Cubic and Quartic Cyclic Homogeneous Inequalities of Three Variables

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Abstract We determine the geometric structures of the families of three variables cubic and quartic cyclic homogeneous inequalities of certain classes. These structures are determined by studying some real algebraic surfaces.

1. Introduction.

Symmetric or cyclic homogeneous polynomial inequalities are one of the most elementary inequalities. But they are not studied well. We may dare say that we know only a few even about three variables cyclic homogeneous inequalities. The aim of this article is to present a geometric method in order to deal with the cubic and quartic cyclic homogeneous inequalities in three variables. We sketch the history. Articles on polynomial inequalities are very few. One of the most important symmetric homogeneous inequalities is Muirhead's inequality published in 1902 ([5]), which says that if

$$\begin{aligned} &l_1 + l_2 + \cdots + l_n = m_1 + m_2 + \cdots + m_n, \\ &l_1 + l_2 + \cdots + l_k \geq m_1 + m_2 + \cdots + m_k \quad (\forall k = 1, 2, \ldots, n-1), \end{aligned}$$

then the inequality

$$\sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)}^{l_1} a_{\sigma(2)}^{l_2} \cdots a_{\sigma(n)}^{l_n} \geq \sum_{\sigma \in \mathfrak{S}_n} a_{\sigma(1)}^{m_1} a_{\sigma(2)}^{m_2} \cdots a_{\sigma(n)}^{m_n}$$

holds for any $a_1 \geq 0, \ldots, a_n \geq 0$.

The following Schur's inequality is also discovered around this age:

$$\begin{aligned} &(a^d + b^d + c^d) + abc(a^{d-3} + b^{d-3} + c^{d-3}) \\ &\qquad \geq (a^{d-1}b + b^{d-1}c + c^{d-1}a) + (ab^{d-1} + bc^{d-1} + ca^{d-1}) \end{aligned}$$

holds for all $a \geq 0$, $b \geq 0$, $c \geq 0$ and integers $d \geq 3$.

It is mystery that no generalization of Schur's inequality is known yet in the case of more than three variables, except the case of degree three (see [2] p.271 Q4). During about a hundred years, there is no essential development. Recently, Cîrtoaje discovered some important theorems about three variable homogeneous inequality. One of them is as the following:

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Theorem. ([4]) (1) Let $f(x, y, z)$ be a quartic symmetric homogeneous polynomial. Then, $f(x, y, z) \geq 0$ for any $x, y, z \in \mathbb{R}$, if and only if

$$f(1,0,0) \geq 0 \quad \text{and} \quad f(x,1,1) \geq 0 \quad (\forall x \in \mathbb{R}).$$

(2) Let $f(x, y, z)$ be a symmetric homogeneous polynomial with $3 \leq \deg f \leq 5$. Then, $f(x, y, z) \geq 0$ for any $x, y, z \geq 0$, if and only if

$$f(x,1,0) \geq 0 \quad \text{and} \quad f(x,1,1) \geq 0 \quad (\forall x \geq 0).$$

He also obtained similar theorem for symmetric homogeneous polynomials with $6 \leq \deg f \leq 8$. But we omit it because its statement is long. The following theorem is also fundamental.

Theorem. ([3]) Let p, q, r be any real numbers. The cyclic inequality

$$\begin{aligned}(a^4+b^4+c^4)+r(a^2b^2+b^2c^2+c^2a^2)+(p+q-r-1)abc(a+b+c)\\ \geq p(a^3b+b^3c+c^3a)+q(ab^3+bc^3+ca^3)\end{aligned}$$

holds for any $a, b, c \in \mathbb{R}$ if and only if $3(1+r) \geq p^2+pq+q^2$.

We analyze the above theorem using a convex cone. Let

$$\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\},$$

$$\mathfrak{C}_d := \left\{ f(a,b,c) \;\middle|\; \begin{array}{l} f \text{ is a cyclic homogeneous polynomial} \\ \text{of degree } d, \text{ such that} \\ f(a,\, b,\, c) \geq 0 \text{ for } \forall a,\, b,\, c \in \mathbb{R}, \\ \text{and that } f(1,\, 1,\, 1) = 0. \end{array} \right\},$$

$$\mathfrak{C}_d^+ := \left\{ f(a,b,c) \;\middle|\; \begin{array}{l} f \text{ is a cyclic homogeneous polynomial} \\ \text{of degree } d, \text{ such that} \\ f(a,\, b,\, c) \geq 0 \text{ for } \forall a,\, b,\, c \in \mathbb{R}_+, \\ \text{and that } f(1,\, 1,\, 1) = 0. \end{array} \right\},$$

$$\mathfrak{S}_d := \{f \in \mathfrak{C}_d \mid f \text{ is symmetric}\},$$

$$\mathfrak{S}_d^+ := \{f \in \mathfrak{C}_d^+ \mid f \text{ is symmetric}\}.$$

Cîrtoaje's inequality implies that $\mathfrak{C}_4$ is an ellipsoid cone in $\mathbb{R}^4$. We can also determine the structures of $\mathfrak{S}_4$ and $\mathfrak{S}_4^+$, using theorems in [3]. $\mathfrak{S}_4$ is an elliptic cone in $\mathbb{R}^3$. The base of $\mathfrak{S}_4^+$ is a domain in $\mathbb{R}^2$ enclosed by a part of the ellipse and two line segments. These are explained later. Note that $\mathfrak{C}_d = \mathfrak{S}_d = 0$ if d is odd. It is easy to see that

$$\mathfrak{C}_2 = \mathfrak{C}_2^+ = \mathfrak{S}_2 = \mathfrak{S}_2^+ = \mathbb{R}_+ \cdot (a^2+b^2+c^2-bc-ca-ab),$$

and these are a half line. In this article, we shall determine the structures of $\mathfrak{C}_3^+$, $\mathfrak{S}_3^+$ and $\mathfrak{C}_4^+$. As consequences, $\mathfrak{S}_3^+$ is a sector on a plane. $\mathfrak{C}_3^+$ is a cone in $\mathbb{R}^3$ whose base is a domain in $\mathbb{R}^2$ enclosed by a part of quartic curve and a line segment. The base of $\mathfrak{C}_4^+ \subset \mathbb{R}^4$ is a domain in $\mathbb{R}^3$ enclosed by three surfaces, one is a part of the ellipsoid, the others are parts of ruled surfaces. The following inequalities can be proved as a direct corollary of this fact. Note that these are analogues of Schur's inequality.

Let $a \geq 0$, $b \geq 0$, $c \geq 0$, then the following hold:

$$\frac{\sqrt[3]{4}}{3}(a^3+b^3+c^3)+\bigl(3-\sqrt[3]{4}\bigr)abc \geq a^2b+b^2c+c^2a, \tag{1.1}$$

$$(a^3+b^3+c^3)+\frac{\sqrt{16\sqrt{2}+13}-1}{2}(a^2b+b^2c+c^2a)$$
$$\geq \frac{\sqrt{16\sqrt{2}+13}+1}{2}(ab^2+bc^2+ca^2), \tag{1.2}$$
$$(a^4+b^4+c^4)+\left(\frac{4\sqrt[4]{3}}{3}-1\right)abc(a+b+c)\geq \frac{4\sqrt[4]{3}}{3}(a^3b+b^2c+c^3a), \tag{1.3}$$
$$(a^4+b^4+c^4)+\alpha(a^3b+b^3c+c^3a)\geq(\alpha+1)(ab^3+bc^3+ca^3), \tag{1.4}$$
here $\alpha = 1.37907443362539958016\cdots$ is a root of
$4\alpha^6+12\alpha^5-48\alpha^4-116\alpha^3+24\alpha^2+84\alpha+229=0$.
$$(a^4+b^4+c^4)+\beta(a^3b^2+b^2c^2+c^2a^2)\geq(\beta+1)(a^3b+b^3c+c^3a), \tag{1.5}$$
here $\beta = 2.18452974131524781307\cdots$ is a root of
$4\beta^5+19\beta^4-32\beta^3+2\beta^2-36\beta-229=0$.
$$(a^4+b^4+c^4)+\gamma(a^3b+b^3c+c^3a)\geq(\gamma+1)(a^2b^2+b^2c^2+c^2a^2), \tag{1.6}$$
here $\gamma = 5.07790940231978661368\cdots$ is a root of
$4\gamma^5+\gamma^4-68\gamma^3-172\gamma^2-192\gamma+144=0$.

