SMALE'S MEAN VALUE CONJECTURE AND THE COEFFICIENTS OF UNIVALENT FUNCTIONS

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Abstract. We study Smale's mean value conjecture and its connection with the second coefficients of univalent functions. We improve the bound on Smale's constant given by Beardon, Minda and Ng.

1. Introduction and results

Let $P(z)$ be a polynomial of degree $d \geq 2$. Then $P(z)$ has $d-1$ critical points counting multiplicities. Denote the critical points by $z_1, z_2, \cdots, z_{d-1}$. If z is not a critical point of P, then we have the following inequality:

$$\min_j \left| \frac{P(z) - P(z_j)}{z - z_j} \right| \leq 4\,|P'(z)|. \tag{1.1}$$

This inequality was first noted by Smale [8]. He also asked if the factor 4 in (1.1) can be replaced by $(d-1)/d$, which is the constant for the polynomial $P(z) = z^d + \lambda z$ ($\lambda \neq 0$). Smale later repeated this problem in [7] and [9], and this is called Smale's mean value conjecture. The conjecture is true for $d = 2, 3, 4$, see [11]. Furthermore, the case where $d = 5$ was recently proved in [2] by using a method based on the results in [3]. The best known result for an arbitrary d is found in [1], where the factor 4 in (1.1) is replaced by $4^{(d-2)/(d-1)}$. In this paper, we improve the estimate. Our main result is the following.

Theorem 1. *Let P be a polynomial of degree $d \geq 2$ and let z_1, z_2, $\cdots$, z_{d-1} be the critical points of P. If z is not a critical point of P, then*

$$\min_j \left| \frac{P(z) - P(z_j)}{z - z_j} \right| \leq 4\,\frac{d-1}{d+1}\,|P'(z)|. \tag{1.2}$$

Let

$$S(P, z) = \min_j \left| \frac{P(z) - P(z_j)}{(z - z_j)P'(z)} \right|.$$

The quotient is invariant under pre- and post- compositions with affine maps; specifically if

$$Q(w) = \frac{P(z + aw) - P(z)}{aP'(z)},$$

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then $Q(0) = 0$, $Q'(0) = 1$ and $w_j = (z_j - z)/a$ are the critical points of $Q(w)$. Therefore,

$$\left|\frac{Q(w_j)}{w_j}\right| = \left|\frac{P(z) - P(z_j)}{(z - z_j)P'(z)}\right|.$$

Thus we may assume that $P(z)$ in (1.1) and (1.2) fixes $z = 0$ and has derivative 1 at 0. Furthermore, we may assume that $\min_j |w_j| = \min_j |(z_j - z)/a|$ is obtained when $j = 1$, and taking

$$a = e^{i\theta}\,\frac{\min_j |P(z) - P(z_j)|}{|P'(z)|} \qquad \text{with} \qquad \theta = -\arg(z - z_1)$$

makes w_1 positive and $\min_j |Q(w_j)| = \min_j |P(z) - P(z_j)| = 1$. All this implies

$$S(P, z) = S(Q, 0) \le \frac{1}{w_1}.$$

We summarize as a definition:

Definition 1. A polynomial $P(z)$ of degree $d \ge 2$ with critical points z_1, z_2, $\cdots$, z_{d-1} is said to be normalized if the following conditions are satisfied: $P(0) = 0$, $P'(0) = 1$, $\min_j |P(z_j)| = 1$ and $\min_j |z_j| = z_1 > 0$.

All polynomials are normalized throughout this paper.

Let f be the inverse branch of P satisfying $f(0) = 0$. Then f is in the class S, that is the set of all normalized univalent functions on the open unit disc Δ. By applying the Koebe covering theorem to f, we see that every critical point z_j of P satisfies $|z_j| \ge 1/4$. Thus

$$\min_j \left|\frac{P(z_j)}{z_j}\right| \le \max_j \frac{1}{|z_j|} = \frac{1}{z_1} \le 4 \tag{1.3}$$

which proves (1.1).

