A New Proof of Shapiro Inequality

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Abstract We present a new proof of Shapiro cyclic inequality. Especially, we treat the case \( n = 23 \) precisely.

§1. Introduction.

Let \( n \geq 3 \) be an integer, \( x_1, x_2, \ldots, x_n \) be positive real numbers, and let

\[
E_n(x_1, \ldots, x_n) := \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}},
\]

here we regard \( x_{i+n} = x_i \) for \( i \in \mathbb{Z} \). In this article, we present a new proof of the following theorem:

**Theorem 1.1.** (1) If \( n \) is an odd integer with \( 3 \leq n \leq 23 \), then

\[
E_n(x_1, \ldots, x_n) \geq \frac{n}{2}.
\]

Moreover, \( E_n(x_1, \ldots, x_n) = \frac{n}{2} \) holds only if \( x_1 = x_2 = \cdots = x_n \).

(2) If \( n \) is an even integer with \( 4 \leq n \leq 12 \), then \((P_n)\) holds. Moreover, the equality holds only if \((x_1, \ldots, x_n) = (a, b, a, b, \ldots, a, b)\) \((\exists a > 0, \exists b > 0)\).

(3) If \( n \) is an even integer with \( n \geq 14 \) or an odd integer with \( n \geq 25 \), then there exists \( x_1 > 0, \ldots, x_n > 0 \) such that \( E_n(x_1, \ldots, x_n) < \frac{n}{2} \).

(3) was proved by [4] in 1979. It is said that (1) was proved by [6] in 1989. (2) was proved by [2] in 2002. Note that [2] treat (1) to be an open problem. The author also thinks we should give a more agreeable proof of (1). In this article, we give more precise proof of (1) than [6].

§2. Basic Facts.

Throughout this article, we use the following notations:

\[
\partial_i E_n(x) := \frac{\partial}{\partial x_i} E_n(x) = \frac{1}{x_{i+1} + x_{i+2}} - \frac{x_{i-2}}{(x_{i-1} + x_i)^2} - \frac{x_{i-1}}{(x_i + x_{i+1})^2}
\]

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\[ \overline{K}_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq 0, \ldots, x_n \geq 0\} \]
\[ K^0_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0, \ldots, x_n > 0\} \]
\[ K^*_n := \left\{(x_1, \ldots, x_n) \in \overline{K}_n \mid (x_1, \ldots, x_n) \notin K^0_n, (x_i, x_{i+1}) \neq (0, 0) \text{ for any } i \in \mathbb{Z}\right\} \]
\[ K_n = K^0_n \cup K^*_n \]

It is easy to see that there exists \( a \in K^*_n \) such that
\[ \inf_{x \in K^*_n} E_n(x) = E_n(a). \]

Thus, we consider \( E_n(x) \) to be a continuous function on \( K^*_n \).

**Proposition 2.1.** ([3]) (1) If \((P_n)\) is false, then \((P_{n+2})\) is also false.
(2) If \((P_n)\) is false for an odd integer \( n \geq 3 \), then \((P_{n+1})\) is also false.

**Proof.** Assume that there exists positive real numbers \( a_1, \ldots, a_n \) such that \( E_n(a_1, \ldots, a_n) < n/2 \).
(1) Since, \( E_{n+2}(a_1, \ldots, a_n, a_1, a_2) = 1 + E_n(a_1, \ldots, a_n) < \frac{n+2}{2} \), \((P_{n+2})\) is false.
(2) Note that
\[
E_{n+1}(a_1, \ldots, a_{r-1}, a_r, a_r, a_{r+1}, \ldots, a_n) - E_n(a_1, \ldots, a_n) - \frac{1}{2}
= \frac{a_{r-1}}{a_r + a_{r+1}} - \frac{a_{r-1}}{a_r + a_{r+1}} - \frac{1}{2}
= \frac{(a_r - a_{r-1})(a_r - a_{r+1})}{2a_r(a_r + a_{r+1})}
\]
for \( 1 \leq r \leq n \). Thus, it is sufficient to show that there exists \( r \) such that \((a_r - a_{r-1})(a_r - a_{r+1}) \leq 0 \).
Assume that \((a_r - a_{r-1})(a_r - a_{r+1}) > 0 \) for all \( 1 \leq r \leq n \). Since \( n \) is odd,
\[
\prod_{r=1}^{n}(a_r - a_{r+1})^2 = \prod_{r=1}^{n}(a_{r-1} - a_r)(a_r - a_{r+1}) < 0.
\]
This is a contradiction. \( \square \)

**Proposition 2.2.** ([4]) (1) \( E_{14}(42, 2, 42, 4, 41, 5, 39, 4, 38, 2, 38, 0, 40, 0) < 7 \). Thus \((P_{14})\) is false.
(2) \( E_{25}(34, 5, 35, 13, 30, 17, 24, 18, 18, 17, 13, 16, 9, 16, 5, 16, 2, 18, 0, 21, 0, 25, 0, 29, 0) < 25/2 \). Thus \((P_{25})\) is false.

Thus, Theorem 1.1 (3) is proved by Proposition 2.1 and 2.2. It is essential to show \((P_{12})\) and \((P_{23})\) for a proof of Theorem 1.1 (2) and (3).

**Definition 2.3.** We say that \( x = (x_1, \ldots, x_n) \in K_n \) and \( y = (y_1, \ldots, y_n) \in K_n \) belong to the same component if \( "x_i = 0 \iff y_i = 0" \) for all \( i = 1, \ldots, n \).

Let \( x = (x_1, \ldots, x_n) \in K^*_n \). If \( x_{i-1} = 0, x_i \neq 0, x_{i+1} \neq 0, \ldots, x_j \neq 0, \) and \( x_{j+1} = 0 \) for \( i < j \in \mathbb{Z} \), then we call \( (x_i, \ldots, x_j) \) to be a segment of \( a \), and we define \( j - i + 1 \) to be the length of this segment. A segment of length \( l \) is called \( l \)-segment.
For a segment $s := (x_1, \ldots, x_j)$ of $x$, we denote
\[
S(s) := \sum_{k=1}^{j-1} \frac{x_k}{x_{k+1} + x_{k+2}}, \quad \text{Head}(s) := x_1, \quad \text{Tail}(s) := x_j.
\]

Here we define $S(s) = 0$, if the length of $s$ is 1.

Let $s_1, \ldots, s_r$ be all the segments of $x$ in this order. Let $l_k$ be the length of $s_k$. Then $(l_1, \ldots, l_r)$ is called the index of $x$. Note that
\[
E_n(a) = \sum_{k=1}^r S(s_k) + \sum_{k=1}^r \frac{\text{Tail}(s_k)}{\text{Head}(s_k)}.
\]

Here we regard $s_{k+r} = s_k$ for $k \in \mathbb{Z}$.

**Theorem 2.4.** Assume that $\min_{x \in K_n^*} E_n(x) = E_n(a)$ at $a = (a_1, \ldots, a_n) \in K_n^*$. Let $s_1, \ldots, s_r$ be all the segments of $a$ in this order, and let $l_k$ be the length of $s_k$. Then the followings hold.

1. \[\frac{\text{Tail}(s_1)}{\text{Head}(s_1)} = \frac{\text{Tail}(s_2)}{\text{Head}(s_2)} = \cdots = \frac{\text{Tail}(s_{r-1})}{\text{Head}(s_{r-1})} = \frac{\text{Tail}(s_r)}{\text{Head}(s_r)}\]
2. Assume that $a = (s_1, 0, s_2, 0, \ldots, s_r, 0)$, and let $\sigma$ be a permutation of $\{1, 2, \ldots, r\}$.