These inequalities are located on the boundary of $\mathfrak{C}_d^+$. Note that Schur's inequality is located on the boundaries of $\mathfrak{S}_d^+$ and $\mathfrak{C}_d^+$.

We shall explain the outline of our theory. Let
$$S_{i,j,k}(a,b,c):=a^ib^jc^k+b^ic^ja^k+c^ia^jc^k.$$
Take an index set I_d so that the set $\{S_{i,j,k} \mid (i,\,j,\,k)\in I_d\}$ is a base of the vector space $\{f(a,\,b,\,c) \mid f \text{ is a cyclic homogeneous polynomial of degree } d\}$. Define the holomorphic map $\varphi_d\colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^N_{\mathbb{R}}$ ($N=\#I-1=\lceil (d+1)(d+2)/6\rceil -1$) by $\varphi(a:b:c)=\big(S_{i,j,k}(a,\,b,\,c) \mid (i,\,j,\,k)\in I\big)$. Then $X_d:=\varphi_d(\mathbb{P}^2_{\mathbb{R}})$ is (a closed domain of) a real projective surface of degree d. Consider X_d in an affine space $\mathbb{R}^N$ with the origin $(1:1:\cdots:1)\in\mathbb{P}^N_{\mathbb{R}}$, and take the convex cone $D_d\subset\mathbb{R}^N$ generated by X_d. Then $\mathfrak{C}_d$ can be identified with the dual convex cone of D_d. It seems that X_d is the whole part of algebraic surfaces, but this is not true for symmetric polynomials. That is, it is only a closed domain of a surface.

Similarly, let $\mathbb{P}^2_+ := \{(a:b:c)\in\mathbb{P}^2_{\mathbb{R}} \mid a\geq 0,\, b\geq 0,\, c\geq 0\}$, $X_d^+ := \varphi_d(\mathbb{P}^2_+)$, and D_d^+ be the convex cone generated by X_d^+. Then $\mathfrak{C}_d^+$ can be identified with the dual convex cone of D_d^+. Thus, if we study an algebraic surface X_d or its closed domain X_d^+, we can determine the structure of the convex cone $\mathfrak{C}_d$ or $\mathfrak{C}_d^+$, and we can obtain the sharpest inequalities.

It may be possible to determine the structures of $\mathfrak{C}_d^+$, $\mathfrak{C}_d$, $\mathfrak{S}_d^+$, $\mathfrak{S}_d$ for $d\geq 5$. But the structure of X_d or X_d^+ is not so simple for $d\geq 5$. It may also possible to do similar observation for more than three variables inequalities, if we study the structure of higher dimensional projective varieties. Theoretically it will be possible, but the calculation is complicated. The author tried this in vain, and expects the research in the future.

2. Main Theorems.

We use the same notation as in the section 1, and we denote
$$S_i = S_i(a,b,c) := a^i+b^i+c^i,$$
$$S_{i,j}=S_{i,j}(a,b,c):=a^ib^j+b^ic^j+c^ia^j,$$

$$
\begin{aligned}
U &= U(a,b,c) := abc,\\
T_{i,j} &= T_{i,j}(a,b,c) := S_{i,j}(a,b,c) + S_{j,i}(a,b,c).
\end{aligned}
$$

Theorem 1. *Let*

$$
\begin{aligned}
\mathfrak{f}_s(a,b,c) &:= s^2 S_3 - (2s^3-1)S_{2,1} + (s^4-2s)S_{1,2}\\
&\qquad\qquad - 3(s^4-2s^3+s^2-2s+1)U,\\
\mathfrak{f}_\infty(a,b,c) &:= S_{1,2} - 3U.
\end{aligned}
$$

Then, the following hold.

(1) The boundary of the convex cone $\mathfrak{C}_3^+$ is

$$
\mathbb{R}_+ \cdot \left\{\mathfrak{f}_s \,\middle|\, s \in [0,\,\infty]\right\} \cup \left(\mathbb{R}_+ \cdot \mathfrak{f}_0 + \mathbb{R}_+ \cdot \mathfrak{f}_\infty\right).
$$

(2) If $f \in \mathfrak{C}_3^+$, then we can find α, β, $s \in \mathbb{R}_+$ such that $f = \alpha\mathfrak{f}_s + \beta\mathfrak{f}_\infty$.

(3) Let $f(a,b,c) = S_3 + pS_{2,1} + qS_{1,2} + rU$ be a cyclic polynomial such that $3+3p+3q+r = 0$. Then, $f \in \mathfrak{C}_3^+$ if and only if

$$
4p^3 + 4q^3 + 27 \geq p^2q^2 + 18pq,
$$

or "$p \geq 0$ and $q \geq 0$".

(4) $\mathfrak{S}_3^+ = \mathbb{R}_+ \cdot (T_{2,1} - 6U) + \mathbb{R}_+ \cdot \left(S_3 + 3U - T_{2,1}\right)$.

Note that $S_3 + 3U - T_{2,1} \geq 0$ is Schur's inequality.

Theorem 2. *Let*

$$
\begin{aligned}
\mathfrak{g}_{p,q}(a,b,c) &:= S_4 - pS_{3,1} - qS_{1,3}\\
&\quad + \left(\frac{p^2+pq+q^2}{3} - 1\right) S_{2,2} + \left(p + q - \frac{p^2+pq+q^2}{3}\right) US_1,\\
\mathfrak{g}_\infty(a,b,c) &= \mathfrak{g}_{p,\infty}(a,b,c) = \mathfrak{g}_{\infty,q}(a,b,c) := S_{2,2} - US_1,\\
\mathfrak{h}_s(a,b,c) &:= S_{3,1} + s^2 S_{1,3} - 2sS_{2,2} - (s-1)^2 US_1,\\
\mathfrak{h}_\infty(a,b,c) &:= S_{1,3} - US_1,\\
\mathfrak{k}_{s,t}(a,b,c) &:= s^2 S_4 - (2s^3 - st)S_{3,1} + (s^3t - 2s)S_{1,3}\\
&\quad + (s^4 - 2s^2t + 1)S_{2,2} + \left(s^2 - (s-1)^2(s^2+st+1)\right)US_1.
\end{aligned}
$$

Then, the following hold.