Let

$$C(d) := \sup_P \frac{1}{z_1},$$

where the supremum is taken over all normalized polynomials P of degree d. Since any polynomial can be normalized, it follows that

$$S(P, z) \le C(d).$$

Thus any upper bound on $C(d)$ less that 4 is obviously an improvement on the constant 4 in (1.1).

By using estimates for the coefficients of univalent functions, we have the following theorem.

Theorem 2. *Let $d \ge 2$. Then*

$$C(d) \le 4\,\frac{d-1}{d+1}.$$

By the above consideration, Theorem 2 implies Theorem 1.

Our method for the proof of Theorem 2 will be applied to some special cases. First we have an extension of the theorem of Ng [6]:

Corollary 1. *If a normalized polynomial P satisfies $P''(0) = 0$, then*

$$S(P, 0) \le 2.$$

Next we show the following.

Corollary 2. *If all the critical points of a normalized polynomial P lie in the right half-plane, then*

$$S(P,0) \leq \frac{4}{3}.$$

Section 2 contains proofs of Theorem 2 and the corollaries. In section 3, we summarize a relationship between the various constants associated with Smale's mean value conjecture.

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2. Univalent functions and the critical points

Let $P(z)$ be a normalized polynomial of degree $d \geq 2$ and let $z_1, z_2, \cdots, z_{d-1}$ be the critical points of $P(z)$. Then the derivative of $P(z)$ is

$$P'(z) = \left(1 - \frac{z}{z_1}\right)\left(1 - \frac{z}{z_2}\right)\cdots\left(1 - \frac{z}{z_{d-1}}\right),$$

and by a simple integration, we have

$$P(z) = z - \left(\frac{1}{2}\sum_{i=1}^{d-1}\frac{1}{z_i}\right)z^2 + \left(\frac{1}{3}\sum_{i\neq j}^{d-1}\frac{1}{z_i z_j}\right)z^3 - \cdots + \frac{(-1)^{d-1}}{d\cdot z_1 z_2\cdots z_{d-1}}z^d. \tag{2.1}$$

Let $f(w) = w + a_2 w^2 + \cdots$ be the inverse branch of P with $f(0) = 0$. Since f is in the class S, we have $|a_2| \leq 2$ by the area theorem (see [4] and [5]). Observe that none of the z_i are in the image $f(\Delta)$. Then the function

$$f_i(w) = \frac{f(w)}{1 - f(w)/z_i}$$

is also in the class S, and the coefficient of w^2 in the expansion of f_i is $a_2 + 1/z_i$. Therefore

$$\left|a_2 + \frac{1}{z_i}\right| \leq 2 \qquad (i = 1, \cdots, d-1) \tag{2.2}$$

by the area theorem again. In particular,

$$\left|a_2 + \frac{1}{z_1}\right| \leq 2, \tag{2.3}$$

and by the triangle inequality,

$$\left|\frac{1}{z_i} - \frac{1}{z_1}\right| \leq 4. \tag{2.4}$$

Our estimate can now be derived as follows.

Proof of Theorem 2. Since $f(P(z)) = z$, it follows that $-P''(0) = f''(0) = 2a_2$. Then by (2.1),

$$a_2 = \frac{1}{2}\sum_{i=1}^{d-1}\frac{1}{z_i}, \tag{2.5}$$

which we substitute into (2.3) to get

$$\left|\frac{1}{2}\sum_{i=1}^{d-1}\frac{1}{z_i}+\frac{1}{z_1}\right|\leq 2.$$

We take the real part of the above inequality and multiply by 2 to find

$$3\,\frac{1}{z_1}+\sum_{i=2}^{d-1}\operatorname{Re}\frac{1}{z_i}\leq 4. \tag{2.6}$$