Then there exist real numbers $t_1 > 0, t_2 > 0, \ldots, t_r > 0$ such that
\[
b := (t_1s_{\sigma(1)}, 0, t_2s_{\sigma(2)}, 0, \ldots, t_rs_{\sigma(r)}, 0)
\]

satisfies $E_n(b) = E_n(a)$.

**Proof.** (1) Since $E_n(a_{1+k}, a_{2+k}, \ldots, a_{n+k}) = E_n(a_1, a_2, \ldots, a_n)$, we may assume $a = (s_1, 0, s_2, 0, \ldots, s_r, 0)$. Let $x_i := \text{Head}(s_i), y_i := \text{Tail}(s_i)$. Define $t_1, \ldots, t_r$ by $t_1 := 1$ and
\[
t_j := \frac{y_1y_2 \cdots y_{j-1}}{x_2x_3 \cdots x_j} \cdot \left(\frac{x_1x_2 \cdots x_r}{y_1y_2 \cdots y_r}\right)^{\frac{1}{r-1}}
\]
for $j = 2, 3, \ldots, r$. It is easy to see that
\[
\frac{t_{j-1}y_{j-1}}{t_jx_j} = \sqrt[\frac{r-1}{r}]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}} = \frac{t_r y_r}{t_1 x_1}
\]

Take $t_1 > 0, \ldots, t_r > 0$, and let
\[
c = (t_1s_1, 0, t_2s_2, 0, \ldots, t_rs_r, 0).
\]

Note that $S(t_is_i) = S(s_i)$. By AM-GM inequality,
\[
E_n(a) = \sum_{i=1}^r S(s_i) + \sum_{i=1}^r \frac{y_i-1}{x_i}
\geq \sum_{i=1}^r S(s_i) + r \cdot \sqrt[\frac{r-1}{r}]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}} = \sum_{i=1}^r S(t_is_i) + \sum_{i=1}^r \frac{t_{i-1}y_{i-1}}{t_ix_i} = E_n(c).
\]

Since $E_n(a)$ is the minimum, we have $E_n(a) = E_n(c)$. By the equality condition of AM-GM inequality, we have $t_1 = t_2 = \cdots = t_r = 1$. Thus
\[
\frac{y_{j-1}}{x_j} = \sqrt[\frac{r-1}{r}]{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}},
\]
and we have (1).
Lemma 3.1. \((1)\) Lemma 3.2, 4.2) The above functions satisfy the followings.

\[
E_n(b) = \sum_{i=1}^{r} S(s_i) + r \cdot \sqrt{\frac{y_1 \cdots y_r}{x_1 \cdots x_r}}.
\]

Thus \(E_n(b) = E_n(a)\).

Remark 2.5. By the above theorem, we may assume that the index \((l_1, \ldots, l_r)\) of \(a\) satisfies
\(l_1 \geq l_2 \geq \cdots \geq l_r\), if \(\min_{x \in K_n} E_n(x) = E_n(a)\). Thus, we always write the index of such \(a\) in descending order.

Definition 2.6. Assume that \(a \in K_n^*\) satisfies the condition of the above theorem. Then we define \(U(a)\) to be

\[
U(a) := \frac{\text{Tail}(s_1)}{\text{Head}(s_2)} = \frac{\text{Tail}(s_2)}{\text{Head}(s_3)} = \cdots = \frac{\text{Tail}(s_{r-1})}{\text{Head}(s_r)} = \frac{\text{Tail}(s_r)}{\text{Head}(s_1)}.
\]

Note that \(E_n(a) = rU(a) + \sum_{k=1}^{r} S(s_k)\), for \(a = (s_1, 0, s_2, 0, \ldots, s_r, 0)\).

§3. Bushell Theorem.

We survey and improve the results of [1]. In this section, we denote

\[
A_i(x) := \frac{x_i}{x_{i+1} + x_{i+2}},
\]

\[
B(x) := (x_2 + x_3, x_3 + x_4, \ldots, x_n + x_1, x_1 + x_2)
\]

\[
R(x) := \left( \frac{1}{x_n}, \frac{1}{x_{n-1}}, \frac{1}{x_{n-2}}, \ldots, \frac{1}{x_1} \right)
\]

\[
T(x) = \left( \frac{x_n}{(x_1 + x_2)^2}, \ldots, \frac{x_{n+1-i}}{(x_n + x_{n+1-i})^2}, \ldots, \frac{x_1}{(x_2 + x_3)^2} \right)
\]

for \(x = (x_1, \ldots, x_n)\). We also denote the \(i\)-th element of \(B(x)\) by \(B(x)_i = x_{i+1} + x_{i+2}\). \(R(x)_i\) and \(T(x)_i\) are also defined similarly. The symbol \(T(x)\) are used throughout this article.

Lemma 3.1. \([1]\) Lemma 3.2, 4.2) The above functions satisfy the followings.

1. \(\partial_i E_n(x) = (R(B(x))_{n+1-i} - (B(T(x)))_{n+1-i})\).
2. \(T^2(x)_i = \frac{x_i}{(1 - (B(x))_i \partial_i E_n(x))}\).
3. \(E_n(T(x)) - E_n(x) = \sum_{i=1}^{n} \frac{x_i(\partial_i E_n(x))^2}{(B(T(x)))_{n+1-i}}\).
4. \(E_n(x) + E_n(y) = E_n(x + y) + E_n(T(x) + T(y))
\]

\[
- \sum_{i=1}^{n} \frac{(T(x) + T(y))_{n+1-i} (\partial_i E_n(x) + \partial_i E_n(y))}{(R(B(x)) + R(B(y)))_{n+1-i}} (B(T(x) + T(y)))_{n+1-i}.
\]
Proof. (1) Let \( E_n(T) = \frac{1}{x_{i+1} + x_{i+2}} - \left( \frac{x_{i-2}}{(x_{i-1} + x_i)^2} + \frac{x_{i-1}}{(x_i + x_{i+1})^2} \right) = (R(B(x)))_{n+1-i} - (B(T(x)))_{n+1-i}. \)

(2) Let \( T(x) := \frac{x_{n+1-i}}{(B(x))^2}. \) Combine this with (1), we obtain

\[
(T^2(x))_i = \frac{(T(x))_{n+1-i}}{(B(T(x)))_{n+1-i}} = \frac{x_i/(B(x))^2}{((R(B(x)))_{n+1-i} - \partial_i E_n(x))^2}. \tag{3.1.1}
\]

Since \( (B(x))_i \cdot (R(B(x)))_{n+1-i} = 1 \), we obtain (2).