(1) $\mathfrak{C}_4$ is an ellipsoid cone whose boundary is

$$
\mathbb{R}_+ \cdot \left(\left\{\mathfrak{g}_{p,q} \,\middle|\, (p,\,q) \in \mathbb{R}^2\right\} \cup \left\{\mathfrak{g}_\infty\right\}\right).
$$

(2) If $f \in \mathfrak{C}_4$, then we can find p, $q \in \mathbb{R}$, and α, $\beta \in \mathbb{R}_+$ such that $f = \alpha\mathfrak{g}_{p,q} + \beta\mathfrak{g}_\infty$. (See [3])

(3) The boundary of $\mathfrak{C}_4^+$ is

$$
\begin{aligned}
&\mathbb{R}_+ \cdot \left(\left\{\mathfrak{g}_{p,q} \,\middle|\, 9(p+q)^2 - (p-q)^2 \geq 6^2,\, p+q \geq 0\right\} \cup \left\{\mathfrak{g}_\infty\right\}\right)\\
&\qquad \cup \mathbb{R}_+ \cdot \left\{\mathfrak{k}_{s,t} \,\middle|\, s \geq 0,\, t \geq 1\right\} \cup \left(\mathbb{R}_+ \cdot \mathfrak{k}_{0,1} + \mathbb{R}_+ \cdot \left\{\mathfrak{h}_s \,\middle|\, s \in [0,\,\infty]\right\}\right).
\end{aligned}
$$

(4) If $f \in \mathfrak{C}_4^+$, then we can find α, β, $t \in \mathbb{R}_+$ and $s \in [0,\,\infty]$ such that

$$
f = \alpha\mathfrak{h}_s + \beta\mathfrak{g}_{\mathfrak{p}(t,s),\mathfrak{q}(t,s)},
$$

here

$$\mathfrak{p}(t,s) := \frac{S_1(t,s,1)\big(T_{2,1}(t,s,1) - 6U(t,s,1) - 3(t-s)(s-1)(1-t)\big)}{2(S_{2,2}(t,s,1) - U(t,s,1)S_1(t,s,1))} - 1,$$
$$\mathfrak{q}(t,s) := \frac{S_1(t,s,1)\big(T_{2,1}(t,s,1) - 6U(t,s,1) + 3(t-s)(s-1)(1-t)\big)}{2(S_{2,2}(t,s,1) - U(t,s,1)S_1(t,s,1))} - 1.$$

Note that $\mathfrak{g}_{p,q} \geq 0$ is Cîrtoaje's inequality.

Corollary 3. *Use the same notation as Theorem 2, and let*

$$\mathfrak{g}_p := \mathfrak{g}_{p,p} = S_4 - pT_{3,1} + (p^2-1)S_{2,2} + (2p-p^2)US_1.$$

Then, the following hold.

(1) $\mathfrak{S}_4$ is an elliptic cone whose boundary is $\mathbb{R}_+ \cdot \big\{\mathfrak{g}_p \bigm| p \in \mathbb{R}\cup\{\infty\}\big\}$. Thus, if $f \in \mathfrak{S}_4$, then we can find α, β, $p \in \mathbb{R}_+$ such that $f = \alpha\mathfrak{g}_p + \beta(S_{2,2} - US_1)$. (See [3])

(2) The boundary of $\mathfrak{S}_4^+$ is

$$\mathbb{R}_+ \cdot \big\{\mathfrak{g}_p \bigm| p \in [1,\,\infty]\big\} \cup \big(\mathbb{R}_+\cdot\mathfrak{g}_1 + \mathbb{R}_+\cdot(T_{3,1} - 2S_{2,2})\big)$$
$$\cup\big(\mathbb{R}_+\cdot\mathfrak{g}_\infty + \mathbb{R}_+\cdot(T_{3,1} - 2S_{2,2})\big).$$

Thus, if $f \in \mathfrak{S}_4^+$, then we can find α, $\beta \in \mathbb{R}_+$ and $p \in [0,\,\infty]$ such that $f = \alpha\mathfrak{g}_p + \beta(T_{3,1} - 2S_{2,2})$. (See [3])

We shall prove the inequalities (1.1)—(1.6) in the section 1, using above theorems. (1.1) and (1.2) are obtained from $\mathfrak{f}_s \geq 0$, putting $s = \sqrt[3]{2}$ or $s = \dfrac{1+\sqrt{2}-\sqrt{2\sqrt{2}-1}}{2}$ respectively. (1.3) is obtained from $\mathfrak{k}_{s,t} \geq 0$ putting $(s,\,t) = \big(\sqrt[4]{3},\,2/\sqrt{3}\big)$. (1.4) is obtained from $\mathfrak{k}_{s,t} \geq 0$, eliminating s and t from $\alpha = -(2s + t/s)$, $s^2 + 1/s^2 - 2t = 0$, $1-(s-1)^2(1+t/s+1/s^2) = 0$. (1.5) and (1.6) can be obtained by the similar way.

3. Proof of Theorem 1.

Throughout this paper, we fix the following notation.

$\mathbb{P}^n_{\mathbb{R}} :=$ (real projective space).

$(x_0 : x_1 : \cdots : x_n)$ the system of homogeneous coordinates of $\mathbb{P}^n_{\mathbb{R}}$.

$\mathbb{P}^2_+ := \big\{(a:b:c) \in \mathbb{P}^2_{\mathbb{R}} \bigm| a \geq 0,\, b \geq 0,\, c \geq 0\big\}$.

It is well known that for any a, b, $c \in \mathbb{R}$, the inequalities $S_4 \geq S_{3,1}$, $S_4 \geq S_{2,2} \geq US_1$ hold. Moreover, if a, b, $c \in \mathbb{R}_+$, then $S_3 \geq S_{2,1} \geq 3U$, $S_{3,1} \geq US_1$, $T_{3,1} \geq 2S_{2,2}$ hold.

Proof of Theorem 1. (1) (i) Ws shall prove that $\mathfrak{f}_s \in \mathfrak{C}_3^+$ for $s \geq 0$.

Since $\mathfrak{f}_s(b,a,c) = s^4\mathfrak{f}_{1/s}(a,b,c)$, we may assume that $0 \leq a \leq b \leq c = 1$. Let $k := (1-b)/(1-a)$. Since $0 \leq a \leq b$, we have $0 \leq k \leq 1$. Then we have

$$\begin{aligned}&\mathfrak{f}_s(a,b,c) = \mathfrak{f}_s(a, 1-k(1-a), 1)\\ &= (1-a)^2\Big\{a(1-ks)^2(k+s^2) + \big(1+(1-k)s^2\big)(1-k-s)^2\Big\}\\ &\geq 0.\end{aligned}$$

Note that $\mathfrak{f}_s(0,s,1) = 0$. We recommend readers to use computer to check some complicated equalities which appear in this article as the above.

(ii) We shall observe X_3^+.

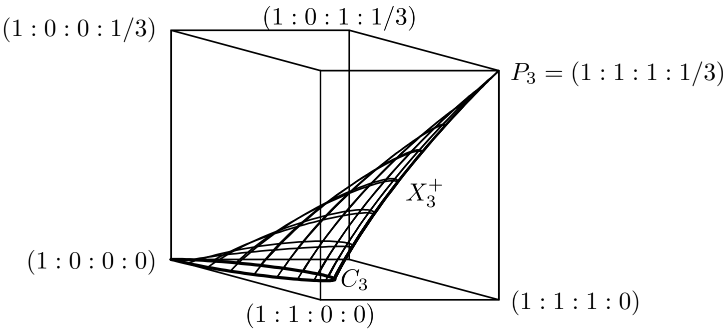
Let $\varphi_3\colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^3_{\mathbb{R}}$ be the holomorphic map defined by

$$\varphi_3(a:b:c) := \big(S_3(a,b,c) : S_{2,1}(a,b,c) : S_{1,2}(a,b,c) : U(a,b,c)\big),$$

and let

$$\begin{aligned}
&X_3^+ := \varphi_3(\mathbb{P}^2_+),\\
&f_0(s) := s^3+1, \quad f_1(s) := s^2, \quad f_2(s) := s, \quad f_3(s) := 0,\\
&C_3 := \big\{(f_0(s) : f_1(s) : f_2(s) : f_3(s)) \in X_3^+ \bigm| s \in \mathbb{R}_+\big\}\\
&\quad\;\; = \big\{(\varphi_3(0:s:1) \in X_3^+ \bigm| s \in \mathbb{R}_+\big\}.
\end{aligned}$$

Note that C_3 is the boundary of X_3^+, and C_3 is a nodal plane cubic curve whose node is at $(1:0:0:0)$.



