

Another approach to the automorphism theorem for Teichmüller spaces

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ABSTRACT. It has been proved that for a Riemann surface of non-exceptional type, every biholomorphic automorphism of the Teichmüller space is induced by a quasiconformal automorphism of the Riemann surface. The proof is a combination of two theorems. One theorem states that the above statement is true if a Riemann surface has the isometry property, which was proved by Earle and Gardiner and is called the automorphism theorem. Another theorem states that every Riemann surface of non-exceptional type has the isometry property, which was finally proved by Markovic. In this paper, we give another approach to a proof of the automorphism theorem.

1. Introduction

The Teichmüller space $T(R)$ of a Riemann surface R is the set of all equivalence classes $[f]$ of quasiconformal homeomorphisms f on R . We say that two quasiconformal homeomorphisms f_1 and f_2 on R are equivalent if there exists a conformal homeomorphism $h : f_1(R) \rightarrow f_2(R)$ such that $f_2^{-1} \circ h \circ f_1$ is homotopic to the identity. All homotopies are considered to be relative to the ideal boundary at infinity. A distance between two points $[f_1]$ and $[f_2]$ in $T(R)$ is defined by $d([f_1], [f_2]) = (1/2) \log K(f)$, where f is an extremal quasiconformal homeomorphism in the sense that its maximal dilatation $K(f)$ is minimal in the homotopy class of $f_2 \circ f_1^{-1}$. Then d is a complete distance on $T(R)$, which is called the Teichmüller distance.

We assume that a Riemann surface R is of hyperbolic type. Namely, it is represented by a quotient space \mathbb{H}^+/Γ of the upper half-plane $\mathbb{H}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by a torsion free Fuchsian group Γ . Let $R' = \mathbb{H}^-/\Gamma$ be the complex conjugate of R where $\mathbb{H}^- = \{z \in \mathbb{C} \mid \operatorname{Im} z < 0\}$, and $B(R')$ the complex Banach space of all bounded holomorphic quadratic differentials on R' with the hyperbolic supremum norm. Then the Teichmüller space $T(R)$ is a complex Banach manifold modeled on $B(R')$. In fact, $T(R)$ is embedded in $B(R')$ as a bounded contractible domain. For details, see [13] and [16].

The *quasiconformal mapping class* $[g]$ on a Riemann surface R is a homotopy equivalence class of quasiconformal automorphisms g of R , and the *quasiconformal mapping class group* $\operatorname{MCG}(R)$ of R is the set of all quasiconformal mapping classes

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on R . Every element $[g] \in \text{MCG}(R)$ induces a biholomorphic automorphism ρ_g of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$. This automorphism ρ_g is also an isometry with respect to the Teichmüller distance. Let $\text{Aut}(T(R))$ be the group of all biholomorphic automorphisms of $T(R)$. Then we have a homomorphism

$$\iota : \text{MCG}(R) \rightarrow \text{Aut}(T(R))$$

by $[g] \mapsto \rho_g$, and define the *Teichmüller modular group* by

$$\text{Mod}(R) = \iota(\text{MCG}(R)).$$

It was proved in [3], [5] and [15] that ι is injective (faithful) for all Riemann surfaces R of non-exceptional type, and then we may identify $\text{Mod}(R)$ with $\text{MCG}(R)$. Here we say that a Riemann surface R is of *exceptional type* if R has finite hyperbolic area and satisfies $2g + n \leq 4$, where g is the genus of R and n is the number of punctures of R .

It was considered whether ι is surjective, namely whether every biholomorphic automorphism of the Teichmüller space is induced by a quasiconformal automorphism of the Riemann surface. Royden [17] proved that ι is surjective for all compact Riemann surfaces R of genus $g \geq 2$, and Earle and Kra [4] proved the statement for all non-exceptional analytically finite Riemann surfaces. After that, it was a problem in researches on infinite dimensional Teichmüller spaces whether ι is surjective for all Riemann surfaces of analytically infinite type. After several partial solutions, finally the following theorem was proved.

THEOREM 1.1. *For every Riemann surface R of non-exceptional type, we have*

$$\text{Aut}(T(R)) = \text{Mod}(R).$$

Theorem 1.1 is proved by a combination of results by Earle and Gardiner [2] and Markovic [14]. To state their results, we begin by the following definitions. We say that a quadratic differential on a Riemann surface R is *integrable* if the norm

$$\|\varphi\| = \iint_R |\varphi|$$

is finite, and let $Q(R)$ be the complex Banach space of integrable holomorphic quadratic differentials on R with the above norm. We say that a Riemann surface R has the *isometry property* if, for any Riemann surfaces S_1 and S_2 that are quasiconformally equivalent to R , and for every surjective \mathbb{C} -linear isometry $G : Q(S_1) \rightarrow Q(S_2)$, there exist $\theta \in \mathbb{C}$ with $|\theta| = 1$ and a conformal homeomorphism $g : S_1 \rightarrow S_2$ such that

$$G(\varphi) = \theta \cdot (\varphi \circ g)(g')^2$$

for all $\varphi \in Q(S_1)$.