(3) By the similar calculation as above, we obtain

\[
E_n(T) - E_n(x) = \sum_{i=1}^{n} \frac{(T(x))_i}{(B(T(x)))_i} - \sum_{i=1}^{n} \frac{x_i}{(B(x))_i}.
\]

\[
= \sum_{i=1}^{n} \left( \frac{(T(x))_{n+1-i} - x_i}{(B(T(x)))_{n+1-i} - (B(x))_i} \right)
\]

\[
= \sum_{i=1}^{n} \left( \frac{x_i}{(B(x))_i(1 - (B(x))_i \partial_i E_n(x))} - \frac{x_i}{(B(x))_i} \right)
\]

\[
= \sum_{i=1}^{n} \frac{x_i \partial_i E_n(x)}{1 - (B(x))_i \partial_i E_n(x)}.
\]

Since,

\[
\sum_{i=1}^{n} x_i \partial_i E_n(x) = \sum_{i=1}^{n} \frac{x_i}{x_{i+1} + x_{i+2}} - \sum_{i=1}^{n} \frac{x_{i-2}x_i}{(x_{i-1} + x_i)^2} - \sum_{i=1}^{n} \frac{x_{i-1}x_i}{(x_i + x_{i+1})^2}
\]

\[
= \sum_{i=1}^{n} \frac{x_{i-1}(x_i + x_{i+1})}{(x_i + x_{i+1})^2} - \sum_{i=1}^{n} \frac{x_{i-1}x_{i+1}}{(x_i + x_{i+1})^2} - \sum_{i=1}^{n} \frac{x_{i-1}x_i}{(x_i + x_{i+1})^2} = 0,
\]

we obtain

\[
E_n(T(x)) - E_n(x) = \sum_{i=1}^{n} x_i \partial_i E_n(x) \left( \frac{1}{1 - (B(x))_i \partial_i E_n(x)} - 1 \right)
\]

\[
= \sum_{i=1}^{n} \frac{x_i (\partial_i E_n(x))^2}{(B(T(x)))_{n+1-i}}.
\]

(4) Let \( a := x_i, b := x_{i+1} + x_{i+2} = (B(x))_i, c := y_i, d := (B(y))_i \).

\[
\frac{x_i + y_i}{(B(x + y))_i} + \frac{(T(x) + T(y))_{n+1-i}}{(R(B(x)) + R(B(y)))_{n+1-i}}
\]

\[
= a + c + \frac{a/b^2 + c/d^2}{1/b + 1/d} = \frac{a + c}{b} = A_i(x) + A_i(y)
\]

By (1), we have

\[
\frac{(T(x) + T(y))_{n+1-i}}{(B(T(x)) + T(y))_{n+1-i}} - \frac{(T(x) + T(y))_{n+1-i}}{(R(B(x)) + R(B(y)))_{n+1-i}}
\]

\[
= \frac{(T(x) + T(y))_{n+1-i} (\partial_i E_n(x) + \partial_i E_n(y))}{(R(B(x)) + R(B(y)))_{n+1-i} \cdot (B(T(x)) + T(y))_{n+1-i}}. \tag{3.1.3}
\]
Take $\sum_{i=1}^{n}$ of (3.1.2) and (3.1.3), we obtain (4). □

**Theorem 3.2.** ([1] Theorem 3.3) (1) $E_n(T(x)) \geq E_n(x)$ holds for $x \in K_n$. Moreover, if $E_n(T(x)) = E_n(x)$, then $T^2(x) = x$ holds.

(2) If $\min_{x \in K_n^*} E_n(x) = E_n(a)$ at $a \in K_n$, then the following holds.

$$T^2(a) = a, \quad E_n(T(a)) = E_n(a).$$

**Proof.** (1) $E_n(T(x)) \geq E_n(x)$ follows from Lemma 3.1 (3). Assume that $E_n(T(x)) = E_n(x)$. Then $x_i(\partial_i E_n(x))^2 = 0$ (\forall i = 1, \ldots, n), by Lemma 3.1 (3). Thus $x_i = 0$ or $\partial_i E_n(x) = 0$. By Lemma 3.1 (2), we obtain $(T^2(x))_i = x_i$.

(2) If $E_n$ is minimum at $a$, then $a_i = 0$ or $\partial_i E_n(a) = 0$. By Lemma 3.1 (2), we have $(T^2(a))_i = a_i$. We also have $E_n(T(a)) = E_n(a)$ by Lemma 3.1 (3). □

**Lemma 3.3.** ([1] Lemma 4.3) Let $a, b, c, d, e$ be positive real numbers, and $p, q$ be real numbers. Assume that

$$p \frac{1 + \lambda a}{(1 + \lambda c)^2} + q \frac{1 + \lambda b}{(1 + \lambda d)^2} = \frac{1}{1 + \lambda e} \quad (3.3.1)$$

for all real numbers $\lambda \geq 0$. Then the followings hold.

(1) If $p = 0$, then $q = 1$ and $b = d = e$.

(2) If $q = 0$, then $p = 1$ and $a = c = e$.

(3) If $p \neq 0$ and $q \neq 0$, then $c = d = e$.

**Proof.** (1) Substitute $\lambda = 0$, $p = 0$ for (3.3.1), we have $q = 1$. In this case, (3.3.1) is equivalent to

$$(1 + \lambda b)(1 + \lambda e) = (1 + \lambda d)^2.$$

As an equality of a polynomial in $\lambda$, we have $b = d = e$.

(2) can be proved similarly as (1).

(3) Let

$$g(\lambda) := p(1 + \lambda a)(1 + \lambda d)^2(1 + \lambda e) + q(1 + \lambda b)(1 + \lambda c)^2(1 + \lambda e) - (1 + \lambda e)^2(1 + \lambda d)^2. \quad (3.3.2)$$

$g(\lambda) = 0$ as a polynomial in $\lambda$. Thus

$$0 = g\left(-\frac{1}{e}\right) = -\left(1 - \frac{c}{e}\right)^2 \left(1 - \frac{d}{e}\right)^2,$$

and we have $c = e$ or $d = e$.

Assume that $d \neq e$. Then $c = e$. From (3.3.2), we obtain

$$p(1 + \lambda a)(1 + \lambda d)^2 + q(1 + \lambda b)(1 + \lambda c)^2 - (1 + \lambda e)(1 + \lambda d)^2 = 0. \quad (3.3.3)$$

Substitute $\lambda = -1/e$ for (3.3.3), we obtain $p(1 - a/e)(1 - d/e)^2 = 0$. Thus $a = e$. Then

$$p(1 + \lambda d)^2 + q(1 + \lambda b)(1 + \lambda e) - (1 + \lambda d)^2 = 0. \quad (3.3.4)$$

Substitute $\lambda = -1/e$ for (3.3.4), we have $d = e$. A contradiction. Thus $d = e$.

Similarly, we have $c = e$. □

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Theorem 3.4. (1) Assume that \( \min_{x \in K_n^e} E_n(x) = E_n(a) = E_n(b) \) at \( a, b \in K_n^e \), and that \( a \) and \( b \) belong to the same component. Then, there exists a real number \( \mu > 0 \) such that \( a = \mu b \).

(2) Assume that \( \min_{x \in K_n^e} E_n(x) = E_n(a) \) at \( a \in K_n^e \). Then \( E_n(a) = n/2 \). Moreover \( a = (a, a, \ldots, a) \) \( (\exists \alpha > 0) \), or \( a = (a, b, a, b, \ldots, a, b) \) \( (\exists \alpha > 0, b > 0) \).

Proof. Assume that \( \min_{x \in K_n^e} E_n(x) = E_n(a) = E_n(b) \) for \( a, b \in K_n \), and that \( a \) and \( b \) belong to the same component. Let \( \lambda > 0 \) be any real number.

