Some Examples of Simple Small Singularities

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Abstract

We provide some examples of simple small singularities of higher dimensional algebraic varieties. One of them is an E_6 type singularity $w^2-z^3+xzw+xy^3-3x^2yz-x^5-x^4y=0$ in $\mathbb{C}^4$. We also treat small contractions of curves with heigher genera whose normal bundles are not negative.

1. Introduction

Let X be a smooth algebraic variety with $\dim X \geqslant 3$, and C be a smooth complete curve in X. If $\varphi\colon X \to Y$ is a morphism such that $P=\varphi(C)$ is a point, and that $\varphi|_{X-C}\colon X-C \cong Y-P$, then φ is called a *small contraction* of C. Such $(Y,\, P)$ is called a *simple small singularity.* φ is also called a *small resolution* of $(Y,\, P)$. In general, an isolated singularity is called *small* if it has a resolution whose exceptional set is one-dimensional. A small singularity is called *simple* if the exceptional set is an irreducible curve.

For an example, the hypersurface

$$Y_1 = \{(z_1,\, z_2,\, z_3,\, z_4) \in \mathbb{C}^4 \;\mid z_1z_2 - z_3z_4 = 0\}$$

has a simple small singularity at the origin $\mathbf{0}$. Y_1 has a small resolution $\varphi\colon X \to Y_1$ such that $C=\varphi^{-1}(\mathbf{0}) \cong \mathbb{P}^1$ with the normal bundle $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

$$Y_{2,n} = \{(z_1,\, z_2,\, z_3,\, z_4) \in \mathbb{C}^4 \;\mid z_1^2 - z_2z_3 - z_2z_4^n = 0\}$$

$(n \geqslant 2)$ has also a simple small singularity at $\mathbf{0}$. $Y_{2,n}$ has a small resolution $\varphi\colon X \to Y_{2,n}$ such that $C=\varphi^{-1}(\mathbf{0}) \cong \mathbb{P}^1$ with $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$.

Laufer([L]) has found a simple small singularity

$$Y_{3,n} = \{(z_1,\, z_2,\, z_3,\, z_4) \in \mathbb{C}^4 \;\mid z_4^2 + z_2^3 - z_1z_3^2 - z_1^{2n+1}z_2 = 0\}.$$

$Y_{3,n}$ has a small resolution $\varphi\colon X \to Y_{3,n}$ such that $C=\varphi^{-1}(\mathbf{0}) \cong \mathbb{P}^1$ with $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

In this paper, we present some more examples of simple small singularities. One of them is

$$Y_4 \;:\; z_4^2 - z_3^3 + z_1z_3z_4 + z_1z_2^3 - 3z_1^2z_2z_3 - z_1^5 - z_1^4z_2 = 0.$$

We also treat the case C has a higher genus in §3.

2. How to construct simple small singularities

We recall that how Y_1, $Y_{2,n}$, $Y_{3,n}$ are constructed by the method of Laufer[L].

Let $U_1 = \mathbb{C}^3$ with the system of coordinate $(s,\, x_1,\, x_2)$, and $U_2 = \mathbb{C}^3$ with $(t,\, y_1,\, y_2)$. We patch U_1 and U_2 and construct $X = U_1 \cup U_2$ by the following transition function.

$$y_1 = sx_1, \quad y_2 = sx_2, \quad t = s^{-1} \tag{2.1}$$

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Let $C \subset X$ be the curve defined by $x_1 = x_2 = 0$ in U_1 and $y_1 = y_2 = 0$ in U_2. The following four functions $z_1, \ldots, z_4$ are holomorphic on X.

$$z_1 = y_1 = sx_1, \quad z_2 = ty_2 = x_2, \quad z_3 = y_2 = sx_2, \quad z_4 = ty_1 = x_1$$

These induce a holomorphic map $\varphi = (z_1, z_2, z_3, z_4)\colon X \to \mathbb{C}^4$. It is easy to see that φ is a small contraction of $C \cong \mathbb{P}^1$, and the image $\varphi(X)$ is the hypersurface defined by $z_1 z_2 = z_3 z_4$. Note that φ is the blowing up of $Y_1 \subset \mathbb{C}^4$ with the center $z_2 = z_4 = 0$.

