ELLIPTIC QUASICONFORMAL MAPPING CLASSES ACTING ON ASYMPTOTIC TEICHMÜLLER SPACES

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ABSTRACT. We consider a problem of determining the group of all quasiconformal mapping classes acting trivially on the asymptotically Teichmüller space. In this note, we focus our attention to elliptic mapping classes which have fixed points on the Teichmüller space. By observing the action of pure mapping classes on the Teichmüller and asymptotic Teichmüller spaces, we seek a simple proof for a fact that every elliptic element of infinite order acts on the asymptotically Teichmüller space non-trivially.

1. PURE AND ESSENTIALLY TRIVIAL MAPPING CLASSES

Throughout this paper, we assume that a Riemann surface R admits a hyperbolic structure. The *Teichmüller space* $T(R)$ of R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R. Here we say that two quasiconformal homeomorphisms f_1 and f_2 on R are *equivalent* if there exists a conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. Here the homotopy is considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d_T([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d_T is a complete distance on $T(R)$ which is called the Teichmüller distance. The Teichmüller space $T(R)$ can be embedded in the complex Banach space of all bounded holomorphic quadratic differentials on R', where R' is the complex conjugate of R. In this way, $T(R)$ is endowed with the complex structure. For details, see [13] and [19].

A *quasiconformal mapping class* is the homotopy equivalence class $[g]$ of quasiconformal automorphisms g of a Riemann surface, and the *quasiconformal mapping class group* $\mathrm{MCG}(R)$ of R is the group of all quasiconformal mapping classes of R. Here the homotopy is again considered to be relative to the ideal boundary at infinity. Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_*$ of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance. Let $\mathrm{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$$

given by $[g] \mapsto [g]_*$, and we define the *Teichmüller modular group* of R by

$$\mathrm{Mod}(R) = \iota_T(\mathrm{MCG}(R)).$$

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It is proved in [2] that the homomorphism ι_T is injective (faithful) for all Riemann surfaces R of non-exceptional type. See also [6] and [15] for other proofs. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R. The homomorphism ι_T is also surjective for every Riemann surface R of non-exceptional type, namely $\mathrm{Mod}(R) = \mathrm{Aut}(T(R))$. The proof is a combination of the results of [1] and [14]. See [8] for a survey of the proof.

In this section, we observe the action of the pure mapping class defined below on the Teichmüller space.

Definition 1.1. We say that a mapping class $[g] \in \mathrm{MCG}(R)$ is *strongly pure* if g fixes all ends of R, and a mapping class $[g] \in \mathrm{MCG}(R)$ is *pure* if g fixes all non-cuspidal ends of R. The *pure mapping class group* $P(R)$ is the group of all pure mapping classes.

We classify mapping classes according to a certain property of the action on the Teichmüller space.

Definition 1.2. We say that a mapping class $[g] \in \mathrm{MCG}(R) - \{id\}$ is *elliptic* if $[g]_* \in \mathrm{Mod}(R)$ has a fixed point on $T(R)$.

Our first observation in this paper is the following.

Proposition 1.3. (i) *Let R be a Riemann surface having more than two ends, and let $[g]$ be a strongly pure and elliptic mapping class. Then $[g]$ is of finite order.* (ii) *Let R be a Riemann surface having more than two non-cuspidal ends, and let $[g]$ be a pure and elliptic mapping class. Then $[g]$ is of finite order.*

Proof. We will prove only statement (ii). Since $[g]$ is elliptic, we may assume that g is a conformal automorphism by considering a quasiconformal deformation of R. Since R has more than two non-cuspidal ends, there exists a pair of pants Y in R with geodesic boundary such that $R-Y$ has three connected components and that each of the connected components has a distinct end of R. See [9, Lemma 3.5]. Since g fixes all non-cuspidal ends, g^n also fixes all non-cuspidal ends for all n. Then $g^n(Y) \cap Y \neq \emptyset$ by the proof of [9, Proposition 3.4]. This implies that $g^n(Y)$ is a subset of the closed δ-neighborhood $\overline{N_\delta(Y)}$ of Y, where $\delta := \mathrm{diam}(Y)$. For a point $y_0 \in Y$, $g^n(y_0) \in \overline{N_\delta(Y)}$. Since $\overline{N_\delta(Y)}$ is compact, we may assume that $g^n(y_0)$ converges to some point $y_\infty \in \overline{N_\delta(Y)}$. Since the group of conformal automorphisms acts on R properly discontinuously, this implies that g is of finite order. □