Earle and Gardiner [2] gives a sufficient condition for Riemann surfaces for which Theorem 1.1 is true.

THEOREM 1.2 ([2] (the automorphism theorem)). *Let R be a Riemann surface having the isometry property. Then $\text{Aut}(T(R)) = \text{Mod}(R)$.*

In the paper [2], they also proved that all Riemann surfaces of topologically finite type have the isometry property. Furthermore, Lakic [12] proved that all Riemann surfaces of finite genus have the isometry property. Finally, Markovic [14] proved the following.

THEOREM 1.3 ([14]). *Every Riemann surface of non-exceptional type has the isometry property.*

In this paper, we give another approach to a proof of the automorphism theorem. We review the proof by Earle and Gardiner in the next section, and state our proof in Section 3.

2. Proof of the automorphism theorem by Earle and Gardiner

In this section, we recall an outline of the proof of the automorphism theorem by Earle and Gardiner [2]. See also Chapter 8 in [11].

Let $E : T(R) \rightarrow T(R)$ be a biholomorphic automorphism of $T(R)$. For a point $p = [f_1] \in T(R)$, set $E(p) = [f_2] \in T(R)$. For each $i = 1, 2$, the quasiconformal homeomorphism $f_i : R \rightarrow S_i$ induces a geometric automorphism $\rho_{f_i} : T(R) \rightarrow T(S_i)$ that maps $[f] \mapsto [f \circ f_i^{-1}]$. Then $F := \rho_{f_2} \circ E \circ \rho_{f_1}^{-1} : T(S_1) \rightarrow T(S_2)$ is a holomorphic isomorphism that preserves the base points.

We fix i and denote S_i by S . The dual space $Q^*(S)$ of $Q(S)$ is identified with the tangent space $T_0(T(S))$ of $T(S)$ at the base point. Indeed, let $\text{Belt}(S)$ be the set of all Beltrami differentials μ on S , which have the norm $\|\mu\| = \text{ess. sup}_{z \in S} |\mu(z)| < \infty$, and set $\text{Belt}(S)_1 = \{\mu \in \text{Belt}(S) \mid \|\mu\| < 1\}$. Let $\Phi_S : \text{Belt}(S)_1 \rightarrow T(S)$ be the Bers projection which maps $\mu \mapsto h^\mu$, where h^μ is a quasiconformal homeomorphism on S whose Beltrami coefficient is μ . The derivative $\Phi'_S(0) : \text{Belt}(S) \rightarrow T_0(T(S))$ of Φ_S at the base point is a surjective homomorphism. Thus $T_0(T(S))$ is isomorphic to the quotient space $\text{Belt}(S)/\text{Ker}(\Phi'_S(0))$. On the other hand, let $P : \text{Belt}(S) \rightarrow Q^*(S)$ be the surjective linear map defined by

$$P\mu(\varphi) = \iint_S \mu \varphi$$

for $\mu \in \text{Belt}(S)$ and for $\varphi \in Q(S)$. Since $\text{Ker } P = \text{Ker}(\Phi'_S(0))$ (see [6]), we see that $Q^*(S)$ is isomorphic to $\text{Belt}(S)/\text{Ker}(\Phi'_S(0))$. Then the map $\Phi'_S(0)\mu \mapsto P\mu$ gives an isomorphism between $T_0(T(S))$ and $Q^*(S)$. The tangent space $T_0(T(S))$ of $T(S)$ at the base point has the Teichmüller norm

$$\|\Phi'_S(0)\mu\| = \lim_{t \rightarrow +0} \frac{d(\Phi_S(0), \Phi_S(t\mu))}{t}.$$

By the above identification, the Teichmüller norm coincides with the standard dual norm on $Q^*(S)$.

We see that $F'([id])$ is a \mathbb{C} -linear isometry of $Q^*(S_1)$ onto $Q^*(S_2)$. Indeed, since F is biholomorphic, $F'([id])$ is an invertible \mathbb{C} -linear map. Furthermore it preserves the Teichmüller norm. This follows from the fact that the biholomorphic isomorphism F preserves the Kobayashi distance, and that the Kobayashi distance coincides with the Teichmüller distance (see [9]).