$Y_{2,n}$ $(n \geqslant 2)$ can be obtained using another transition function instead of (2.1).

$$y_1 = s^2 x_1 + s x_2^n, \quad y_2 = x_2, \quad t = s^{-1} \tag{2.2}$$

In this case we find four holomorphic functions

$$\begin{cases} z_1 = ty_1 = sx_1 + x_2^n \\ z_2 = y_1 = s^2 x_1 + s x_2^n \\ z_3 = t^2 y_1 - y_2^n = x_1 \\ z_4 = y_2 = x_2 \end{cases}$$

Then we obtain $Y_{2,n} : z_1^2 - z_2 z_3 - z_2 z_4^n = 0$.

This singularity is studied in [R] §5. The defining ideal of C in X is $I_C = (x_1, x_2) = (y_1, y_2)$. I_C/I_C^2 decompose as $(x_1) \oplus (x_2) = (y_1) \oplus (y_2)$, and transformed as $y_1 = s^2 x_1$, $y_2 = x_2$. Thus I_C/I_C^2 has the bidegree (2, 0), and $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$. Y_1 and $Y_{2,n}$ are not isomorphic since whose normal bundles are not isomorphic.

It will be interesting to consider the sequence of normal bundles ([R], [P]). Let $\mu_1\colon X_1 \to X$ be the blowing up along C and $E_1 = \mu_1^{-1}(C)$. If $E_1 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$, choose C_1 as the minimal section of the Hirzeburch surface E_1. Again let $\mu_2\colon X_2 \to X_1$ be the blowing up along C_1, $E_2 = \mu_2^{-1}(C_1)$, and E_1' be the strict transform of E_1. If $E_2 \not\cong \mathbb{P}^1 \times \mathbb{P}^1$, choose C_2 as the minimal section of E_2. If $E_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $C_2 \subset E_1'$, we stop here. Otherwise we continue this process. The sequence of the bidegrees of $N_{C/X}$, N_{C_1/X_1}, $\ldots$ is called *the sequence of normal bundles*.

Theorem 2.1. ([P], [A4]) *If (Y, P) is a simple small singularity with $C \cong \mathbb{P}^1$ and $(K_X \cdot C)_X = 0$, then the sequence of normal bundles is one of the followings:*

(i) $(-1, -1)$.
(ii) $(-2, 0), \ldots, (-2, 0), (-1, -1)$.
(iii) $(-3, 1), (-2, -1), (-1, -1)$.
(iv) $(-3, 1), (-3, 0), (-2, -1), (-1, -1)$.
(v) $(-3, 1), (-3, 0), (-3, 0), (-2, -1), (-1, -1)$.

Let (a_0, b_0), $(a_1, b_1), \ldots$ be the sequence of normal bundles of $Y_{2,n}$. Then $(a_0, b_0) = (-2, 0)$. We can choose the unique ideal J such that $I_C \supset J \supset I_C^2$ and that $I_C/J = \mathcal{O}_C(-b_0)$. Then $J_C/I_C J \cong I_1/I_1^2 \otimes I_C/J$ via $\mu_1\colon C_1 \to C$, here C_1 is the defining ideal of C_1 in X_1. It is easy to see that $J = (y_1, y_2^2) = (x_1, x_2^2)$. This observation implies

$$I_k/I_k^2 \cong (x_1) \oplus (x_2^{k+1}) = (y_1) \oplus (y_2^{k+1}) \cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \quad (k \leqslant n-2)$$
$$I_{n-1}/I_{n-1}^2 \cong (x_1) \oplus (x_2^2 + sx_1) = (y_2^n - ty_1) \oplus (y_1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$

Thus we have $(a_0, b_0) = \cdots = (a_{n-2}, b_{n-2}) = (-2, 0)$ and $(a_{n-1}, b_{n-1}) = (-1, -1)$. Therefore, if $m \neq n$, then $Y_{2,m}$ and $Y_{2,n}$ are not isomorphic. Note that this consideration is useful to find transition functions.