Taking the real part of (2.4) gives

$$-4\leq \operatorname{Re}\frac{1}{z_i}-\frac{1}{z_1},$$

that is,

$$\frac{1}{z_1}-4\leq \operatorname{Re}\frac{1}{z_i}.$$

Substituting this last inequality for the remaining terms in (2.6) yields

$$3\,\frac{1}{z_1}+(d-2)\left(\frac{1}{z_1}-4\right)\leq 4$$

which is easily solved for $1/z_1$. Therefore

$$C(d)=\sup_P\frac{1}{z_1}\leq 4\,\frac{d-1}{d+1},$$

which completes a proof. □

Certain restrictions on polynomials permit modifications of the proof of Theorem 2. If $P''(0)=0$, then $a_2=0$. Applying this to (2.3), we have

$$S(P,0)\leq\frac{1}{z_1}=\left|a_2+\frac{1}{z_1}\right|\leq 2,$$

and Corollary 1 is proved.

Next we prove Corollary 2. Suppose that all the critical points lie in the right half-plane. Then all the quantities in the sum in (2.6) are positive, and we have

$$\frac{3}{z_1}\leq 4,$$

which yields the assertion.

3. Various constants

Consider a normalized polynomial $P(z)$ with critical points $z_1,z_2,\cdots,z_{d-1}$. Then $\min_j|P(z_j)|=1$ and $\min_j|z_j|=z_1>0$. Let $i:=i(P)$ be an index satisfying $|P(z_{i(P)})|=1$. Then

$$\min_j\left|\frac{P(z_j)}{z_j}\right|\leq\left|\frac{P(z_{i(P)})}{z_{i(P)}}\right|=\frac{1}{|z_{i(P)}|}.$$

We choose $i(P)$ so that $|z_{i(P)}|$ is as large as possible and we define

$$N(d)=\sup_P\frac{1}{|z_{i(P)}|},$$

where the supremum is taken over all polynomials P of degree d. Since $|z_1| \leq |z_{i(P)}|$, we have

$$N(d) \leq C(d).$$

We conjecture that $C(d) \leq 1$. The factor 4 in (1.1) is the inverse of the Koebe constant 1/4. Tishler asked in [10] if the covering constant for the class of normalized inverse branches is larger than 1/4. Let κ be this constant. Then $\kappa = \inf_d \kappa(d)$, where $\kappa(d)$ is the covering constant for the class of normalized inverse branches of polynomials of degree d. Since $|z_j| \geq \kappa(d)$ for any critical point z_j, the following sequence of inequalities hold:

$$S(P,z) \leq N(d) \leq C(d) \leq \frac{1}{\kappa(d)}.$$

Note that each constant represents different problems. The Koebe covering theorem implies

$$S(P,z) \leq N(d) \leq C(d) \leq \frac{1}{\kappa} \leq 4.$$

The referee of this paper points out that our approach may be used to show that

$$\kappa(d) \geq \frac{d+1}{4d}.$$

To see this, observe that, for $f \in S$ and for $w \notin f(\Delta)$, the proof of inequalities (2.2) together with the triangle inequality implies

$$|w| \geq \frac{1}{2+|a_2|}.$$

Furthermore, inequalities (2.2) imply

$$\left|\sum_{i=1}^{d-1}\left(a_2 + \frac{1}{z_i}\right)\right| \leq \sum_{i=1}^{d-1}\left|a_2 + \frac{1}{z_i}\right| \leq 2(d-1).$$

On the other hand, (2.5) implies

$$\left|\sum_{i=1}^{d-1}\left(a_2 + \frac{1}{z_i}\right)\right| = |(d-1)a_2 + 2a_2|.$$

Thus

$$|a_2| \leq 2\,\frac{d-1}{d+1}.$$

Therefore,

$$|w| \geq \frac{1}{2+|a_2|} \geq \frac{d+1}{4d},$$

where d is the degree of P. This yields that

$$S(P,z) \leq N(d) \leq C(d) \leq \frac{1}{\kappa(d)} \leq \frac{4d}{d+1},$$

while our main result of this paper can be stated as

$$S(P,z) \leq N(d) \leq C(d) \leq 4\,\frac{d-1}{d+1}.$$

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