The defining equation of $X_3 := \varphi_3(\mathbb{P}^4_{\mathbb{R}})$ is

$$x_1^3 + x_2^3 + 9x_3^3 - 6x_1x_2x_3 - x_0x_1x_2 + 3x_0x_3^2 + x_0^2x_3 = 0, \tag{3.1}$$

and X_3 has a rational double point of the type A_1 at $P_3 := (1:1:1:1/3)$. Let

$$V^3 := \big\{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_{\mathbb{R}} \bigm| x_0 \neq 0\big\} \cong \mathbb{R}^3,$$

and we choose a system of coordinates $(x,\ y,\ z)$ of V^3 as

$$\mathbf{x} = (x,y,z) = \left(\frac{x_1}{x_0} - 1,\ \frac{x_2}{x_0} - 1,\ \frac{x_3}{x_0} - \frac{1}{3}\right).$$

Note that the coordinate of P_3 is $(x,\ y,\ z) = (0,\ 0,\ 0)$. Let $D_3^+ \subset \mathbb{R}^3$ be the convex cone in V^3 generated by $X_3^+ \subset V^3 = \mathbb{R}^3$.

(iii) We shall show that $\mathfrak{C}_3^+$ can be identified with the dual convex cone of D_3^+, i.e.

$$(D_3^+)^\perp := \big\{\mathbf{f} \in \mathbb{R}^3 \bigm| (\mathbf{f}\cdot\mathbf{x}) \geq 0 \text{ for } \forall \mathbf{x} \in D_3^+.\big\}.$$

Any three variables cyclic cubic homogeneous polynomial can be written as

$$f(a,b,c) = p_0S_3 + p_1S_{2,1} + p_2S_{1,2} + p_3U, \quad (\exists p_0,\ldots,\ p_3 \in \mathbb{R}).$$

For this f, we denote

$$\begin{aligned}
&F_f(x_0,x_1,x_2,x_3) := p_0x_0 + p_1x_1 + p_2x_2 + p_3x_3,\\
&\mathbf{n}_f := (p_1,p_2,p_3) \in \mathbb{R}^3.
\end{aligned}$$

Assume that $f \in \mathfrak{C}_3^+$. Then $3p_0 + 3p_1 + 3p_2 + p_3 = 0$, and $p_0 = f(1,\ 0,\ 0) \geq 0$. Let $\mathbf{x} \in X_3^+ \subset D_3^+$, and assume that $\mathbf{x}$ corresponds to $(x_0 : x_1 : x_2 : x_3) = \varphi_3(a:b:c)$ $(\exists(a:b:c) \in \mathbb{P}^2_+)$. Since $f \in \mathfrak{C}_3^+$,

$$(\mathbf{n}_f \cdot \mathbf{x}) = \frac{F_f(x_0,x_1,x_2,x_3)}{x_0} = \frac{f(a,b,c)}{S_3(a,b,c)} \geq 0.$$

Since X_3^+ generates D_3^+, $(\mathbf{n}_f \cdot \mathbf{x}) \geq 0$ for $\forall \mathbf{x} \in D_3^+$. Thus $\mathbf{n}_f \in (D_3^+)^\perp$, and $\mathfrak{C}_3^+$ can be identified with $(D_3^+)^\perp$, corresponding f to $\mathbf{n}_f$.

(iv) We shall show that $\mathfrak{f}_s$ is located on the boundary of $\mathfrak{C}_3^+$.

Let $\mathbf{F}_s$ be the plane in $\mathbb{P}^3_{\mathbb{R}}$ which tangents to C_3 at $Q_s := \varphi(0:s:1) = (f_0(s) : f_1(s) : f_2(s) : f_3(s))$ and which passes through P_3. The defining equation of $\mathbf{F}_s$ is given by.

$$\begin{aligned} &-3\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ f_0(s) & f_1(s) & f_2(s) & f_3(s) \\ \frac{d}{ds}f_0(s) & \frac{d}{ds}f_1(s) & \frac{d}{ds}f_2(s) & \frac{d}{ds}f_3(s) \\ 1 & 1 & 1 & 1/3 \end{vmatrix} \\ &= s^2x_0 - (2s^2-1)x_1 + (s^4-2s)x_2 - 3(s^4-2s^3+s^2-2s+1)x_3. \end{aligned}$$

This corresponds to $\mathfrak{f}_s$. By (i), $\mathfrak{f}_s \in \mathfrak{C}_3^+ = (D_3^+)^\perp$. Thus $\mathfrak{f}_s$ lies on the boundary of $\mathfrak{C}_3^+$. This fact also implies that $\{Q_s \mid s \in [0,\infty]\} \subset C_3$ generates D_3^+, and

$$D_3^+ = \{\mathbf{x} \in V^3 \mid (\mathfrak{f}_s \cdot \mathbf{x}) \geq 0 \text{ for } \forall s \in [0,\infty]\}.$$

Let $B_3 := \mathbb{R}_+ \cdot \{\mathfrak{f}_s \mid s \in [0,\infty]\}$, and $\mathfrak{C}_3^{+b}$ be the boundary of $\mathfrak{C}_3^+$. Above observation implies that $B_3 \subset \mathfrak{C}_3^{+b}$.

(v) We shall determine $\mathfrak{C}_3^{+b} - B_3$, and shall prove (1).

Note that $Q_0 = Q_\infty$ is the node of C_3, and C_3 is smooth at Q_s if $s \in (0,\infty)$. The boundary of B_3 is $\mathbb{R}_+ \cdot \mathfrak{f}_0 \cup \mathbb{R}_+ \cdot \mathfrak{f}_\infty$. Let $B_3' := \mathbb{R}_+ \cdot \mathfrak{f}_0 + \mathbb{R}_+ \cdot \mathfrak{f}_\infty$. A point on B_3' corresponds to a surface which tangents X_3^+ at Q_0 and which passes through P_3. Thus $B_3' \subset \mathfrak{C}_3^{+b}$, and we conclude that $\mathfrak{C}_3^{+b} = B_3 \cup B_3'$. Therefore, we obtain (1).

(2) If f lies on the boundary of $\mathfrak{C}_3^+$, then (2) is trivial. Assume that f is an interior point of $\mathfrak{C}_3^+$. The half line from $\mathfrak{f}_\infty$ to f crosses $\mathbb{R}_+ \cdot B_3$ at a point $\beta'\mathfrak{f}_s$ ($\exists \beta' > 0$, $\exists s \in [0,\infty)$). Then, we can write f in the form $f = \alpha\mathfrak{f}_s + \beta\mathfrak{f}_\infty$.

(3) Eliminate s from $p = -\dfrac{2s^3-1}{s^2}$, $q = \dfrac{s^4-2s}{s^2}$, we obtain $27+4p^3+4q^2 = p^2q^2+18pq$. If observe the graph of this curve, we have the conclusion. Note that since the dual curve of a plane nodal cubic curve is a quartic curve, B_3 is generated by a part of a plane quartic curve.

(4) Let $\psi_3 \colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow Z^2 := \mathbb{P}^2_{\mathbb{R}}$ be the holomorphic map defined by

$$\psi_3(a:b:c) := (S_3(a,b,c) : T_3(a,b,c) : U(a,b,c)),$$

and let $\pi_3 \colon \mathbb{P}^3_{\mathbb{R}} \longrightarrow Z^2 = \mathbb{P}^2_{\mathbb{R}}$ be the rational map defined by

$$\pi_3(x_0 : x_1 : x_2 : x_3) := (x_0 : x_1 + x_2 : x_3).$$

Let $Y_3^+ := \psi_3(\mathbb{P}^2_+) = \pi_3(X_3^+) \subset Z^2$, and denote $y_1 := x_1 + x_2$, $y_2 := x_1 - x_2$,

$$\eta_3(x_0, y_1, x_3) := 4x_0^2x_3 + 12x_0x_3^2 + 36x_3^3 - 6x_3y_1^2 + y_1^3 - x_0y_1^2.$$

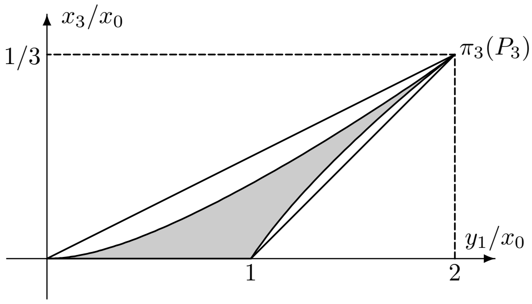
Then, (3.1) can be written as $\eta_3(x_0,y_1,x_3) + y_2^2(x_0 + 6x_3 + 3y_1) = 0$. Thus,

$$Y_3^+ = \left\{ (x_0 : y_1 : x_3) \in Z^2 \;\middle|\; \begin{array}{l} \eta_3(x_0,y_1,x_3) \leq 0, \\ x_0 \geq 0,\ y_1 \geq 0,\ x_3 \geq 0. \end{array} \right\}.$$

Note that $\eta_3(x_0,y_1,x_3) = 0$ defines the cubic curve which has the cusp at $(1 : 2 : 1/3)$, and which has a parameterization

$$\big((4-6m^2+3m^3) : (8-16m+12m^2-3m^3) : (-4+8m-5m^2+m^3)\big),$$

$(1 \le m \le 2)$.