Laufer ([L]) has found a transition function

$$y_1 = s^3x_1 + x_2^2 + s^2x_2^{2n+1}, \quad y_2 = s^{-1}x_2, \quad t = s^{-1}$$

($n \geqslant 1$). He fond four holomorphic functions

$$z_1 = y_1, \quad z_2 = t^2y_1 - y_2^2 = sx_1 + x_2^{2n+1}, \quad z_3 = tz_2 - z_1^ny_2, \quad z_4 = y_2z_2 - sz_1^{n+1},$$

and he obtained $Y_{3,n} : z_4^2 + z_2^3 - z_1z_3^2 - z_1^{2n+1}z_2 = 0$ with $N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Since $J/I_CJ \cong I_1/I_1^2 \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$ ($J = (x_1) + I_C^2 = (y_1) + I_C^2$), and J/I_CJ splits as

$$J/I_CJ = (x_1) \oplus (x_2^2 + s^3x_1) = (y_2^2 - t^2y_1) \oplus (y_1) \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1},$$

we have $(a_1, b_1) = (-2, -1)$. Thus, $Y_{3,n}$ is of type (iii) of Theorem 2.1. This singularity with its family

$$z_4^2 + z_2^3 - z_1z_3^2 - z_1^3z_2 + \lambda(z_1z_2^2 - z_1^4) = 0$$

is studied in [P] Example 10.

We consider a transition function

$$y_1 = s^3x_1 + s^2x_2^2 + s^{-1}x_2^3 + s^{-3}x_2^4, \quad y_2 = s^{-1}x_2, \quad t = s^{-1}.$$

The inverse map is given by

$$x_1 = t^3y_1 - t^{-1}y_2^2 - ty_2^3 - t^2y_2^4, \quad x_2 = t^{-1}y_2, \quad s = t^{-1}.$$

It is not hard to check that the following four functions are holomorphic on X.

$$\begin{cases} z_1 = sx_1 + x_2^2 \\ z_2 = -s^2z_1 + x_1x_2 + x_2z_1 \\ z_3 = x_1^2 + sx_2z_1 + x_1z_1 \\ z_4 = sz_1z_2 + x_1z_3 + x_2z_1^2 \end{cases}$$

Mathematica will show that

$$Y_4 : z_4^2 - z_3^3 + z_1z_3z_4 + z_1z_2^3 - 3z_1^2z_2z_3 - z_1^5 - z_1^4z_2 = 0.$$

Since

$$\begin{cases} t = \dfrac{z_3^3 + 3z_1^2z_2z_3 - z_1z_2^3 + z_1^5}{z_1z_3z_4 + z_1^4z_2 + z_1^2z_3^2 - z_2^2z_4}, \\ y_1 = \dfrac{z_3^2z_4 + z_1^2z_2z_4 - z_1^3z_2z_3 - z_1^6}{z_1z_3z_4 + z_1^4z_2 + z_1^2z_3^2 - z_2^2z_4} \\ y_2 = \dfrac{z_2z_3z_4 + z_1^3z_4 + z_1^4z_3 - z_1^3z_2^2}{z_1z_3z_4 + z_1^4z_2 + z_1^2z_3^2 - z_2^2z_4} \end{cases}$$

we know that $\varphi\colon X \to Y_5$ is a small contraction.

Since J/I_CJ splits as

$$J/I_CJ = (x_1) \oplus (x_2^2 + sx_1) = (y_2^2 - t^4y_1) \oplus (y_1) \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

we have $(a_1, b_1) = (-3, 0)$. We choose an ideal $J \supset L \supset I_CJ$ such that $J/L \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. That is $L = (x_2^2 + sx_1,\, x_1x_2,\, x_1^2) = (y_1,\, y_2^3)$. Then, there exists an injection $I_2/I_2^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-2) \longrightarrow L/(I_CL + J^2)$. Since $y_1 = s^2x_2^2 + s^3x_1 + s^{-1}x_2^3 = s^2x_2^2 + s^3x_1 - x_1x_2$ in $I_CL + J^2$, we have

$$\begin{aligned} L/(I_CL + J^2) &= (-x_1x_2 + s^2x_2^2 + s^3x_1) \oplus (x_2^2 + sx_1) \\ &= (y_1) \oplus (-y_2^3 + t^2y_1) \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}. \end{aligned}$$

Thus $I_2/I_2^2 \cong \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2)$, and, Y_4 is of type (iv) of Theorem 2.1.