If \( a_i \neq 0 \), then \( \partial_i E_n(a) = \partial_i E_n(\lambda b) = 0 \). If \( a_i = 0 \), then \( b_i = 0 \) and \( (T(a))_{n+1-i} = 0 \), \( (T(\lambda b))_{n+1-i} = 0 \). Thus we have

\[
(T(a) + T(\lambda b))_{n+1-i} : (\partial_i E_n(a) + \partial_i E_n(\lambda b)) = 0
\]

(\( \forall i \in \mathbb{Z} \)). We use the Lemma 3.1 (4) with \( x = T(a) \), \( y = \lambda b \). Since the numerators of the fractions in \( \sum \) in Lemma 3.1 (4) are zero, we have

\[
E_n(a) + E_n(\lambda b) = E_n(a + \lambda b) + E_n(T(a) + T(\lambda b)).
\]

Since \( E_n(\lambda b) = E_n(b) = E_n(a) \) is minimum, we have

\[
E_n(a + \lambda b) = E_n(T(a) + T(\lambda b)) = E_n(a).
\]

Since \( E_n(x) \) is minimum at \( x = a + \lambda b \) for any \( \lambda > 0 \), we have

\[
0 = \partial_i E_n(a + \lambda b) = \frac{1}{(B(a + \lambda b))_i} - \frac{a_{i-2} + \lambda b_{i-2}}{(B(a + \lambda b))_{i-2}^2} - \frac{a_{i-1} + \lambda b_{i-1}}{(B(a + \lambda b))_{i-1}^2} \quad (3.4.1)
\]

when \( a_i \neq 0 \). Let

\[
a := \frac{b_{i-2}}{a_{i-2}}, \quad b := \frac{b_{i-1}}{a_{i-1}}, \quad c := \frac{(B(b))_{i-2}}{(B(a))_{i-2}}, \quad d := \frac{(B(b))_{i-1}}{(B(a))_{i-1}},
\]

\[
e := \frac{(B(b))_i}{(B(a))_i}, \quad p := \frac{a_{i-2}(B(a))_i}{(B(a))_{i-2}^2}, \quad q := \frac{a_{i-1}(B(a))_i}{(B(a))_{i-1}^2}.
\]

Then, (3.4.1) become (3.3.1). It is easy to see that the cases (1) and (2) of Lemma 3.3 do not occur. Lemma 3.3 (3) implies

\[
\frac{(B(b))_{i-2}}{(B(a))_{i-2}} = \frac{(B(b))_{i-1}}{(B(a))_{i-1}} = \frac{(B(b))_i}{(B(a))_i} = \frac{1}{\mu} > 0.
\]

Thus

\[
a_{i+1} + a_{i+2} = B(u) = \mu B(v) = \mu(b_{i+1} + b_{i+2}) \quad (3.4.2)
\]

(\( \forall i \in \mathbb{Z} \)). If \( n \) is odd, then \( a_i = \mu b_i \) (\( \forall i \in \mathbb{Z} \)) from (3.4.2). Thus \( a = \mu b \).

We treat the case \( n \) is even. Let \( w = (1, -1, 1, -1, \ldots, -1) \in \mathbb{R}^n \). By elementary linear algebra, we conclude that the solutions of the system of equations (3.4.2) is of the form

\[
a - \mu b = \nu w \quad (\exists \nu \in \mathbb{R}).
\]

If \( a = K_n^e \), then \( a \) and \( b \) have zeros at the same place. Thus, \( \nu \) must be zero. Thus we obtain (1).

We shall prove (2). Apply above argument to \( b = (a_2, a_3, \ldots, a_n, a_1) \). If \( n \) is odd, then \( a = \mu b \). Thus \( \mu = 1 \), and \( a_1 = a_2 = \cdots = a_n \). In this case, \( E_n(a) = n/2 \).

If \( n \) is even, \( a - \mu b = \nu w \). Thus \( a = (a_1, a_2, a_1, a_2, \ldots, a_1, a_2) \). Then \( E_n(a) = n/2 \). \( \square \)

Corollary 3.5. Assume that \( \min_{x \in K_n^e} E_n(x) = E_n(a) \) at \( a \in K_n^e \). Let \( s \) and \( t \) be segments of \( a \) with the same length \( l \). Then, there exists a real number \( c > 0 \) such that \( s = ct \).
Proof. We construct a vector \( \mathbf{b} \) as in the proof of Theorem 2.4 (2), where \( \sigma \) is the transposition of \( \mathbf{s} \) and \( \mathbf{t} \). Then \( E_n(\mathbf{a}) = E_n(\mathbf{b}) \). By Theorem 3.4, \( \mathbf{a} = \mu \mathbf{b} (\exists \mu > 0) \). Thus \( \mathbf{s} = ct (\exists c > 0) \).

Corollary 3.6. Assume that \( \min_{x \in K_n} E_n(x) = E_n(\mathbf{a}) \) at \( \mathbf{a} \in K_n^* \). Let \( \mathbf{s} = (a_1, \ldots, a_l) \) be a \( l \)-segment of \( \mathbf{a} \) with \( l \geq 2 \). Let \( U := U(\mathbf{a}) \). Then there exists a real number \( \mu > 0 \) such that

\[
\begin{pmatrix}
U^2 & a_{l-1} & a_{l-2} & \cdots & a_1 \\
a_l & a_l^2 & (a_{l-1} + a_l)^2 & \cdots & (a_2 + a_3)^2 \\
 & & (a_{l-2} + a_{l-1})^2 & \cdots & (a_3 + a_4)^2 \\
 & & & & (a_2 + a_3)^2 \\
 & & & & (a_3 + a_4)^2 \\
\end{pmatrix}
= \mu(a_1, a_2, a_3, a_4, \ldots, a_{l-1}, a_1).
\]

(3.6.1)

Proof. We may assume that \( \mathbf{a} = (s, 0, \ldots) \). Rotate the elements of \( T(\mathbf{a}) \) so that the segment corresponding to \( \mathbf{s} \) comes to be the same place with \( \mathbf{s} \), and we denote this vector by \( \mathbf{b} \). Then the top segment of \( \mathbf{b} \) is

\[
\begin{pmatrix}
a_1 & a_{l-1} & a_{l-2} & \cdots & a_1 \\
& a_l^2 & (a_{l-1} + a_l)^2 & \cdots & (a_2 + a_3)^2 \\
& & (a_{l-2} + a_{l-1})^2 & \cdots & (a_3 + a_4)^2 \\
& & & & (a_2 + a_3)^2 \\
& & & & (a_3 + a_4)^2 \\
\end{pmatrix}
.
\]

By Theorem 3.2 (2), \( E_n(\mathbf{b}) = E_n(T(\mathbf{a})) = E_n(\mathbf{a}) \). By Theorem 3.4, \( \mathbf{b} = \mu \mathbf{a} (\exists \mu > 0) \). Since \( U = a_l/a_{l+2}, a_l/a_{l+2}^2 = U^2/a_1 \). Thus, we have (3.6.1).


The aim of this section is to explain Theorem 4.3, according to [2]. In this section, we denote

\[
K_n^\Delta := \{(x_1, \ldots, x_n) \in K_n^* | x_{n-1} = 1, x_n = 0\}
\]

\[
y_i := \frac{x_i}{x_i + x_{i+1}} = A_i(\mathbf{x}).
\]

Note that \( y_n = 0, y_{n-1} = x_{n-1}/x_1, \) and \( y_{n-2} = x_{n-2} \) for \( \mathbf{x} = (x_1, \ldots, x_n) \in K_n^\Delta \). The map \( \Phi: K_n^\Delta \rightarrow \Phi(K_n^\Delta) \) defined by \( \Phi(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \) is bijective. The inverse map \( \Phi^{-1} \) is obtained as the solution of the system of equations \( y_i(x_{i+1} + x_{i+2}) - x_i = 0 \) \( (i = 1, \ldots, n - 2) \). Let

\[
P_k(z_1, z_2, \ldots, z_k) :=
\begin{vmatrix}
z_1 & z_2 & \cdots & z_k \\
-1 & z_2 & \cdots & z_k \\
& -1 & z_3 & \cdots \\
& & \ddots & \ddots \\
& & & -1 & z_{k-2} \\
& & & & -1 \end{vmatrix}
\]

Inductively, we can prove that \( x_i = P_{n-i-1}(y_i, y_{i+1}, \ldots, y_{n-2}) \). By the properties of determinant, we can prove the following lemma.