As the above figure,

$$Y_3^+ \subset \left\{(x_0 : y_1 : x_3) \in Z^2 \;\middle|\; y_1 - 6x_3 \ge 0,\ x_0 + 3x_3 - y_1 \ge 0.\right\}.$$

Thus, the boundary of $\mathfrak{S}_3^+$ is $\mathbb{R}_+ \cdot (T_{2,1} - 6U) + \mathbb{R}_+ \cdot \big(S_3 + 3U - T_{2,1}\big)$. $\square$

4. Proof of Theorem 2.

Proof of Theorem 2. (1) We denote

$$\begin{aligned}
G_{p,q}(x_0,x_1,x_2,x_3,x_4) &:= x_0 - px_1 - qx_2 \\
&\quad + \left(\frac{p^2+pq+q^2}{3} - 1\right)x_3 + \left(p + q - \frac{p^2+pq+q^2}{3}\right)x_4. \\
G_\infty(x_0,x_1,x_2,x_3,x_4) &:= x_3 - x_4. \\
H_s(x_0,x_1,x_2,x_3,x_4) &:= x_1 + s^2x_2 - 2sx_3 - (s-1)^2x_4. \\
H_\infty(x_0,x_1,x_2,x_3,x_4) &:= x_2 - x_4, \\
K_{s,t}(x_0,x_1,x_2,x_3,x_4) &:= s^2x_0 - (2s^3 - st)x_1 + (s^3t - 2s)x_2 \\
&\quad + (s^4 - 2s^2t + 1)x_3 + \big(s^2 - (s-1)^2(s^2+st+1)\big)x_4.
\end{aligned}$$

(i) We shall show that $\mathfrak{g}_{p,q} \in \mathfrak{C}_4$ for $\forall p$, $\forall q \in \mathbb{R}$.
As Cîrtoaje([3]) had shown,

$$3\mathfrak{g}_{p,q}(a,b,c) = \sum_{\text{cyclic}} (2a^2 - b^2 - c^2 - pab + (p+q)bc - qca)^2 \ge 0.$$

(ii) We shall show that $\mathfrak{h}_s$, $\mathfrak{k}_{s,t} \in \mathfrak{C}_4^+$ for $s \ge 0$ and $t \ge 1$.
Since $S_{1,3} \ge US_1$ for a, b, $c \in \mathbb{R}_+$, we have

$$\begin{aligned}
\mathfrak{h}_s(a,b,c) &= s^2(S_{1,3} - US_1) - 2s(S_{2,2} - US_1) + (S_{3,1} - US_1) \\
&= (S_{1,3} - US_1)\left(s - \frac{S_{2,2} - US_1}{S_{1,3} - US_1}\right)^2 + \frac{US_1(S_2 - S_{1,1})^2}{S_{1,3} - US_1} \\
&\ge 0, \\
\mathfrak{k}_{s,t}(a,b,c) &= s^2\mathfrak{g}_{2s-1/s,2/s-s}(a,b,c) + s(t-1)\mathfrak{h}_s(a,b,c) \in \mathfrak{C}_4^+.
\end{aligned}$$

Note that $\mathfrak{p}(0,\,s) = 2s - 1/s$, $\mathfrak{q}(0,\,s) = 2/s - s$, and

$$\mathfrak{g}_{\mathfrak{p}(s,t),\mathfrak{q}(s,t)}(s,t,1) = 0, \quad \mathfrak{h}_s(0,s,1) = 0, \quad \mathfrak{k}_{s,t}(0,s,1) = 0.$$

(iii) We shall show that $\mathfrak{k}_{s,t} \notin \mathfrak{C}_4^+$ if $t < 1$, $s \ge 0$.

Since $\mathfrak{k}_{s,t}(b,\ a,\ 1) = s^4\mathfrak{k}_{1/s,t}(a,\ b,\ 1)$, we may assume that $0 < s \le 1$. Let $t > 0$ and $p > \max\{2,\ 12t/s\}$. Then,

$$\begin{aligned}
&\mathfrak{k}_{s,1-t}(st/p, s, 1) \\
&= -\frac{s^2t^2}{p^4}\Big(p^2(p-1)(1-s) + p^3s^2(1-s^2) + p^2s^5(1-s) \\
&\qquad + p^3s^5 + p(2p-1)s^3t + 2ps^5t + ps^3(1-s)t^2 + (2p-1)s^4t^2 \\
&\qquad + \big\{p^2(p-1)s^3 - p(3p-2)s^2t - 3p^2s^4t - ps^4t\big\}\Big).
\end{aligned}$$

Since $t < ps/12$ and $p/2 < p-1$, we have

$$\begin{aligned}
&p(3p-2)s^2t + 3p^2s^4t + ps^4t = ps^2t((3p-2) + (3p+1)s^2) \\
&< \frac{p^2s^3}{12}\big((3p-2) + (3p+1)\big) < \frac{p^3s^3}{2} < p^2(p-1)s^3.
\end{aligned}$$

Thus $\mathfrak{k}_{s,1-t}(st/p,\ s,\ 1) < 0$.

(iv) We shall observe X_4.

Let $\varphi_4 \colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{P}^4_{\mathbb{R}}$ be the holomorphic map defined by

$$\varphi_4(a : b : c) := (S_4 : S_{3,1} : S_{1,3} : S_{2,2} : US_1),$$

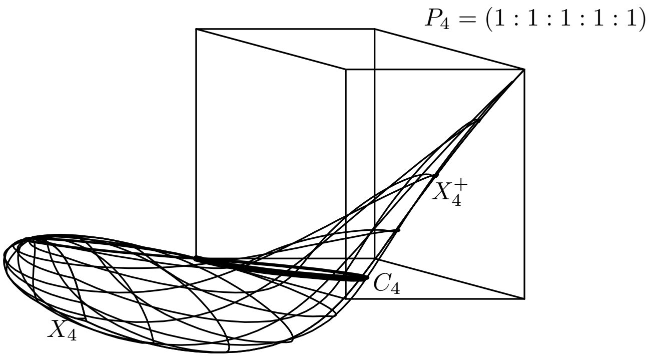
and let $X_4 := \varphi_4(\mathbb{P}^2_{\mathbb{R}})$. It is easy to see the following equalities hold.

$$\begin{aligned}
&(S_{3,1} + S_{1,3} + US_1)^2 - (S_4 + 2S_{2,2})(S_{2,2} + 2US_1) = 0 \\
&(S_{3,1} + S_{1,3} - 2US_1)^2 + 3(S_{3,1} - S_{1,3})^2 + (S_4 - 2S_{2,2} + US_1)^2 - (S_4 - US_1)^2 = 0
\end{aligned}$$

Thus, the defining equations of the quartic surface X_4 is

$$\begin{aligned}
&(x_1 + x_2 + x_4)^2 - (x_0 + 2x_3)(x_3 + 2x_4) = 0, \\
&(x_1 + x_2 - 2x_4)^2 + 3(x_1 - x_2)^2 + (x_0 - 2x_3 + x_4)^2 - (x_0 - x_4)^2 = 0.
\end{aligned} \tag{4.1}$$

We knows that X_4 has a rational double point of the type A_1 at $P_4 := (1 : 1 : 1 : 1 : 1)$, from the above equations.