The following transition function also provides a simple small singularity.

$$y_1 = s^3 x_1 + s^2 x_2^2 + s^{-1} x_2^3 + s^{-5} x_2^6, \quad y_2 = s^{-1} x_2, \quad t = s^{-1}$$
$$x_1 = t^3 y_1 - t^{-1} y_2^2 - t y_2^3 - t^2 y_2^6, \quad x_2 = t^{-1} y_2, \quad s = t^{-1}$$

Four holomorphic functions are

$$\begin{cases} z_1 = x_2^2 + s x_1 = -y_2^3 - t y_2^6 + t^2 y_1, \\ z_2 = x_1 x_2 - s x_2^4 - s^2 (x_2^2 + 2 x_1 x_2^2) - s^3 (x_1 + x_1^2) \\ \quad = (-y_1 - y_2^4 - 2 y_2^9) + t(2 y_1 y_2^3 - y_2^7 - y_2^{12}) + t^2 (y_1^2 y_2 + 2 y_1 y_2^6) - t^3 y_1^2, \\ z_3 = x_1^2 + s x_2 z_1 + x_2 z_1^2, \\ z_4 = s z_1 z_2 + x_1 z_3 + x_2 z_1^2 \end{cases}$$

This present a simple small singularity

$$Y_5 : z_4^2 - z_3^3 + z_1 z_2^3 - 3 z_1^2 z_2 z_3 + z_1^2 z_2 z_4 - z_1^5 + z_1^5 z_3 = 0.$$

But this is isomorphic to Y_4.

Note that general hyperplane sections of $(Y_i, \mathbf{0})$ passing through the origin are the rational double points. Those of Y_1, $Y_{2,n}$, $Y_{3,n}$, Y_4 are of type A_1, A_1, D_4, E_6 respectively. It is known there exist also those of the type E_7 (length 4) and E_8 (length 5, 6) (see [K], [KM], [R], [M]). But I don't succeed to construct such equations in the above method.

3. Exceptional curves of higher genus

Now we study a small contraction $\varphi: X \to Y$ whose exceptional set is a smooth curve C of the genus $g(C) \geqslant 1$ and the normal bundle of C is not negative. To begin with, we construct an easy example to understand our method.

Let C be a smooth curve of genus g, D be a base point free effective divisor on C. Take two effective divisors D_1 and D_2 such that $D_1 \sim D_2 \sim D$ and $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \phi$. Put $C_i = C - \operatorname{Supp} D_i$ and $U_i = C_i \times \mathbb{C}^2$. We represent a point in U_1 by (x_C, x_1, x_2) where $x_C \in C_1$, and $x_1, x_2 \in \mathbb{C}$. Similarly, $(y_C, y_1, y_2) \in U_2$.

There exists a rational function s on C such that $D_1 = D_2 + \operatorname{div}(s)$. D_1 and D_2 are zeros and poles of s respectively. We extend the rational function s on C to X by

$$s(x_C, x_1, x_2) = s(x_C), \quad s(y_C, y_1, y_2) = s(y_C).$$

Note that s is holomorphic on U_1 and $t = 1/s$ is holomorphic on U_2.

Let $y_C = \tau_C(x_C)$ be the transition function on $C_1 \cap C_2 \subset C$ which patches C_1 and C_2. We patch U_1 and U_2 and construct $X = U_1 \cup U_2$ by the following transition function.

$$y_1 = s^4 x_1 + s x_2^2, \quad y_2 = s^{-1} x_2, \quad y_C = \tau_C(x_C)$$

We identify C with the zero section of X defined by $x_1 = x_2 = 0$ in U_1 and by $y_1 = y_2 = 0$ in U_2. Since $I_C/I_C^2 \cong \mathcal{O}_C(4D) \oplus \mathcal{O}_C(-D)$. We have $N_{C/X} = \mathcal{O}_C(-4D) \oplus \mathcal{O}_C(D)$.