Lemma 4.1. ([2] Lemma 3.1) The followings hold. Here we put \( P_0 := 1 \) and \( P_{-1} = 1 \).

1. \( P_k(z_1, \ldots, z_k) = z_kP_{k-1}(z_1, \ldots, z_{k-1}) + z_{k-1}P_{k-2}(z_1, \ldots, z_{k-2}) \).
(2) For $1 \leq j < k$,
\[ P_k(z_1, \ldots, z_k) = P_j(z_1, \ldots, z_j)P_{k-j}(z_{j+1}, \ldots, z_k) \]
\[ + z_j P_{j-1}(z_1, \ldots, z_{j-1})P_{k-j-1}(z_{j+2}, \ldots, z_k). \]

**Lemma 4.2.** ([2] Lemma 3.2) Let $x = (x_1, \ldots, x_n) \in K_+^n$, and $(y_1, \ldots, y_n) = \Phi(x_1, \ldots, x_n)$. Assume that $x_i \partial_i E_n(x) = 0$ for all $i = 1, 2, \ldots, n$. Then the followings hold.

1. $y_i = y_i^2 P_{i-1}(y_1, \ldots, y_{i-1}) P_{n-i-1}(y_i, \ldots, y_n)$
2. $y_1 - y_i = y_i^2 y_{i-1} P_{i-2}(y_1, \ldots, y_{i-2}) P_{n-i-2}(y_{i+1}, \ldots, y_n)$

*Proof.* Put $p_i := P_i(y_1, \ldots, y_i)$. Then (1), (2) can be written as (1) $y_i = y_i^2 p_{i-1} x_i$, and (2) $y_1 - y_i = y_i^2 y_{i-1} p_{i-2} x_{i+1}$.

(1) As a formal rational function
\[ x_i \partial_i E_n(x) = \frac{x_i}{x_{i+1} + x_{i+2}} - \frac{x_i (x_{i-1} + x_i)^2}{x_{i+1} x_{i+2}} - \frac{x_i - x_{i+1}}{x_{i+1} x_{i+2}} \]
\[ = y_i - y_i^2 \frac{x_{i-2} x_i}{x_{i-2}} - y_i^2 \frac{x_{i-1}}{x_{i-1}}. \]

So, the condition $x_i \partial_i E_n(x) = 0$ can be represented as
\[ \frac{y_i}{x_i} = y_i^2 - \frac{y_i^2}{x_{i-2}} + \frac{y_i^2}{x_{i-1}} \quad (4.2.1) \]
as an equation in the field $\mathbb{R}(x_1, \ldots, x_{n-2})$. Here, we regard $x_0 = x_n = 0$, $x_{-1} = x_{n-1} = 1$, $y_0 = y_n = 0$, and $y_{-1} = y_{n-1} = 1/x_1$. It is enough to show
\[ \frac{y_i}{x_i} = y_i^2 p_{i-1} \quad (4.2.2) \]
in $\mathbb{R}(x_1, \ldots, x_{n-2})$.

Consider the case $i = 1$. Then, $p_0 = 1$. (4.2.1) can be written as $y_1/x_1 = 1/x_1^2$. Multiply $x_1^2 y_1$, then we have (4.2.2).

Consider the case $i = 2$. By (4.2.1) and $x_1 y_1 = 1$, $y_1 = P_1(y_1) = p_1$, we have
\[ \frac{y_2}{x_2} = \frac{y_1^2}{x_1} = y_1^3 = y_1^2 p_1. \]
Thus we obtain (4.2.2).

Consider the case $i \geq 3$. We shall prove (4.2.2) by induction on $i$. By induction assumption, $y_j/x_j = y_j^2 p_{j-1}$ for $1 \leq j < i$. By Lemma 4.1 (1), $p_i = y_{i-1} p_{i-2} + y_{i-2} p_{i-3}$.

Thus
\[ \frac{y_i}{x_i} = \frac{y_i^2}{x_{i-2}} + \frac{y_{i-1}^2}{x_{i-1}} - y_i (y_{i-2} p_{i-3} + y_{i-1} p_{i-2}) = y_i^2 p_{i-1}. \]

(2) Apply Lemma 4.1 (5) with $k = n - 2$, $j = i - 1$, then we obtain $x_i = p_{i-1} x_i + y_{i-1} p_{i-2} x_{i+1}$. Since $x_1 = 1/y_1$, after multiplying $y_1^2$ to the both hand sides, we obtain
\[ y_1 = y_i^2 p_{i-1} x_i + y_i^2 y_{i-1} p_{i-2} x_{i+1}. \]

By (1),
\[ y_1 - y_i = y_1 - y_i^2 p_{i-1} x_i = y_i^2 y_{i-1} p_{i-2} x_{i+1}. \]
Thus we obtain (2). \(\square\)

**Theorem 4.3.** ([2] Proposition 3.3) If $\min_{x \in K_n} E_n(x) = E_n(a)$ at $a \in K_n^*$, then $U(a) \geq 1/2$. 

\[ 9 \]
Proof. We may assume a = (x₁, ..., xₙ) ∈ Kₙ. By Lemma 4.2 (1), (2), we have 0 ≤ xᵢ/(xᵢ₊₁ + xᵢ₊₂) = yᵢ ≤ 1/y₁ = 1/x₁ = U(a) (i = 1, ..., n). Assume that U(a) < 1/2. Then x₁ > 2, and 2xᵢ ≤ xᵢ₊₁ + xᵢ₊₂. Take ∑, we obtain

\[ 2 \sum_{i=1}^{n} x_i < \sum_{i=1}^{n} (x_{i+1} + x_{i+2}) = 2 \sum_{i=1}^{n} x_i. \]

A contradiction. □

§5. Short segments.


Theorem 5.1. Assume that \( \min_{x \in K_n} E_n(x) = E_n(a) \) at \( a \in K_n \). Then \( a \) does not contain segments of length 2, 3, 4, 5, 7, or 9.

Proof. Let \( s = (a₁, ..., a_l) \) be a \( l \)-segment of \( a \) (\( l \geq 2 \)). Put \( U := U(a) \), \( V := \frac{a_{l-1} + a_l}{a_l} > 1 \).

Note that \( a_{l+1} = 0 \), \( a_{l+2} = a_l/U \) by Theorem 2.4 (1). By Theorem 4.3, \( U \geq 1/2 \).

Since \( a_{l+2} + a_{l+3} \geq a_{l+2} = a_l/U \), we have

\[ 0 \leq \partial_{l+1} E_{n}(a) = \frac{1}{a_{l+2} + a_{l+3}} - \frac{a_{l-1}}{a_l^2} - \frac{a_l}{a_{l+2}^2} \leq \frac{1}{a_l} (U - (V - 1) - U^2). \]

Thus, we have \( V \leq 1 + U - U^2 \). Since \( 1 < V \leq 1 + U - U^2 \), we have \( U < 1 \) and \( 1 < V \leq \frac{5}{4} - \left( U - \frac{1}{2} \right)^2 \leq \frac{5}{4} \). Thus \( (U, V) \) is included in the set \( D := \{(u, v) \in \mathbb{R}^2 \mid 1/2 \leq u < 1, 1 < v \leq 1 + u - u^2 \} \).