Let $V^4 := \big\{(x_0 : x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^4_{\mathbb{R}} \ \big|\ x_0 \neq 0\big\} \cong \mathbb{R}^4$, and we choose a system of coordinates $(x,\ y,\ z,\ w)$ of V^4 as

$$\mathbf{x} = (x, y, z, w) = \left(\frac{x_1}{x_0} - 1,\ \frac{x_2}{x_0} - 1,\ \frac{x_3}{x_0} - 1,\ \frac{x_4}{x_0} - 1\right).$$

By (4.1), the defining equations of $X_4 \cap V^4$ are

$$\begin{aligned}
&(x + y + w + 3)^2 - (2z + 3)(z + 2w + 3) = 0, \\
&(x + y - 2w)^2 + 3(x - y)^2 + (2z - w)^2 - w^2 = 0.
\end{aligned} \tag{4.2}$$

Let $W^3 = \{(x,\, y,\, z) \mid x,\, y,\, z \in \mathbb{R}\}$ be the hyperplane defined by $w = -1$ in V^4, and let $\rho\colon \mathbb{P}^4_{\mathbb{R}} \longrightarrow W^3$ be the projection from the center P_4. By (4.2), $E := \rho\bigl(X_4 - \{P_4\}\bigr)$ is an ellipsoid

$$E = \bigl\{(x, y, z) \in W^3 \bigm| (x+y+2)^2 + 3(x-y)^2 + (2z+1)^2 = 1.\bigr\}.$$

Let D_4 be the convex cone generated by X_4 in V^4. The boundary of D_4 is the cone whose base is E. By the same argument as (iii) of the proof of Theorem 1, we conclude that $\mathfrak{C}_4$ can be identified with the dual convex cone of D_4.

(v) We shall determine the boundary of $\mathfrak{C}_4$, and shall prove (1).

Let

$$\begin{aligned} &g_0(s,t) := S_4(s,t,1), \quad g_1(s,t) := S_{3,1}(s,t,1), \quad g_2(s,t) := S_{1,3}(s,t,1), \\ &g_3(s,t) := S_{2,2}(s,t,1), \quad g_4(s,t) := U(s,t,1)S_1(s,t,1), \end{aligned}$$

and let $\mathbf{G}_{s,t}$ (resp. $\mathbf{G}_\infty$) be the hyperplane in $\mathbb{P}^4_{\mathbb{R}}$ which tangents to X_4 at the point $\varphi_4(s : t : 1) = (g_0(s,t) : \cdots : g_4(s,t))$ (resp. (1:0:0:0)) and which passes through P_4. Since

$$\begin{aligned} &\begin{vmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ g_0(s,t) & g_1(s,t) & g_2(s,t) & g_3(s,t) & g_4(s,t) \\ \frac{\partial}{\partial s} g_0(s,t) & \frac{\partial}{\partial s} g_1(s,t) & \frac{\partial}{\partial s} g_2(s,t) & \frac{\partial}{\partial s} g_3(s,t) & \frac{\partial}{\partial s} g_4(s,t) \\ \frac{\partial}{\partial t} g_0(s,t) & \frac{\partial}{\partial t} g_1(s,t) & \frac{\partial}{\partial t} g_2(s,t) & \frac{\partial}{\partial t} g_3(s,t) & \frac{\partial}{\partial t} g_4(s,t) \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \\ &= -S_1(s,t,1)\bigl(S_2(s,t,1) - S_{1,1}(s,t,1)\bigr)^2\bigl(S_{2,2}(s,t,1) - U(s,t,1)S_1(s,t,1)\bigr) \\ &\qquad \times G_{\mathfrak{p}(s,t),\mathfrak{q}(s,t)}(x_0, x_1, x_2, x_3, x_4), \end{aligned}$$

the defining equation of $\mathbf{G}_{s,t}$ is given by $G_{\mathfrak{p}(s,t),\mathfrak{q}(s,t)} = 0$. Note that the range of $\bigl(\mathfrak{p}(s,t), \mathfrak{q}(s,t)\bigr)$ is $\mathbb{R}^2$ ($(s,\, t) \in \mathbb{R}^2$). When $s^2 + t^2 \to \infty$, defining equation of $\mathbf{G}_{s,t}$ tends to $G_\infty = 0$. Thus $\mathfrak{g}_{p,q}$ and $\mathfrak{g}_\infty$ are on the boundary of $\mathfrak{C}_4$.

Let $\psi\colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow W^3$ be the rational map defined by

$$\psi(a : b : c) = \left(-\frac{S_4 - S_{3,1}}{S_4 - US_1},\ -\frac{S_4 - S_{1,3}}{S_4 - US_1},\ -\frac{S_4 - S_{2,2}}{S_4 - US_1}\right).$$

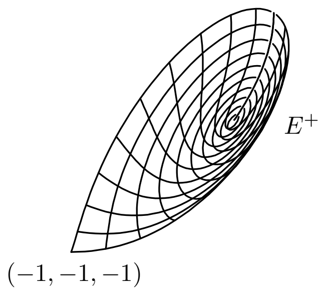
Take any point $Q \in E$. Since $\psi = \rho \circ \varphi_4$, we have $\psi(\mathbb{P}^2_{\mathbb{R}}) = E$. Thus, there exists $s,\, t \in \mathbb{R}$ such that $\psi(s : t : 1) = Q$, or $\psi(1 : 0 : 0) = Q$. Then $\mathbf{G}_{s,t}$ or $\mathbf{G}_\infty$ tangents to E at Q. Thus, we conclude that

$$B_4 := \mathbb{R}_+ \cdot \bigl(\{\mathfrak{g}_{p,q} \bigm| p,\, q \in \mathbb{R}\} \cup \{\mathfrak{g}_\infty\}\bigr)$$

is the boundary of $D_4^\perp = \mathfrak{C}_4$.

(2) can be obtained by the similar argument as the proof of (2) of Theorem 1.

(3) Let $X_4^+ := \varphi_4(\mathbb{P}^2_+)$, $E^+ := \psi(\mathbb{P}^2_+)$, and let D_4^+ be the convex cone generated by X_4^+. If we obtain the convex closure $\overline{E}^+$ of E^+, we can determine D_4^+ as the convex cone whose base is $\overline{E}^+$. Let $\mathfrak{C}_4^{+b}$ be the boundary of $\mathfrak{C}_4^+$.



(vi) We shall determine $B_4 \cap \mathfrak{C}_4^{+b}$.