We find the following holomorphic functions on X.

$$\begin{cases} z_1 = s^4x_1 + sx_2^2 = y_1 \\ z_2 = s^3x_1 + x_2^2 = ty_1 \\ z_3 = sx_1 = t^3y_1 - y_2^2 \\ z_4 = x_1 = t^4y_1 - ty_2^2 \\ z_5 = x_1x_2 = t^3y_1y_2 - y_2^3 \\ z_6 = s^3x_1x_2 + x_2^3 = y_1y_2 \end{cases}$$

Let $h\colon X \to \mathbb{C}^6$ be the holomorphic map defined by $(z_1,\ldots, z_6)$, and let $Z = h(X)$. It is easy to see that $Q = h(C)$ is a point, and that $h\colon (X - C) \longrightarrow (Z - Q)$ is a finite map. Let $X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$ be the Stein factorization of h. Then $\varphi\colon X \to Y$ is a small contraction of C.

Recall that a small singularity is Cohen-Macaulay if and only if $\mathbf{R}^1\varphi_*\mathcal{O}_X = 0$. Since there exists the natural surjection $\mathbf{R}^1\varphi_*\mathcal{O}_X \to H^1(C, I_C/I_C^2)$, (Y, P) can not be Cohen-Macauley if $g(C) \geqslant 1$. Thus Y is never complete intersection.

Now, remember the following theorem.

Theorem 3.1. ([A5]) *Let C be a smooth exceptional curve of genus g in a smooth variety X with $\dim X \geqslant 3$, and let M be a subbundle of $N_{C/X}$ of the maximal degree. Put $b = \deg M \geqslant 0$ and $a = \deg N_{C/X} - \deg M < 0$. (i.e. a is the degree of the negative part of $N_{C/X}$, and b is the degree of the positive part of $N_{C/X}$.) Then*

$$a + 2b < 0$$

We present two examples as theorems. These examples encourage the above theorem.

Theorem 3.2. *We use the same notation as above. Define the transition function by*

$$\begin{cases} y_1 = s^{2m+1}x_1 + x_2^2 + s^{2m}x_2^3 \\ y_2 = s^{-m}x_2 \\ y_C = \tau_C(y_C) \end{cases}$$

Then C admits a small contraction, and

$$N_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-(2m+1)D) \oplus \mathcal{O}_{\mathbb{P}^1}(mD).$$

Proof. It is easy to check the following z_1 and z_2 are holomorphic on X.

$$z_1 = y_1 = s^{2m+1}x_1 + x_2^2 + s^{2m}x_2^3$$
$$z_2 = s^{-2m}y_1 - y_2^2 = sx_1 + x_2^3.$$

Since

$$u = z_2^2 - z_1^3 = 2sx_2^3x_1 + s^2x_1^2 + s^{2m}u_0(x_C, x_1, x_2)$$

can be divided by s, the following z_3 and z_4 are holomorphic on X.

$$z_3 = y_2u^m$$
$$z_4 = s^{-1}u$$

For $r \geqslant 0$, We define inductively

$$f_{0,r} = \begin{cases} s^{-m} y_1^{r/2} & \text{if } r \text{ is even} \\ y_2 y_1^{(r-1)/2} & \text{if } r \text{ is odd} \end{cases}$$

$$f_{q,r} = f_{0,r} z_2^q - \sum_{i=0}^{q-1} \binom{q}{i} f_{i,3q-3i+r} \quad (q = 1, 2, \ldots, m)$$

$$\sigma_i^{(q,r)} = \begin{cases} t^m y_1^{(3q+r-3i)/2} z_2^i & \text{if } q+r+i \text{ is even} \\ y_2 y_1^{(3q+r-3i-1)/2} z_2^i & \text{if } q+r+i \text{ is odd.} \end{cases}$$

Since $\sigma_i^{(k,3q-3k+r)} = \sigma_i^{(q,r)}$, and since $\displaystyle\sum_{i=a}^{b} (-1)^{i-a} \binom{b}{i} \binom{i}{a} = 0$, we obtain

$$f_{q,r} = \sum_{i=0}^{q} (-1)^{q-i} \binom{q}{i} \sigma_i^{(q,r)}.$$

Since $f_{q,r}$ is a polynomial on t, y_1, y_2, $f_{q,r}$ is holomorphic on U_2. On the other hand, by construction,

$$f_{q,r} = s^{q-m} x_1^q x_2^r + s^m g_{q,r}(s, x_1, x_2),$$

where $g_{q,r}$ is a suitable polynomial. So, $z_5 = f_{m,0}$ is a holomorphic on X.