By (3.6.1), \( \frac{a_1 a_l}{U^2} = \frac{1}{\mu} = \frac{a_2 a_l^2}{a_l-1} \). Thus we have

\[ a_2 = \frac{a_1 a_{l-1}}{a_l U^2} = \frac{V - 1}{U^2} a_1. \]

Since \( \partial_{l-2} E_{n}(a) = 0 \) (\( i = 3, 4, ..., l + 2 \)), we have

\[ a_i = \frac{1}{a_{i-4} + a_{i-3}} - a_{i-1}. \]

Here \( a_{-1} = a_{n-1} = U a_1 \) and \( a_0 = a_n = 0 \). Inductively, we obtain

\[ a_3 = \frac{1}{a_{n-1}/a_1^2} - a_2 = \frac{U - V + 1}{U^2} a_1 \quad \text{(if } l \geq 3\text{)} \]

\[ a_4 = \frac{V - U}{U^2} a_1 \quad \text{(if } l \geq 4\text{)} \]

\[ a_5 = \frac{1 + UV - V^2}{U^2 V} a_1 \quad \text{(if } l \geq 5\text{).} \]
Thus, we define a series of rational functions by
\[
\begin{align*}
 f_1(u, v) &:= 1, \quad f_2(u, v) := \frac{v - 1}{u^2}, \quad f_3(u, v) := \frac{u - v + 1}{u^2}, \quad f_4(u, v) := \frac{v - u}{u^2} \\
 f_i(u, v) &:= \frac{1}{(f_{i-4}(u, v))^2 + \frac{f_{i-3}(u, v)}{(f_{i-2}(u, v) + f_{i-1}(u, v))^2}} - f_{i-1}(u, v)
\end{align*}
\]

(i ≥ 5). Then, \(a_i = f_i(U, V)a_1\) for 1 ≤ i ≤ l + 2. Especially, \(f_{i+1}(U, V) = a_{i+1}/a_1 = 0\).

Since \(u - v + 1 > 0, u - v^2 > 0\) on \(D\), we obtain \(f_i(u, v) > 0\) on \(D\) for \(i = 3, 4, 5\). Thus \(a_{i+1} ≠ 0\) for \(l = 2, 3, 4\). Therefore, \(a\) does not contain segments of length 2, 3, or 4.

Similarly, \(f_i(u, v) > 0\) on \(D\) for \(i = 6, 8, 10\). We need numerical analysis to prove this. If you have ‘Mathematica’, execute the following.

```
<< Graphics'ImplicitPlot'
fi[i_, u_, v_] := (a = 1; b = (v-1)/u^2;
 c = (1+u-v)/u^2; d = (v-u)/u^2;
 Do[(e=1/(a/(b+c)^2 + b/(c+d)^2) - d; a=b; b=c; c=d; d = e),
 {k, 5, i, 1}];
{e, 1/(a/(b+c)^2 + b/(c+d)^2) - d; a=b; b=c; c=d; d = e},
{k, 5, i, 1});
G1[i_]:=Plot3D[fi[i, u, v], {u, 1/2, 1}, {v, 1, 1 + u - u^2}])
G2[i_]:=ImplicitPlot[(u^2 - u + v - 1) fi[i, u, v] == 0,
 {u, 1/2, 1}, {v, 1, 5/4})]
```

For example, you can observe the graph of \(f_{10}(u, v)\) by \(G1[10]\). You can also draw the graph of \(f_{10}(u, v) = 0\) by \(G2[10]\).

\[
f_{10}(u, 1 + u - u^2)\] have a zero of the order 2 at \(u = 1\). Thus, as the above figure, the graph of \(f_{10}(u, v) = 0\) tangents to the parabola \(v = 1 + u - u^2\) at \((1, 1)\), but have no common point with \(D\). Thus we know that \(f_{10}(u, v) > 0\) on \(D\).

We know also \(f_8(u, v) > 0\) on \(D\) similarly.

It is possible to prove \(f_6(u, v) > 0\) on \(D\) directly. \(f_6(u, v)\) can be written as \(f_6(u, v) = \frac{f_{6,1}(u, v)f_{6,2}(u, v)}{u^2v f_{6,3}(u, v)}\), here
\[
\begin{align*}
 f_{6,1}(u, v) &:= 1 - v + v^3 - uv^2 \\
 f_{6,2}(u, v) &:= (1 + v - v^2) + uv \\
 f_{6,3}(u, v) &:= -1 + v + v^3 - v^3 + uv^2.
\end{align*}
\]

It is easy too see that \(f_{6,1}(u, v) > 0, f_{6,2}(u, v) > 0, f_{6,3}(u, v) > 0\) on \(D\). Thus \(f_6(u, v) > 0\) on \(D\). Since \(f_6(u, v) > 0, f_8(u, v) > 0\) and \(f_{10}(u, v) > 0\) on \(D\), we conclude that \(a\) does not contain segments of length 5, 7, or 9.
Corollary 5.2. Assume that \( \min_{x \in K_n} E_n(x) = E_n(a) \) at \( a \in K_n^* \).

1. If \( n = 12 \), then the index of \( a \) must be (11).
2. If \( n = 23 \), then the index of \( a \) must be one of the following 17 indexes: (22), (20, 1),
   (18, 1, 1), (16, 1, 1, 1), (15, 6), (14, 1, 1, 1, 1), (13, 8), (13, 6, 1),
   (12, 1, 1, 1, 1, 1), (11, 10), (11, 8, 1), (11, 6, 1, 1), (10, 1, 1, 1, 1, 1, 1),
   (8, 6, 6), (8, 1, 1, 1, 1, 1, 1, 1), (6, 6, 6, 1), (6, 1, 1, 1, 1, 1, 1, 1, 1).

Definition 5.3. Assume that \( \min_{x \in K_n} E_n(x) = E_n(a) \) at \( a \in K_n^* \), and that \( s = (s_1, s_2, \ldots, s_l) \)
is a \( l \)-segment of \( a \) with \( l \geq 2 \). Then, we define

\[
V_l(a) := 1 + \frac{s_{l-1}}{s_l},
\]

\[
R_l(a) := \frac{s_1}{s_l} = \frac{\text{Head}(s)}{\text{Tail}(s)}.
\]

If there are no segment of length \( l \) in \( a \), we define \( R_l(a) := 1 \). Moreover we define \( R_1(a) := 1 \). By Corollary 3.5, \( V_l(a) \) and \( R_l(a) \) do not depend the choice of \( s \).

Theorem 5.4. Assume that \( \min_{x \in K_n} E_n(x) = E_n(a) \) at \( a \in K_n^* \).