The following proposition states that $F'([id])$ induces a \mathbb{C} -linear isometry of $Q(S_2)$ onto $Q(S_1)$. Note that $Q(S_i)$ is not reflexive if it is infinite dimensional.

PROPOSITION 2.1 (the adjointness theorem). *Let S_1 and S_2 be Riemann surfaces and $I : Q^*(S_1) \rightarrow Q^*(S_2)$ a \mathbb{C} -linear isometry. Then there exists a \mathbb{C} -linear isometry $G : Q(S_2) \rightarrow Q(S_1)$ such that $G^* = I$.*

By applying the adjointness theorem to our case, there exists a \mathbb{C} -linear isometry $G : Q(S_2) \rightarrow Q(S_1)$ such that $F'([id]) = G^*$. Since R has the isometry property, there exist $\theta \in \mathbb{C}$ with $|\theta| = 1$ and a conformal homeomorphism $g : S_1 \rightarrow S_2$ such that $G(\varphi) = \theta \cdot (\varphi \circ g)(g')^2$ for all $\varphi \in Q(S_2)$. Then g induces a geometric isomorphism $\rho_g : T(S_1) \rightarrow T(S_2)$, which preserves the base points.

Set $\omega_p := (\rho_{f_2})^{-1} \circ \rho_g \circ \rho_{f_1} = \rho_{f_2^{-1} \circ g \circ f_1} : T(R) \rightarrow T(R)$, which is an element of $\text{Mod}(R)$. Then $\omega_p(p) = E(p)$. To complete a proof, we show that ω_p is independent of p , namely $\omega_p = E$ on $T(R)$.

PROPOSITION 2.2 (the uniqueness theorem). *Let R be a Riemann surface having the isometry property, and $H : T(R) \rightarrow T(R)$ a holomorphic automorphism satisfying $H(q) = q$ and $H'(q) = \theta \cdot id$ ($\theta \in \mathbb{C}$) for some $q \in T(R)$. Then H is the identity.*

We would like to apply the uniqueness theorem to $H = \omega_p \circ E^{-1}$. It is clear that $\omega_p \circ E^{-1}(p) = p$. By another consideration, we see that $\rho'_g([id]) : Q^*(S_1) \rightarrow Q^*(S_2)$ is the adjoint of $\theta^{-1}G$ (see Lemma 4.2 in [2]), and thus $F'([id]) = \theta \rho'_g([id])$. Then we have

$$E'(p) = ((\rho_{f_2})^{-1} \circ F \circ \rho_{f_1})'(p) = \theta((\rho_{f_2})^{-1} \circ \rho_g \circ \rho_{f_1})'(p) = \theta \omega'_p(p),$$

where the derivatives are at the base point. Hence we have $(\omega_p \circ E^{-1})'(p) = \theta \cdot id$. Then by the uniqueness theorem, we conclude that $\omega_p = E$ on $T(R)$. \square

3. Main theorem and proof

In the proof of the automorphism theorem by Earle and Gardiner, the uniqueness theorem was used to conclude that $\omega_p = E$ on $T(R)$. The purpose of this paper is to give another proof of the automorphism theorem by considering the discontinuity of Teichmüller modular groups, instead of using the uniqueness theorem. However we have to assume that Riemann surfaces have the bounded geometry.

We say that a Riemann surface R has the *bounded geometry* if R satisfies the following three conditions:

- (i) *the lower bound condition*: the injectivity radius at any point of R except cusp neighborhoods are uniformly bounded away from zero.
- (ii) *the upper bound condition*: there exists a subdomain R^* of R such that the injectivity radius at any point of R^* is uniformly bounded from above and that the simple closed curves in R^* carry the fundamental group of R .
- (iii) R has no ideal boundary at infinity, namely the Fuchsian model of R is of the first kind.

The bounded geometry condition is quasiconformally invariant, and every non-universal normal cover of a Riemann surface of analytically finite type has the bounded geometry. Our theorem is the following.