Now we have a holomorphic mapping $h = (z_1, \ldots, z_5)\colon X \longrightarrow \mathbb{C}^5$. Let $X \xrightarrow{\varphi} Y \xrightarrow{g} \mathbb{C}^5$ be the Stein factorization of h. Since $C = h^{-1}(\mathbf{0})$, $\varphi(C)$ is a point. We shall show that $h\colon (X - C) \longrightarrow h(X - C)$ is a finite map. Let $z = (z_1, \ldots, z_5) \in h(U_1) - \{0\}$.

If $z_2^2 - z_1^3 \neq 0$, then

$$y_1 = z_1, \quad y_2 = \frac{z_3}{(z_2^2 - z_1^3)^m}, \quad t = \frac{z_4}{z_2^2 - z_1^3}.$$

Thus $h^{-1}(z)$ is a finite set.

Assume $z_2^2 - z_1^3 = 0$. If $z_2 = 0$, then $z_1 = z_2 = z_3 = z_4 = z_5 = 0$. Thus we assume $z_2 \neq 0$. Let

$$\alpha = \begin{cases} \displaystyle\sum_{j=0}^{m/2} \binom{m}{2j} z_1^{(3m-6j)/2} z_2^{2j} & \text{if } m \text{ is even} \\ \displaystyle\sum_{j=0}^{(m-1)/2} \binom{m}{2j+1} z_1^{(3m-6j-3)/2} z_2^{2j+1} & \text{if } m \text{ is odd} \end{cases}$$

$$\beta = \begin{cases} \displaystyle\sum_{j=0}^{(m/2)-1} \binom{m}{2j+1} z_1^{(3m-6j-4)/2} z_2^{2j+1} & \text{if } m \text{ is even} \\ \displaystyle\sum_{j=0}^{(m-1)/2} \binom{m}{2j} z_1^{(3m-6j-1)/2} z_2^{2j} & \text{if } m \text{ is odd.} \end{cases}$$

Then we have $\alpha t^m - \beta y_2 = f_{m,0} = z_5$. Since $z_1^3 = z_2^2 \neq 0$, we have $(\alpha, \beta) \neq (0, 0)$. Thus the system of equations on t and y_2

$$\alpha t^m - \beta y_2 = z_5, \quad z_1 t^{2m} - y_2^2 = z_2$$

has only finite solutions. Thus h is finite on $U_1 - C$.

If $z = (z_1, \ldots, z_5) \in h(U_2 - U_1) - \{0\}$, then $s = 0$. Thus

$$z_1 = x_2^2, \quad z_2 = x_2^3, \quad z_3 = 2^m x_1^m x_2^{3m+1}, \quad z_4 = 2x_1 x_2^3, \quad z_5 = x_1^m.$$

Therefore h is finite on $X - C$. □

Theorem 3.3. *Let C be a smooth projective curve of any genus, and $n \geqslant 3$ be an integer. Let q and r are non-negative integers with $q+r=n-1$, and let $n_1, \dots, n_q$ and $p_1, \dots, p_r$ be any integers such that $n_i \geqslant 1$, $p_j \geqslant 0$ for $1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant r$, and that*

$$-(n_1+\cdots+n_q)+2(p_1+\cdots+p_r) \leqslant -n+1.$$

If D is a base point free effective divisor on C, we can construct a smooth n-dimensional variety $X \supset C$ which satisfies the following conditions.

(i) *There exists a small contraction $\varphi\colon X \to Y$ whose exceptional set is C,*

(ii) $N_{C/X} \cong \mathcal{O}_C(-n_1D) \oplus \cdots \oplus \mathcal{O}_C(-n_qD) \oplus \mathcal{O}_C(p_1D) \oplus \cdots \oplus \mathcal{O}_C(p_rD)$.