1. If \( a \) contains segment of length 6, then the following holds.
   \[
   1/2 \leq U(a) < 0.63894, \quad R_6(a) < 1/2
   \]
2. If \( a \) contains a segment of length 8, then the following holds.
   \[
   1/2 \leq U(a) < 0.73254, \quad R_8(a) < 0.65994
   \]
3. If \( a \) contains a segment of length 10, then the following holds.
   \[
   0.63893 < U(a) < 0.78332, \quad R_{10}(a) < 0.90213
   \]
4. If \( a \) contains a segment of length 11, then the following holds.
   \[
   0.94197 < U(a) < 1
   \]
5. If \( a \) contains a segment of length 12, then the following holds.
   \[
   0.73253 < U(a) < 0.81295, \quad R_{12}(a) < 1.20768
   \]
6. If \( a \) contains a segment of length 13, then the following holds.
   \[
   0.90868 < U(a) < 1
   \]
7. If \( a \) contains a segment of length 14, then the following holds.
   \[
   0.78331 < U(a) < 0.83098, \quad R_{14}(a) < 1.61530
   \]
8. If \( a \) contains a segment of length 15, then the following holds.
   \[
   1/2 \leq U(a) < 0.63894 \quad \text{or} \quad 0.88942 < U(a) < 0.94198
   \]
9. If \( a \) contains a segment of length 16, then the following holds.
   \[
   0.81294 < U(a) < 0.84220, \quad R_{16}(a) < 2.20409
   \]

Proof. We use the same notation with the proof of Theorem 5.1. Moreover put \( U := U(a) \),
\( V := V_l(a) \), and

\[
D'_i := \{(u, v) \in D \mid f_i(u, v) > 0\},
\]

\[
D_i := D'_2 \cap D'_3 \cap D'_4 \cap \cdots \cap D'_l.
\]
Note that $D'_2 = D'_3 = D'_4 = D'_6 = D'_8 = D'_{10} = D$. 

(1) Consider the case $l = 6$. The graph $\Gamma_7$ of $f_7(u, v) = 0$ on $D$ is as following.

This curve $\Gamma_7$ is the hyper elliptic curve defined by 

$$(2v - 2v^2 - v^3 + v^4) + u(-1 + 2v + v^2 - 2v^3) + u^2 v^2 = 0.$$ 

Thus, we put 

$$f_{7,1}(v) := \frac{(v^2 - 1)(2v - 1) + \sqrt{(v - 1)(v^3 + v^2 + 3v - 1)}}{2v^2}. $$ 

We obtain the intersection of $\Gamma_7$ and the parabola $v = 1 + u - u^2$ on $D$ by solving $f_7(u, 1 + u - u^2) = 0$. This root is $u \approx 0.6389355101$ (rounded up). If $a$ has a 6-segment, then $f_7(U, V) = 0$. Thus $1/2 \leq U < 0.6389355101$. Since $f_6(f_7, 1(v), v)$ is monotonically increasing on $1.15239 < v < 1.23070$, we have 

$$R_6(a) \leq \frac{1}{f_6}(f_7, 1(1.23070), 1.23070) < 0.42657 < 1/2$$

(2) Consider the case $l = 8$. The graph $\Gamma_9$ of $f_9(u, v) = 0$ on $D$ is as following.

We can calculate the root of $f_9(u, 1 + u - u^2) = 0$ with $1/2 \leq u < 1$ by

$$\text{FindRoot}[f_9[u, 1+u-u^2] == 0, \{u, 0.7\}]$$

and we have $u \approx 0.7325361425$ (rounded up). Thus $1/2 \leq U < 0.7325361425$. Execute

$$\text{Plot3D}[1/f_8[u, v], \{u, 1/2, 0.7325361425\}, \{v, 1, 1 + u - u^2\}]$$

Maximize[{1/f_8[0.7325361425, v], 1 < v <= 5/4}, v] // N

and we conclude that

$$\frac{1}{f_8(u, v)} < \frac{1}{f_8(0.73254, 1.10735)} < 0.65994$$
on \( I_9 \cap D \). Thus \( R_8(a) < 0.65994 \).

(3) Consider the case \( l = 10 \). The graph \( \Gamma_{11} \) of \( f_{11}(u,v) = 0 \) on \( D \) is as following.

Thus, \( 0.6389355100 < U < 0.7833151924 \). Since \( 1/f_{10} < 1/f_{10}(0.78332, 1.09863) < 0.90213 \) on \( I_{11} \cap D \), we have \( R_{10}(a) < 0.90213 \).

(4) Consider the case \( l = 11 \). The graph of \( f_{12}(u,v) = 0 \) on \( D \) is a curve connecting \((1, 1)\) and \((0.94197, 1.05466)\) as following.

Thus, \( 0.9419748741 < U < 1 \).

(5) Consider the case \( l = 12 \). The graph \( \Gamma_{13} \) of \( f_{13}(u,v) = 0 \) on \( D \) is as following.

Thus, \( 0.7325361424 < U < 0.8129451277 \). Since \( 1/f_{13}(u, v) < 1/f_{13}(0.81295, 1.08843) < 1.20768 \) on \( I_{13} \cap D \), we have \( R_{12}(a) < 1.20768 \).

(6) Consider the case \( l = 13 \). The graph of \( f_{14}(u,v) = 0 \) on \( D \) is as following. But the curve connecting \((1/2, 1.19728)\) and \((0.55413, 1.24707)\) is included in \( D - D_6 \) on which \( a_6 < 0 \). Thus, we omit this curve.
Thus we have $0.9086897811 < U < 1$.

(7) Consider the case $l = 14$. The graph $\Gamma_{15}$ of $f_{15}(u,v) = 0$ on $D$ is as following.

Thus, $0.7833151923 < U < 0.8309779815$. Since $1/f_{14}(u,v) < 1/f_{14}(0.83098, 1.08039) < 1.61530$, we have $R_{14}(a) < 1.61530$.

(8) Consider the case $l = 15$. The graph $\Gamma_{16}$ of $f_{16}(u,v) = 0$ on $D$ is as following.

Thus, $1/2 \leq U < 0.6389355101$ or $0.8894259160 < U < 0.9419748742$.

(9) Consider the case $l = 16$. The graph $\Gamma_{17}$ of $f_{17}(u,v) = 0$ on $D$ is as following.
Thus, \(0.8129451276 < U < 0.8421985095\). Since \(1/f_{16}(u, v) < 1/f_{16}(0.84220, 1.07460) < 2.20409\) on \(\Gamma_{17} \cap D\), we have \(R_{16}(a) < 2.20409\). 

§6. Proof of Theorem 1.1.

**Theorem 6.1.** Assume that \(\min_{x \in K_{23}} E_{23}(x) = E_{23}(a)\) at \(a \in K_{23}^{*}\). Then the index of \(a\) can not be any of the following values.

1. \((6, 6, 6, 1), (6, 1, 1, 1, 1, 1, 1, 1, 1)\).
2. \((8, 6, 6), (8, 1, 1, 1, 1, 1, 1, 1, 1)\).
3. \((10, 1, 1, 1, 1, 1, 1, 1)\).
4. \((11, 10), (11, 8, 1), (11, 6, 1, 1)\).
5. \((13, 8), (13, 6, 1)\).
6. \((15, 6)\).
7. \((12, 1, 1, 1, 1, 1)\).
8. \((14, 1, 1, 1)\).
9. \((16, 1, 1, 1)\).

**Proof.** We use the same notation with the proof of Theorem 5.1. Let \(U := U(a), R_l := R_l(a),\) and let \(m_i\) be the number of \(l_i\)-segments in \(a\) \((i = 1, \ldots, q)\), and let \(r := m_1 + m_2 + \cdots + m_q\) be the number of segments in \(a\). Then,

\[
U^r R_{m_1} \cdots R_{m_q} = 1. \tag{6.1.1}
\]

(1) In these cases, \(U < 1\), \(R_6 < 1\) by Theorem 5.4 (1). Thus (6.1.1) can not hold.

(2) In these cases, \(U < 1\), \(R_6 < 1\), \(R_8 < 1\) by Theorem 5.4 (1), (2). Thus (6.1.1) can not hold.