THEOREM 3.1. *Let R be a Riemann surface that is of non-exceptional type and has the bounded geometry, and Ψ a biholomorphic automorphism of $T(R)$. Suppose that, for every point $p \in T(R)$, there exists an element $\omega_p \in \text{Mod}(R)$ such that $\Psi(p) = \omega_p(p)$. Then $\Psi \in \text{Mod}(R)$.*

We say that the Teichmüller modular group $\text{Mod}(R)$ acts at a point $p \in T(R)$ *discontinuously* if there exists a neighborhood U of p such that $\chi(U) \cap U = \emptyset$ for all

$\chi \in \text{Mod}(R) - \text{Stab}(p)$ and that the stabilizer subgroup $\text{Stab}(p) = \{\chi \in \text{Mod}(R) \mid \chi(p) = p\}$ consists of finitely many elements. Set

$$\Omega(\text{Mod}(R)) = \{p \in T(R) \mid \text{Mod}(R) \text{ acts at } p \text{ discontinuously}\}.$$

For a Riemann surface R of analytically finite type, the Teichmüller modular group $\text{Mod}(R)$ acts on $T(R)$ discontinuously, namely $\Omega(\text{Mod}(R)) = T(R)$ (see Section 8 in [10]). However, for a Riemann surface of analytically infinite type, the action of $\text{Mod}(R)$ is not discontinuous, in general. In a series of our papers [6], [7] and [8], we studied the discontinuity of Teichmüller modular groups, and proved the following.

PROPOSITION 3.2 ([6]). *If a Riemann surface R is of non-exceptional type and has the bounded geometry, then $\Omega(\text{Mod}(R)) \neq \emptyset$.*

We will prove Theorem 3.1 by using Proposition 3.2.

PROOF OF THEOREM 3.1. By Proposition 3.2, we take a point $p \in \Omega(\text{Mod}(R))$. Then there exists an open ball U_δ centered at p with radius $\delta > 0$ such that $\chi(U_\delta) \cap U_\delta = \emptyset$ for all elements $\chi \in \text{Mod}(R) - \text{Stab}(p)$, where $\text{Stab}(p)$ consists of finitely many elements. Since the action of $\text{Mod}(R)$ is isometric and any points p and p' in $T(R)$ can be joined by a segment of length $d(p, p')$, we have $d(p, \chi(p)) \geq 2\delta$ for all $\chi \in \text{Mod}(R) - \text{Stab}(p)$. We take an arbitrary point $q \in U_{\delta/2}$, namely $d(p, q) < \delta/2$. Then $d(\chi(p), \chi(q)) < \delta/2$ for all $\chi \in \text{Mod}(R)$. Thus $d(q, \chi(q)) > \delta$ for all $\chi \in \text{Mod}(R) - \text{Stab}(p)$. For every point $q \in U_{\delta/2}$,

$$\begin{aligned} d(q, \omega_p^{-1} \circ \omega_q(q)) &= d(\omega_p(q), \omega_q(q)) \\ &\leq d(\omega_p(q), \omega_p(p)) + d(\omega_p(p), \omega_q(q)) \\ &= d(p, q) + d(\Psi(p), \Psi(q)) \\ &= 2d(p, q) < \delta \end{aligned}$$

by the fact that the biholomorphic automorphism Ψ preserves the Teichmüller distance. Thus $\omega_p^{-1} \circ \omega_q \in \text{Stab}(p)$ for every $q \in U_{\delta/2}$. We denote the finitely many elements of $\text{Stab}(p)$ by $\{\chi_i\}_{i=1}^n$, and set

$$Q_i = \{q \in U_{\delta/2} \mid \omega_p^{-1} \circ \omega_q(q) = \chi_i(q)\} = \{q \in U_{\delta/2} \mid \Psi(q) = \omega_p \circ \chi_i(q)\}.$$

Then $U_{\delta/2} = \bigcup_{i=1}^n Q_i$. Since all Q_i are closed in $U_{\delta/2}$, at least one of them, say Q_1 , has an interior point. For all points $q \in Q_1$, we have $\Psi(q) = \omega_p \circ \chi_1(q)$. Since $T(R)$ is connected, the uniqueness theorem for holomorphic functions implies that $\Psi = \omega_p \circ \chi_1$ on $T(R)$. Thus $\Psi \in \text{Mod}(R)$. \square

Theorem 3.1 with its proof gives another approach to the automorphism theorem. This proof is a natural generalization of the proof for finite dimensional cases in [4] and [17].

In Theorem 3.1, we use the assumption that R has the bounded geometry to apply Proposition 3.2. Thus Theorem 3.1 with its proof is true for all Riemann surfaces R satisfying $\Omega(\text{Mod}(R)) \neq \emptyset$. We have an example of a Riemann surface R such that R does not satisfy the upper bound condition but satisfies $\Omega(\text{Mod}(R)) \neq \emptyset$. On the other hand, if R does not satisfy the lower bound condition, then $\Omega(\text{Mod}(R)) = \emptyset$ (see Theorem 1 in [6]). We conjecture that $\Omega(\text{Mod}(R)) \neq \emptyset$ if and only if R satisfies the lower bound condition and has no ideal boundary at infinity, and our proof of the automorphism theorem is valid for all such Riemann surfaces.

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