Proof. Take two effective divisors D_1 and D_2 such that $D_1 \sim D_2 \sim D$ and $\operatorname{Supp} D_1 \cap \operatorname{Supp} D_2 = \phi$. Put $C_i = C - \operatorname{Supp} D_i$ and $U_i = C_i \times \mathbb{C}^{n-1}$ $(i=1, 2)$. We represent points in U_1 and U_2 by $x=(x_C, x_1, \dots, x_q, u_1, \dots, u_r) \in U_1$ and $y=(y_C, y_1, \dots, y_q, v_1, \dots, v_r) \in U_2$ as before. Let s be a rational function on C such that $D_1 = D_2 + \operatorname{div}(s)$. We extend s to U_1, U_2 by $s(x_C, x_1, \dots, u_r) = s(x_C)$ and $s(y_C, y_1, \dots, v_r) = s(y_C)$. Note that s is holomorphic on U_1, and $t=1/s$ is holomorphic on U_2. We identify C with the curve in X defined by $x_1=\cdots=x_q=u_1=\cdots=u_r=0$ in U_1 and $y_1=\cdots=y_q=v_1=\cdots=v_r=0$ in U_2.

Let $P_j = 2(p_1+\cdots p_j)+j$ $(0 \leqslant j \leqslant r, P_0=0)$ and $N_i = n_1+\cdots+n_i-i$ $(0 \leqslant i \leqslant q, N_0=0)$. Note that $N_i \geqslant 0$. Let

$$\sigma = u_1^2 + s^{P_1}u_2^2 + s^{P_2}u_3^2 + \cdots + s^{P_{r-1}}u_r^2$$

(if $r=0$ then $\sigma=0$). We patch U_1 and U_2 and construct $X=U_1 \cup U_2$ by the following transition function.

$$\begin{cases} y_i = s^{n_i}x_i + s^{1-N_{i-1}}\sigma \\ v_j = s^{-p_j}u_j \\ y_C = \tau_C(x_C) \end{cases}$$

Then

$$N_{C/X} \cong \mathcal{O}_C(-n_1D) \oplus \cdots \oplus \mathcal{O}_C(-n_qD) \oplus \mathcal{O}_C(p_1D) \oplus \cdots \oplus \mathcal{O}_C(p_rD).$$

Let

$$\begin{aligned} f_1 &= y_1 = s^{n_1}x_1 + s\sigma \\ f_i &= y_i - t^{n_{i-1}-1}y_{i-1} = s^{n_i}x_i - sx_{i-1} \quad (i=2, 3, \dots, q) \end{aligned}$$

It is easy to see that $f_1, \dots, f_q$ are holomorphic on X. Let

$$\sigma_j = \sum_{k=j}^{r} s^{P_{k-1}-P_{j-1}}u_k^2$$

for $1 \leqslant j \leqslant r$. Formally, put $\sigma_{r+1}=0$. Note that $\sigma_1=\sigma$. For $1 \leqslant j \leqslant r$, let $I(j)$ be an integer such that $N_{I(j)}+1 > P_j \geqslant N_{I(j)-1}$ ($I(j)$ is not always unique). Since $N_q \geqslant P_r$, we have $1 \leqslant I(1) \leqslant \cdots \leqslant I(r) \leqslant q$. Let

$$g_j = t^{P_j - N_{I(j)-1}}y_{I(j)} - \sum_{k=1}^{j} t^{P_j-P_k}v_k^2 = s^{1+N_{I(j)}-P_j}x_{I(j)} + s\sigma_{j+1}.$$

Then $g_1, \dots, g_r$ are holomorphic functions on X which vanish on C. Moreover, f_i and g_j can be divided by s on U_1. Thus tf_i, tg_j, $v_kf_i^{p_k}$ and $v_kg_j^{p_k}$ are also such functions ($1 \leqslant i \leqslant q$, $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant r$).

Now we have $(q+r)(2+r)$ holomorphic functions f_i, g_j, tf_i, tg_j, $v_kf_i^{p_k}$ and $v_kg_j^{p_k}$. By these functions, we have the holomorphic generically finite map $h\colon X \longrightarrow \mathbb{C}^{(n-1)(2+r)}$. Let $X \xrightarrow{\varphi} Y \xrightarrow{g} \mathbb{C}^{(n-1)(2+r)}$ be the Stein factorization of h. Then $\varphi\colon X \to Y$ gives a small contraction of C. □

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