Let

$$k_1(s) := \frac{s^3}{s^4+1} - 1, \quad k_2(s) := \frac{s}{s^4+1} - 1, \quad k_3(s) := \frac{s^2}{s^4+1} - 1,$$
$$\Gamma := \{(k_1(s), k_2(s), k_3(s)) \in W^3 \mid s \in \mathbb{R}_+\} = \{\psi(0 : s : 1) \in W^3 \mid s \in \mathbb{R}_+\}.$$

Γ is the boundary of E^+. Note that Γ has a node at $\psi(0:0:1) = (-1, -1, -1)$. Since

$$\begin{aligned}\{(\mathfrak{p}(s,0),\, \mathfrak{q}(s,0)) \mid s \in \mathbb{R}_+\} &= \{(\mathfrak{p}(0,s),\, \mathfrak{q}(0,s)) \mid s \in \mathbb{R}_+\} \\ &= \{(p,q) \in \mathbb{R}^2 \mid 9(p+q)^2 - (p-q)^2 = 6^2 \text{ and } p+q \geq 0\},\end{aligned}$$

we know that the plane defined by $G_{p,q}(0, x, y, z, -1) = 0$ tangents to E at a point on Γ if and only if $9(p+q)^2 - (p-q)^2 = 6^2$ and $p + q \geq 0$. By the above observation, we know that $B_4 \cap \mathfrak{C}_4^{+b}$ is the following B_4^+:

$$\begin{aligned}B_4^+ &:= \mathbb{R}_+ \cdot \left(\{\mathfrak{g}_{p,q} \mid 9(p+q)^2 - (p-q)^2 \geq 6^2,\, p+q \geq 0\} \cup \{\mathfrak{g}_\infty\}\right) \\ &= \mathbb{R}_+ \cdot \left(\mathbb{R}_+ \cdot \{\mathfrak{g}_{\mathfrak{p}(s,t),\, \mathfrak{q}(s,t)} \mid s,\, t \in \mathbb{R}_+\} \cup \{\mathfrak{g}_\infty\}\right).\end{aligned}$$

(vii) We shall determine one of the another parts of $\mathfrak{C}_4^{+b}$.

Let $\mathbf{K}_s$ ($s \in [0, \infty]$)be the plane in W^3 which tangents to Γ at $\psi(0 : s : 1)$, and which passes through $(-1, -1, -1)$. The equation of $\mathbf{K}_s$ is given by

$$\begin{aligned}&\begin{vmatrix} x & y & z & 1 \\ k_1(s) & k_2(s) & k_3(s) & 1 \\ \frac{d}{ds}k_1(s) & \frac{d}{ds}k_2(s) & \frac{d}{ds}k_3(s) & 0 \\ -1 & -1 & -1 & 1 \end{vmatrix} \\ &= \frac{s^2}{(s^4+1)^2} H_s(0, x, y, z, -1).\end{aligned}$$

Thus, we know that $\mathfrak{h}_s$ lies on the boundary of $\mathfrak{C}_4^+$, by (ii). Let

$$\ell_s := \{(1-\tau)(-1,-1,-1) + \tau \cdot \psi(0 : s : 1) \mid 0 \leq \tau \leq 1\}$$

be the line segments connecting $(-1, -1, -1)$ and $\psi(0 : s : 1)$, and let $E_2 := \bigcup_{s \geq 0} \ell_s$. The plane defined by $H_s = 0$ tangents $\overline{E}^+$ at the line segment ℓ_s on E_2. Thus, the boundary of $\overline{E}^+$ is $E^+ \cup E_2$.

The plane defined by $G_{\mathfrak{p}(s,t),\, \mathfrak{q}(s,t)} = 0$ tangents $\overline{E}^+$ at the point $\psi(s : t : 1)$ on E^+. Since

$$K_{s,t} = s^2 G_{2s-1/s, 2/s-s} + s(t-1)H_s = s^2 G_{\mathfrak{p}(0,s), \mathfrak{q}(0,s)} + s(t-1)H_s,$$

the plane defined by $K_{s,t} = 0$ tangents $\overline{E}^+$ at the point $\psi(0 : s : 1)$ on Γ ($s \geq 0$, $t \geq 1$). Thus, $\mathfrak{k}_{s,t}$ ($s \geq 0$, $t \geq 1$) lies on the boundary of $\mathfrak{C}_4^+$, and

$$B_4' := \mathbb{R}_+ \cdot \{\mathfrak{k}_{s,t} \mid s \geq 0,\, t \geq 1\}$$

is a part of $\mathfrak{C}_4^{+b}$.

(viii) We shall determine $\mathfrak{C}_4^{+b} - (B_4^+ \cup B_4')$, and shall prove (3).

Let B_4^{+b}, $B_4'^b$ be the boundaries of B_4^+, B_4'. Note that we can identify $\mathfrak{k}_{0,t} = \mathfrak{k}_{0,1}$ with $\mathfrak{k}_{\infty,t}$. As is shown in the above,

$$\begin{aligned}B_4^{+b} &= \mathbb{R}_+ \cdot \{\mathfrak{g}_{\mathfrak{p}(0,s), \mathfrak{q}(0,s)} \mid s \geq 0\} = \mathbb{R}_+ \cdot \{\mathfrak{k}_{s,1} \mid s \geq 0\}, \\ B_4'^b &= \mathbb{R}_+ \cdot \{\mathfrak{h}_s \mid s \in [0,\infty]\} \cup \mathbb{R}_+ \cdot \{\mathfrak{k}_{s,1} \mid s \geq 0\} \\ &\quad \cup (\mathbb{R}_+ \cdot \mathfrak{h}_0 + \mathbb{R}_+ \cdot \mathfrak{k}_{0,1}) \cup (\mathbb{R}_+ \cdot \mathfrak{h}_\infty + \mathbb{R}_+ \cdot \mathfrak{k}_{0,1}).\end{aligned}$$

An element of $B_4'' := \mathbb{R}_+ \cdot \mathfrak{k}_{0,1} + \mathbb{R}_+ \cdot \{\mathfrak{h}_s \mid s \in [0, \infty]\}$ corresponds to a plane which tangents to $\overline{E}^+$ at the point $(-1, -1, -1)$. Thus $B_4'' \subset \mathfrak{C}_4^{+b}$. Therefore, $\mathfrak{C}_4^{+b} = B_4^+ \cup B_4' \cup B_4''$, and we complete the proof of (3).

(4) For $s \geq 0$, let $M_s := \mathbb{R}_+ \cdot \mathfrak{h}_s + \mathbb{R}_+ \cdot \{\mathfrak{g}_{\mathfrak{p}(t,s),\mathfrak{q}(t,s)} \mid t \geq 1\}$, and $M_\infty := \mathbb{R}_+ \cdot \mathfrak{h}_\infty + \mathbb{R}_+ \cdot \mathfrak{g}_\infty$. By the above observation, we conclude that $\bigcup_{s \in [0,\infty]} M_s = \mathfrak{C}_4^+$. Thus, we have (4). □

Remark 4. The polynomials H_s and $K_{s,t}$ appear in the defining equation of the hyperplane which tangents to the boundary of X_4^+.

Let

$$\begin{aligned}
&l_0(s) := s^4 + 1, \quad l_1(s) := s^3, \quad l_2(s) := s, \quad l_3(s) := s^2, \quad l_4(s) := 0, \\
&C_4 := \left\{ (l_0(s) : l_1(s) : l_2(s) : l_4(s)) \in \mathbb{P}^4_{\mathbb{R}} \;\middle|\; s \in \mathbb{R}_+ \right\} \\
&\quad\;\; = \left\{ \varphi_4(0 : s : 1) \in \mathbb{P}^4_{\mathbb{R}} \;\middle|\; s \in \mathbb{R}_+ \right\}.
\end{aligned}$$

C_4 is the boundary of X_4^+. Let $\mathbf{L}_s \subset \mathbb{P}^4_{\mathbb{R}}$ be a hyperplane which tangents to C_4 at $\varphi_4(0 : s : 1)$ $(s \geq 0)$ and which passes through $P_4 = (1 : 1 : 1 : 1 : 1)$. But these conditions do not determine $\mathbf{L}_s$ uniquely. Moreover we assume that $\mathbf{L}_s$ passes through a point $(t_0 : t_1 : t_2 : t_3 : t_4)$. Then the defining equation of $\mathbf{L}_s$ is

$$\begin{aligned}
&\begin{vmatrix}
x_0 & x_1 & x_2 & x_3 & x_4 \\
l_0(s) & l_1(s) & l_2(s) & l_3(s) & l_4(s) \\
\frac{d}{ds} l_0(s) & \frac{d}{ds} l_1(s) & \frac{d}{ds} l_2(s) & \frac{d}{ds} l_3(s) & \frac{d}{ds} l_4(s) \\
1 & 1 & 1 & 1 & 1 \\
t_0 & t_1 & t_2 & t_3 & t_4
\end{vmatrix} \\
&= -s^2(t_1 + s^2 t_2 - 2st_3 - (1-s)^2 t_4) G_{\mathfrak{p}(0,s),\mathfrak{q}(0,s)} \\
&\quad + \left\{ s^2 t_0 + (s - 2s^3) t_1 + (-2s + s^3) t_2 \right. \\
&\qquad\qquad \left. + (1 - s^2)^2 t_3 - (1 - s - s^2 - s^3 + s^4) t_4 \right\} H_s.
\end{aligned}$$

Thus the defining equation of $\mathbf{L}_s$ can be written as $K_{s,t}(x_0, x_1, x_2, x_3, x_4) = 0$, if we take a suitable t.

