g^{n_i}]_{**}(\hat{p}), \hat{p}) \to 0$ as $i \to \infty$ for the projection $\hat{p} = \pi(p)$. Since g is a conformal automorphism of infinite order, the element $[g]_{**} \in \mathrm{Mod}_A(R)$ is of infinite order. Indeed, if the order of $[g]_{**}$ is n, then $g^n \in \mathrm{Ker}\, \iota_A$. Since g is of infinite order, so is g^n. This contradicts Theorem 3.7. Hence we conclude that $\hat{p} \in \Lambda_A(G)$. □

We have another partial solution of the above problem.

Theorem 4.4. *Let R be a Riemann surface that has more than two non-cuspidal ends and satisfies the bounded geometry condition. Then the inclusion $\pi(\Lambda(\mathrm{MCG}(R))) \subset \Lambda_A(\mathrm{MCG}(R))$ holds.*

We will prove Theorem 4.4. First we note the following.

Lemma 4.5. *Let R be a Riemann surface that satisfies the bounded geometry condition and has more than two non-cuspidal ends. Then $\mathrm{Ker}\, \iota_A$ acts on $T(R)$ discontinuously.*

Proof. By Theorem 3.4, we have $\mathrm{Ker}\, \iota_A \subset P(R)$. Hence the assertion follows from Theorem 2.4. □

Proof of Theorem 4.4. We take a limit point $p \in \Lambda(\mathrm{MCG}(R))$ arbitrarily. Then there exists a sequence $[g_n]$ of distinct elements of $\mathrm{MCG}(R)$ such that $d([g_n]_*(p), p) \to 0$ as $n \to \infty$. Then $d_A([g_n]_{**}(\hat{p}), \hat{p}) \to 0$ for the projection $\hat{p} = \pi(p)$. We will show that $\{[g_n]_{**}\}_{n\in\mathbb{Z}} \subset \mathrm{Mod}_A(R)$ contains infinitely many elements. Then we conclude that $\hat{p} \in \Lambda_A(\mathrm{MCG}(R))$. Suppose to the contrary that $\{[g_n]_{**}\}_{n\in\mathbb{Z}}$ is a finite set $\{[g_1]_{**}, \ldots, [g_k]_{**}\}$ for some $k \geq 1$. Then there exists an integer i $(1 \leq i \leq k)$, say 1, such that $[g_n]_{**} = [g_1]_{**}$ for infinitely many n. Let $\gamma_n := g_n \circ g_1^{-1}$. Then $[\gamma_n] \in \mathrm{Ker}\, \iota_A$ and $d([\gamma_n]_*([g_1]_*(p)), p) = d([g_n]_*(p), p) \to 0$. This means that the point $p \in T(R)$ is a limit point for the subgroup $\mathrm{Ker}\, \iota_A$. This contradicts Lemma 4.5. Thus $\{[g_n]_{**}\}_{n\in\mathbb{Z}}$ contains infinitely many elements. □

In general, each inclusion in the relation in Theorem 3.4 is proper. However, in the forthcoming paper, we will prove that $E(R) = \mathrm{Ker}\, \iota_A$ if R satisfies the strong bounded geometry condition. Then Theorem 4.4 is true for all Riemann surfaces R satisfying the strong bounded geometry condition. Indeed, by Theorem 2.12, the essentially trivial mapping class group $E(R)$ acts on $T(R)$ discontinuously under the bounded geometry condition. Then so does $\mathrm{Ker}\, \iota_A$ if R satisfies the strong bounded geometry condition. By using this fact instead of Lemma 4.5 in the proof of Theorem 4.4, we have the assertion.

As we have seen in Proposition 4.1, the projection of a point in the region of discontinuity on $T(R)$ can be a limit point on $AT(R)$. However, under a certain condition, the region of discontinuity on $AT(R)$ is not empty as the following theorem says (cf. Theorem 2.8).

Theorem 4.6. *Let R be a Riemann surface that admits a conformal automorphism g of infinite order. Then $\Omega_A(\langle[g]\rangle) \neq \emptyset$.*

Remark. Under the assumption in Theorem 4.6, let $\hat{p}$ be a point in $\Omega_A(\langle[g]\rangle)$. Then by Theorem 4.3, all the preimages of $\hat{p}$ to $T(R)$ under π belong to $\Omega(\langle[g]\rangle)$.

We will prove Theorem 4.6. In the proof, we use the following estimate which is proved in [9]. Let $\ell(c)$ be the geodesic length for the free homotopy class of a simple closed curve c on a Riemann surface.