We have another proof of Proposition 1.3 by using an advanced research. We say that a subgroup G of $\mathrm{MCG}(R)$ is *stationary* if there exists a compact subsurface W of R such that $g(W) \cap W \neq \emptyset$ for every representative g of every element of G, and a mapping class $[g] \in \mathrm{MCG}(R)$ is stationary if the cyclic group generated by $[g]$ is stationary. Moreover, we say that a mapping class $[g]$ is of divergent type if the orbit $\langle \gamma_* \rangle(p) = \{\gamma_*^n(p)\}_{n \in \mathbb{Z}}$ of each point $p \in T(R)$ diverges to the point at infinity of $T(R)$ as $n \to \pm\infty$. Then it was proved in [17, Theorem 6] that every stationary mapping class of infinite order is of divergent type, and a pure mapping class is stationary as we have seen in the proof above. Since an elliptic mapping class is not of divergent type, it should be of finite order.

In Proposition 1.3, we cannot replace the conclusion with the statement that $[g]$ is the identity as the following example says.

Example 1.4. There exists a Riemann surface R having more than two ends such that there is a strongly pure and elliptic mapping class $[g]$ on R and that $[g]$ is not the identity. Indeed, we consider a torus S and a hyperelliptic involution ϕ of S. Then ϕ has four fixed points x_i $(i = 1, \ldots, 4)$ on S. For $i = 1, 2$ and a small constant $r > 0$, we take a disk D_i with center x_i and radius r. We remove D_i $(i = 1, 2)$ and x_i $(i = 3, 4)$ from S and we obtain a Riemann surface S' of type $(1, 2, 2)$. We make infinitely many copies of S' and we glue the copies along the boundaries of the disks. Then we obtain a Riemann surface R of infinite type having infinitely many cuspidal ends. The hyperelliptic involution ϕ on S induces an elliptic involution g of R, which is clearly not the identity.

On the other hand, for an essentially trivial mapping class defined below, we have a strong conclusion.

Definition 1.5. We say that a mapping class $[g] \in \mathrm{MCG}(R)$ is *essentially trivial* if there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. Here the homotopy is considered to be relative to the ideal boundary at infinity. The *essentially trivial mapping class group $E(R)$* is the group of all essentially trivial mapping classes.

Then we have the following.

Proposition 1.6. *Let R be a Riemann surface of analytically infinite type. If an elliptic mapping class $[g] \in \mathrm{MCG}(R)$ is essentially trivial, then $[g]$ is the identity.*

Proof. Since $[g]$ is elliptic, we may assume that g is a conformal automorphism by considering a quasiconformal deformation of R. Since $[g]$ is essentially trivial, there exists a compact subsurface V_g of R such that, for each connected component W of $R - V_g$ that is not a cusp neighborhood, the restriction $g|_W : W \to R$ is homotopic to the inclusion map $id|_W : W \hookrightarrow R$. We take such a connected component W that is not relatively compact. If W is doubly connected, then the statement is easily proved. Thus we may assume that W is not doubly connected. Let Γ be a Fuchsian model of R, namely $R = \mathbb{H}/\Gamma$, and let $\tilde{g}$ be a lift of g to $\mathbb{H}$. Let Γ_W be a subgroup of Γ such that it corresponds to W. Then $\tilde{g}$ is the identity on the limit set $\Lambda(\Gamma_W)$. Since $\Lambda(\Gamma_W)$ contains more than two points and $\tilde{g}$ is conformal, we conclude that $\tilde{g}$ is the identity. Thus we have the assertion. □

2. Asymptotically trivial mapping classes

In this section, we apply our observation in the previous section to the action of mapping classes on the asymptotic Teichmüller space.

The asymptotic Teichmüller spaces was introduced in [12] when R is the hyperbolic plane and in [2], [3] and [11] when R is an arbitrary hyperbolic Riemann surface. We say that a quasiconformal homeomorphism f on R is *asymptotically conformal* if for every $\epsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f|_{R-V})$ of the restriction of f to $R - V$ is less than $1 + \epsilon$. We say that two quasiconformal homeomorphisms f_1 and f_2 on R are *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \to f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. Here the homotopy is considered to be relative to the ideal boundary at infinity. The *asymptotic Teichmüller space* $AT(R)$ of a Riemann surface R is the set of all asymptotic equivalence classes $[[f]]$ of quasiconformal homeomorphisms f on R. The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. In fact, it is infinite dimensional. Since a conformal homeomorphism is asymptotically conformal, there is a natural projection $\pi : T(R) \to AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic Teichmüller equivalence class $[[f]] \in AT(R)$. The asymptotic Teichmüller space $AT(R)$ has a complex manifold structure such that π is holomorphic. See also [4] and [5].