Remark 5. When we eliminate s and t from

$$p = -\frac{2s^3 - st}{s^2}, \quad q = \frac{s^3 t - 2s}{s^2}, \quad r = \frac{s^4 - 2s^2 t + 1}{s^2},$$

we obtain

$$\begin{aligned}
p^2q^2r^2 &- 4p^3q^3 + 18p^3qr + 18pq^3r - 4p^2r^3 - 4q^2r^3 \\
&- 27p^4 - 27q^4 + 16r^4 - 6p^2q^2 - 80pqr^2 \\
&+ 144p^2r + 144q^2r - 192pq - 128r^2 + 256 = 0.
\end{aligned}$$

But the singularity of the surface defined by the above equation is so complicated to state the similar proposition like (3) of Theorem 1.

Proof of Corollary 3. We use the same notation as the above proof.

(1) Let $\psi_4 \colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow Z^3 := \mathbb{P}^3_{\mathbb{R}}$ be the holomorphic map defined by

$$\psi_4(a : b : c) := (S_4 : T_{3,1} : S_{2,2} : US_1),$$

and let $\pi_4\colon \mathbb{P}^4_{\mathbb{R}} \longrightarrow Z^3 = \mathbb{P}^3_{\mathbb{R}}$ be the rational map defined by

$$\pi_4(x_0 : x_1 : x_2 : x_3 : x_4) := (x_0 : x_1 + x_2 : x_3 : x_4).$$

Put $Y_4 := \psi_4(\mathbb{P}^2_{\mathbb{R}}) = \pi_4(X_4)$. We choose a system of coordinates of $\pi_4(V^4) \cong \mathbb{R}^3$ as

$$(u, z, w) = \left(\frac{x_1 + x_2}{x_0} - 2,\ \frac{x_3}{x_0} - 1,\ \frac{x_4}{x_0} - 1 \right).$$

Let $W^2 := \pi_4(W^3) \cong \mathbb{R}^2$, and we choose a system of coordinates of W^2 as $(u,\ z)$. Note that

$$\pi_4(E) = \left\{ (u, z) \in W^2 \;\middle|\; (u + 2)^2 + (2z + 1)^2 \leq 1. \right\}$$

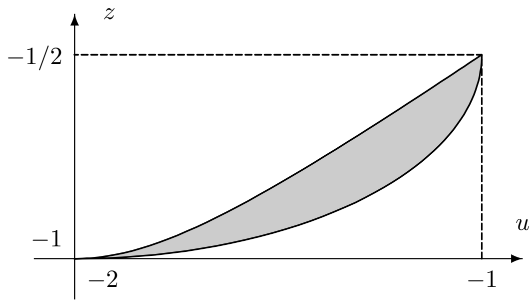
is an ellipse. $\pi_4(D_4)$ is the convex cone in $\pi_4(V^4)$ whose base is $\pi_4(E)$ and whose vertex is $(0,\ 0,\ 0)$. Since the line defined by $-pu + (p^2 - 1)z = 2p - p^2$ (resp. $z = -1$) tangents to the ellipse $(u + 2)^2 + (2z + 1)^2 = 1$ at $\left(\frac{2p}{1 + p^2} - 2,\ -\frac{p^2}{1 + p^2} \right)$ (resp. at $(-2,\ -1)$), we conclude that $\mathfrak{g}_p$ (resp. $\mathfrak{g}_\infty$) lies on the boundary of $\mathbf{S}_4 = (\pi_4(D_4))^\perp$. It is easy to see that these surround $\mathbf{S}_4$. Thus $\mathbb{R}_+ \cdot \left\{ \mathfrak{g}_p \;\middle|\; p \in \mathbb{R} \cup \{\infty\} \right\}$ is the boundary of $\mathbf{S}_4$, and we have (1).

(2) Let $\psi_2\colon \mathbb{P}^2_{\mathbb{R}} \longrightarrow W^2$ be the rational map defined by

$$\psi_2(a : b : c) = \left(-\frac{2S_4 - T_{3,1}}{S_4 - US_1},\ -\frac{S_4 - S_{2,2}}{S_4 - US_1} \right),$$

and let $C_2 := \left\{ \psi_2(0 : s : 1) \;\middle|\; s \in [0,\ \infty] \right\}$. Since $\psi_2 = \pi_4 \circ \psi$, $\pi_4(E^+) = \psi_2(\mathbb{P}^2_+)$. Since the defining equation of C_2 is $2(z + 5/4)^2 - (u + 2)^2 = 1/8$, we have that

$$\pi_4(E^+) = \left\{ (u, z) \in W^2 \;\middle|\; \begin{array}{l} (u + 2)^2 + (2z + 1)^2 \leq 1, \\ 2(z + 5/4)^2 - (u + 2)^2 \leq 1/8 \end{array} \right\}.$$



Thus, the convex closure of $\pi_4(E^+)$ is

$$\left\{ (u, z) \in W^2 \;\middle|\; (u + 2)^2 + (2z + 1)^2 \leq 1 \text{ and } u - 2z \geq 0 \right\}.$$

Let D'_4 be the convex cone generated by $\pi_4(E^+)$. By the above observation, we conclude that the boundary of the dual convex cove $(D'_4)^\perp \cong \mathbf{S}^+_4$ is the union of a surface

$$B'_1 := \mathbb{R}_+ \cdot \left\{ (-p, p^2 - 1, 2p - p^2) \;\middle|\; p \in [1,\ \infty] \right\} = \mathbb{R}_+ \cdot \left\{ \mathfrak{g}_p \;\middle|\; p \in [1,\ \infty] \right\},$$

and two faces

$$B'_2 := \mathbb{R}_+ \cdot (-1, 0, 1) + \mathbb{R}_+ \cdot (1, -2, 0) = \mathbb{R}_+ \cdot \mathfrak{g}_1 + \mathbb{R}_+ \cdot (T_{3,1} - 2S_{2,2}),$$
$$B''_2 := \mathbb{R}_+ \cdot (0, 1, -1) + \mathbb{R}_+ \cdot (1, -2, 0) = \mathbb{R}_+ \cdot \mathfrak{g}_\infty + \mathbb{R}_+ \cdot (T_{3,1} - 2S_{2,2}).$$

Thus we have (2). □

References

[1] T. Ando, Some homogeneous cyclic inequalities of three variables of degree three and four, Aust. J. Math. Anal. Appl. 7 (2) Art. 12 (2010)

[2] V. Cîrtoaje, Algebraic Inequalities — Old and New Methods, GIL Publishing House (Romania), (2006)

[3] V. Cirtoaje, On the Cyclic Homogeneous Polynomial Inequalities of Degree Four, J. Inequal. Pure and Appl. Math., 10(3) Art. 67 (2009)

[4] V. Cirtoaje, Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities, Preprint.

[5] R. Muirhead, Some methods applicable to identities and inequalities of symmetric algebraic functions of n letters, Proc. Edinb. Math. Soc., 21 (1902) 144-157.