Lemma 4.7. *Let R be a Riemann surface and let c be a simple closed geodesic on R. Let E be a subset on R and $d = d_R(c, E)$ be the hyperbolic distance between c and E. If g is a K-quasiconformal homeomorphism of R onto another Riemann surface such that the restriction of g to $R - E$ is H-quasiconformal* $(1 \leq H \leq K)$*, then the inequality*

$$(1/\alpha) \cdot \ell(c) \leq \ell(g(c)) \leq \alpha \cdot \ell(c)$$

holds for a constant

$$\alpha = \alpha(K, H, d) = K - (K - H)\frac{2\arctan(\sinh d)}{\pi}.$$

We also use the following fundamental fact for a conformal automorphism of infinite order.

Lemma 4.8. *Let R be a Riemann surface that admits a conformal automorphism g of infinite order. Then there exist a simple closed geodesic c on R and a positive integer k such that the images of c under the cyclic group $\langle g^k \rangle$ generated by g^k are mutually disjoint.*

Proof. Take a simple closed geodesic c_0 on R arbitrarily. Since the group of all conformal automorphisms of R acts on R properly discontinuously, only finitely many of $\{g^n(c_0)\}_{n \in \mathbb{N}}$ have non-empty intersection with c_0. Let k be a positive integer that is greater than any power n of g^n satisfying $g^n(c_0) \cap c_0 \neq \emptyset$. Then the geodesics $\{g^{kn}(c_0)\}_{n \in \mathbb{N}}$ are mutually disjoint. Thus the simple closed geodesic $c := g^k(c_0)$ is a desired one. $\square$

We begin to prove Theorem 4.6.

Proof of Theorem 4.6. We will deform the Riemann surface R by a quasiconformal homeomorphism and construct a point $\hat{p} \in AT(R)$ as follows. By Lemma 4.8, we can take a simple closed geodesic c on R such that the images of c under $\langle g^k \rangle$ are mutually disjoint. We rename γ for g^k and set $c_n = \gamma^n(c)$. Then c_n have the same hyperbolic length ℓ and the hyperbolic distance $d_R(x, c_n)$ between any point $x \in R$ and c_n tends to ∞ as $n \to \infty$. See the proof of [22, Proposition 1]. Consider a subsequence $\{c_{2^m}\}_{m \in \mathbb{N}}$. We can take a quasiconformal homeomorphism f of R and a positive constant ℓ' $(< \ell)$ such that $\ell(f(c_{2^m})) \leq \ell'$ for all m and that $\ell(f(c_n)) = \ell$ for $n \neq 2^m$. Let c'_n be the geodesic that is freely homotopic to $f(c_n)$.

We will show that the point $\hat{p} := [[f]] \in AT(R)$ belongs to $\Omega_A(\langle[g]\rangle)$. We have only to prove that $\hat{p} \in \Omega_A(\langle[\gamma]\rangle)$. Set $\gamma_p = f \circ \gamma \circ f^{-1}$. Suppose to the contrary that $\hat{p} \in \Lambda_A(\langle[\gamma]\rangle)$. Then there exists a sequence $\{n_i\}$ of integers such that $d_A([\gamma^{n_i}]_*(\hat{p}), \hat{p}) \to 0$ as $i \to 0$. We take a constant H_0 such that $1 < H_0 < \ell/\ell'$ and may assume that the boundary dilatation of $\gamma_p^{n_i}$ satisfies $H(\gamma_p^{n_i}) < H_0$ for sufficiently large all i. Then for each i, there exists a compact subset E_i on $f(R)$ such that $K(\gamma_p^{n_i} |_{f(R)-E_i}) < H_0$. Let $\alpha = \alpha(K, H, d)$ be the constant in Lemma 4.7, which tends to H as $d \to \infty$. For each i, take a positive constant d_i such that $\alpha_i := \alpha(K(\gamma_p^{n_i}), H_0, d_i) < \ell/\ell'$. Then we take an integer m_i such that $d(E_i, c'_{2^{m_i}}) \geq d_i$ and that $2^{m_i} + n_i$ is not a power of 2. Then we have the inequality

$$\frac{\ell(\gamma_p^{n_i}(c'_{2^{m_i}}))}{\ell(c'_{2^{m_i}})} = \frac{\ell(c'_{2^{m_i}+n_i})}{\ell(c'_{2^{m_i}})} \geq \frac{\ell}{\ell'} > \alpha_i,$$

which contradicts Lemma 4.7. $\square$

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