For a quasiconformal homeomorphism f of R, the *boundary dilatation* of f is defined by $H^*(f) = \inf K(f|_{R-E})$, where the infimum is taken over all compact subsets E of R. Furthermore, for a Teichmüller equivalence class $[f] \in T(R)$, the *boundary dilatation* of $[f]$ is defined by $H([f]) = \inf H^*(g)$, where the infimum is taken over all elements $g \in [f]$. A distance between two points $[[f_1]]$ and $[[f_2]]$ in $AT(R)$ is defined by $d_{AT}([[f_1]], [[f_2]]) = (1/2) \log H([f_2 \circ f_1^{-1}])$, where $[f_2 \circ f_1^{-1}]$ is a Teichmüller equivalence class of $f_2 \circ f_1^{-1}$ in $T(f_1(R))$. Then d_{AT} is a complete distance on $AT(R)$, which is called the asymptotic Teichmüller distance. For every point $[[f]] \in AT(R)$, there exists an asymptotically extremal element $f_0 \in [[f]]$ in the sense that $H([f]) = H^*(f_0)$.

Every element $[g] \in \mathrm{MCG}(R)$ induces a biholomorphic automorphism $[g]_{**}$ of $AT(R)$ by $[[f]] \mapsto [[f \circ g^{-1}]]$, which is also isometric with respect to d_{AT}. See [4]. Let $\mathrm{Aut}(AT(R))$ be the group of all biholomorphic automorphisms of $AT(R)$. Then we have a homomorphism

$$\iota_{AT} : \mathrm{MCG}(R) \to \mathrm{Aut}(AT(R))$$

given by $[g] \mapsto [g]_{**}$, and we define the *asymptotic Teichmüller modular group* of R by

$$\mathrm{Mod}_{AT}(R) = \iota_{AT}(\mathrm{MCG}(R)).$$

It is different from the case of $\iota_T : \mathrm{MCG}(R) \to \mathrm{Aut}(T(R))$ that the homomorphism ι_{AT} is not injective, namely $\mathrm{Ker}\, \iota_{AT} \neq \{[id]\}$, unless R is either the unit disc or a once-punctured disc. We call an element of $\mathrm{Ker}\, \iota_{AT}$ asymptotically trivial and call $\mathrm{Ker}\, \iota_{AT}$ the *asymptotically trivial mapping class group.*

In [9], we have proved the following property of the asymptotically trivial mapping class group.

Proposition 2.1. *The inclusion relation* $E(R) \subset \mathrm{Ker}\, \iota_{AT} \subset P(R)$ *holds.*

On the other hands, we would like to know conditions for mapping classes that is not in the asymptotically trivial mapping class group. In [7], we proved the following.

Proposition 2.2. *Let R be a normal cover of a compact Riemann surface whose covering transformation group is a cyclic group $\langle g \rangle$ generated by a conformal automorphism g of R of infinite order. Then* $[g] \notin \mathrm{Ker}\, \iota_{AT}$.

We extend Proposition 2.2 to the following theorem.

Theorem 2.3. *Let R be a Riemann surface of analytically infinite type having more than two non-cuspidal ends. Then every elliptic mapping class* $[g] \in \mathrm{MCG}(R)$ *of infinite order does not belong to* $\mathrm{Ker}\, \iota_{AT}$.

Proof. By Proposition 2.1, $\mathrm{Ker}\,\iota_{AT}$ is a subgroup of the pure mapping class group $P(R)$. Thus we have the assertion by proving that $[g] \notin P(R)$. Suppose to the contrary that $[g] \in P(R)$. Then $[g]$ is of finite order by Proposition 1.3. This contradicts the assumption. $\square$

In [18], we extend the statement of Theorem 2.3 for all Riemann surfaces of analytically infinite type.

Each inclusion in Proposition 2.1 is proper, in general. In [16], a Riemann surface R satisfying $E(R) = \mathrm{Ker}\,\iota_{AT} = P(R) = \mathrm{MCG}(R)$ have been constructed. In [10], we prove that $E(R) = \mathrm{Ker}\,\iota_{AT} \subsetneq P(R)$ if R satisfies a certain condition on hyperbolic geometry.

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