(3) In this case, \(U < 1\), \(R_{10} < 1\) by Theorem 5.4 (3). Thus (6.1.1) can not hold.

(4) In these cases, \(0.94197 < U < 1\) by Theorem 5.4 (4). But if \(a\) have a segment of length 10, 8 or 6, then \(0.63893 < U < 0.78332, 1/2 \leq U < 0.73254, 1/2 \leq U < 0.63894\) respectively. There exists no such \(U\).

(5) is similar to (4).

(6) Consider the case \((15, 6)\). \(1/2 \leq U < 0.63894\) and \(R_6(a) < 1/2\) by Theorem 5.4 (1), (8). Execute

\[
\text{Plot3D}[R_{15}[u, v], \{u, 1/2, 0.6389355101\}, \{v, 1, 1 + u - u^2\}]\]

\[
\text{Maximize}[\{R_{15}[0.6389355101, V], 1 <= V <= 5/4, V\}] \quad // \quad N
\]

Thus we have \(1/f_{15}(u, v) < 1/f_{15}(0.63894, 1.09583) < 0.08952\) on the set \(\Gamma_{16} \cap \{(u, v) \in D \mid 1/2 \leq u \leq 0.63894\}\). Thus \(R_{15} < 0.08952\) and (6.1.1) can not hold.

(7) In this case, \(1 = U^6 R_{12} < 0.81295^6 \times 1.20768 < 1\). A contradiction.

(8) In this case, \(1 = U^5 R_{14} < 0.83098^5 \times 1.61530 < 1\). A contradiction.

(9) In this case, \(1 = U^4 R_{16} < 0.84220^4 \times 2.20409 < 1\). A contradiction. 

The left cases are (11) when \(n = 12\), and (22), (20, 1), (18, 1, 1) when \(n = 23\).

**Theorem 6.2.** (1) Assume that \(\min_{x \in K_{12}} E_{12}(x) = E_{12}(a)\) at \(a \in K_{12}^*\). Then the index of \(a\) can not be (11). Thus, Theorem 1.1 (2) holds.
(2) Assume that $\min_{x \in K_{23}} E_{23}(x) = E_{23}(a)$ at $a \in K_{23}$. Then the index of $a$ can not be (22).

**Proof.** We use the same notation with the proof of Theorem 6.1.

(1) We may assume $a = (1, a_2, \ldots, a_{11}, 0)$. Note that $a_{11} = U a_1 = U$. We draw the graph of $f_{11}(u, v) - u = 0$ on $D$.

```
Plot3D[Ai[[11,u,v]]-u, {u, 0.5, 1}, {v, 1, 1.25}]
ImplicitPlot[(u^2-u+v-1) (Ai[[11,u,v]]-u)==0, {u, 0.5, 1}, {v, 1, 1.25}]
```

We obtain the following.

Thus $0.6082388995 < U < 0.6893774937$. But $0.94197 < U < 1$ by Theorem 5.4 (4). Thus the index (11) can not occur.

(2) We may assume $a = (1, a_2, \ldots, a_{21}, 0)$, here $a_{21} = U$. The graph of $f_{23}(u, v) = 0$ and the graph of $f_{22}(u, v) - u = 0$ on $D$ are as following.

The graph $\Gamma_{23}$ of $f_{23}(u, v) = 0$ consists of five parts. The first is the curve connecting $(1/2, 1.20417)$ and $(0.51615, 1.24974)$, the second is $(1/2, 1.12731) - (0.51615, 1)$, the third is $(1/2, 1.02526) - (0.51615, 1)$, the fourth is $(0.84484, 1) - (0.85369, 1)$, and the fifth is $(0.85369, 1) - (0.85369, 1.12491)$. The graph $\Gamma'_{22}$ of $f_{22}(u, v) - u = 0$ consists of three parts. The first is $(0.68507, 1) - (0.72164, 1.20088)$, the second is $(0.75947, 1) - (0.84925, 1)$.
(0.81969, 1.14780), and the third is (0.84484, 1) — (0.83898, 1.13510). As the above figure, 
$I_{23} \cap I'_{22} \cap D = \emptyset$. Thus, $(U, V_{23})$ can not exists if the index of $a$ is (23).

**Theorem 6.3.** Assume that \( \min_{x \in K_{23}} E_{23}(x) = E_{23}(a) \) at \( a \in K_{23}^* \). Then, the index of $a$ can not be any of the following values. Thus, Theorem 1.1 (1) holds.

(1) (18, 1, 1).
(2) (20, 1).

**Proof.** (1) We may assume that $a = (1, a_2, \ldots, a_{18}, 0, a_{20}, 0, a_{22}, 0)$. Let $U := U(a)$ and $V := V_{18}(a)$. Then, $a_{22} = U$, $a_{20} = U^2$, $a_{18} = U^3$, $f_{19}(U, V) = 0$ and $f_{18}(U, V) = U^3$.

The graph of $f_{19}(u, v) = 0$ and the graph of $f_{18}(u, v) - u^3 = 0$ on $D$ are as following. 

![Graph of equations](image)

The graph $I_{19}$ of $f_{19}(u, v) = 0$ consists of two parts. The first is the curve $C_1$ connecting (0.83098, 1) and (0.84925, 1), and the second is (0.84220, 1) — (0.84220, 1.13290). The graph $I'_{18}$ of $f_{18}(u, v) - u^3 = 0$ consists of three parts. The first is (0.55362, 1) — (0.63606, 1.23149), the second is (0.64255, 1) — (0.70658, 1.20733), and the third is the curve $C_2$ connecting (0.84496, 1) and (0.84454, 1.13129). As the above figure, $I_{19} \cap I'_{18} \cap D = C_1 \cap C_2 \sim (0.8391429974, 1.0981287467)$. Thus $U \sim 0.8391429974$ and $V \sim 1.0981287467$. In this case $E_{23}(a) > 11.511 > 23/2 = E_{23}(1, 1, \ldots, 1)$. So, $E_{23}(a)$ can not be minimum.

(2) We may assume $a = (1, a_2, \ldots, a_{20}, 0, a_{22}, 0)$. Let $U := U(a)$ and $V := V_{18}(a)$. Then $a_{22} = U$, $a_{20} = U^2$, $f_{21}(U, V) = 0$ and $f_{20}(U, V) = U^3$.

The graph of $f_{21}(u, v) = 0$ and the graph of $f_{20}(u, v) - u^2 = 0$ on $D$ are as following.
The graph $\Gamma_{21}$ of $f_{21}(u, v) = 0$ consists of three parts. The first is $(1/2, 1.23198) - (0.51615, 1.24974)$, the second is the curve $C_3$ connecting $(0.84220, 1)$ and $(0.85369, 1.12491)$, and the third is $(0.84925, 1) - (0.84925, 1.12803)$. The graph $\Gamma_{20}$ of $f_{20}(u, v) - u^2 = 0$ consists of three parts. The first is $(0.63606, 1) - (0.68507, 1.21575)$, the second is $(0.70658, 1) - (0.75947, 1.18268)$, and the third is the curve $C_4$ connecting $(0.84454, 1)$ and $(0.84484, 1.13108)$. As the above figure, $\Gamma_{21} \cap \Gamma_{20} \cap D = C_3 \cap C_4 \sim (0.8388196493, 1.0346467269)$. Thus $U \sim 0.8388196493$, and $V \sim 1.0346467269$. Then $E_{23}(a) > 11.512 > 23/2 = E_{23}(1, \ldots, 1)$. Thus $E_{23}(a)$ can not be minimum. \hfill \box

References