Theory of PSD Cones on Semialgebraic Varieties.

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Abstract. Let $\mathcal{H} \subset \mathcal{H}_{n,d} := \mathbb{R}[x_1, \ldots, x_n]_d$ be a vector space, and $A$ be a compact semialgebraic subset of $\mathbb{P}^{n-1}_\mathbb{R}$. We shall study some PSD cones $\mathcal{P} = \mathcal{P}(A, \mathcal{H}) := \{ f \in \mathcal{H} \mid f(a) \geq 0 \ (\forall a \in A) \}$. Our interests are (1) to determine the extremal elements of $\mathcal{P}$, and (2) to determine discriminants of $\mathcal{P}$, and (3) to describe $\mathcal{P}$ as a union of basic semialgebraic subsets. For $\mathcal{P}^+_{3,3} = \mathcal{P}(\mathbb{P}^2_+, \mathcal{H}_{3,3})$, we determine all the extremal elements. We complete (1), (2) and (3) for $\mathcal{P}^+_{3,4} = \mathcal{P}(\mathbb{P}^2_+, \mathcal{H}_{3,4}^\mathbb{C})$. We also provide (1) and (2) for $\mathcal{P}^+_{3,4} = \mathcal{P}(\mathbb{P}^2_+, \mathcal{H}_{3,4}^\mathbb{C})$. Let $\mathcal{H}^0_{n,d} := \{ f \in \mathcal{H}_{n,d} \mid f(1, \ldots, 1) = 0 \}$. About four variable inequalities, we also complete (1), (2) and (3) for $\mathcal{P}^0_{4,3} = \mathcal{P}(\mathbb{P}^3_+, \mathcal{H}_{4,3}^\mathbb{C})$. We also prove that an extremal element of $\mathcal{P}$ is determined by the equality conditions considering infinitely near zeros. We also present a basic part of theory of semialgebraic varieties. Our method is based on algebraic geometry.

Section 0. Introduction.

0.1. Aim of this article.

Let $\mathcal{H}_{n,d} := \mathbb{R}[x_1, \ldots, x_n]_d$, $\mathcal{H} \subset \mathcal{H}_{n,d}$ be a vector subspace, and let $A = \mathbb{R}^n$, $\mathbb{R}^n_+$ or semialgebraic set. $\mathcal{P}(A, \mathcal{H}) := \{ f \in \mathcal{H} \mid f(a) \geq 0 \ \text{for all} \ \ a \in A \}$ is called the PSD cone on $A$ in $\mathcal{H}$. Our interests are:

(I1) To determine all the extremal elements of $\mathcal{P} := \mathcal{P}(A, \mathcal{H})$.

(I2) To determine all the discriminants of $\mathcal{P}$.

(I3) To describe $\mathcal{P}$ as a union of basic semialgebraic subsets using some inequalities.

(I4) Find a test set for $(A, \mathcal{H})$.

Historically, the PDS cone $\mathcal{P}_{n,d} := \mathcal{P}(\mathbb{R}^n, \mathcal{H}_{n,d})$ was studied well. Hilbert proved that every extremal element of $\mathcal{P}_{4,4}$ is the square of a quadric polynomial ([19]). Zeros of $f \in \mathcal{P}_{n,d}$ also plays important roll. For $f \in \mathcal{H}_{n,d}$ and $K = \mathbb{R}$ or $\mathbb{C}$, we denote $V_K(f) := \{ (x_1, \ldots, x_n) \in \mathbb{P}^{n-1}_K \mid f(x_1, \ldots, x_n) = 0 \}$.

In [2] and [8], the following theorem was proved.

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Theorem 0.1. If \( f \in \mathcal{P}_{3,6} \) is an exposed extremal element which is not the square of a cubic polynomial, then \( V_C(f) \) is a rational curve which has 10 nodes \( P_1, \ldots, P_{10} \), and \( V_2(f) = \{ P_1, \ldots, P_{10} \} \). Conversely, if \( f \in \mathcal{P}_{3,6} \) and \( V_C(f) \) is an irreducible curve which has 10 nodes in \( \mathbb{P}^2_\mathbb{R} \), then \( f \) is extremal in \( \mathcal{P}_{3,6} \).

It is essential to consider the zero locus of \( f \) on \( \mathbb{P}^2_\mathbb{C} \) not on \( \mathbb{R}^3 \). For this reason, it is better to consider \( \mathcal{P}_{3,6} \) as \( \mathcal{P}_3 = \mathcal{P}(\mathbb{P}^2_\mathbb{R}, \mathcal{H}_{3,6}) \) instead of \( \mathcal{P}_{3,6} = \mathcal{P}(\mathbb{R}^3, \mathcal{H}_{3,6}) \). Moreover, we need to assume that \( A \) is compact with respect to Euclidian topology, otherwise our theory does not proceed. In our process of some proofs, we need to blow up \( A \), or to take a quotient \( A/G \) by a finite group \( G \). So, we need to treat general compact semialgebraic set \( A \). An element \( f \in \mathcal{H}_{n,d} \) is not a function on \( \mathbb{P}^n_\mathbb{R} \). So, a vector space \( \mathcal{H} \) must be considered to be a signed linear system on \( A \). A signed linear system \( \mathcal{H} \) is a linear system on \( A \) such that every \( f(a) \) has a signature for all \( f \in \mathcal{H} \) and \( a \in A \). To define these, we need to construct a certain structure sheaf \( \mathcal{R}_A \) on \( A \), and we call \( (A, \mathcal{R}_A) \) to be a semialgebraic variety. The linear system \( \mathcal{H} \) defines a rational map \( \Phi_A : A \rightarrow \mathcal{P}(\mathcal{H}) \). The closure of its image \( X := \text{Cls}(\Phi_A(A)) \) is called the characteristic variety of \( (A, \mathcal{H}) \). Essentially, \( \mathcal{P} = \mathcal{P}(A, \mathcal{H}) \) is a ‘dual’ of \( X \). We explain this duality. Let \( \Delta(X) = \{ D_1, \ldots, D_r \} \) be the critical decomposition of \( X \) (see Definition 1.2). Then \( X = D_1 \sqcup \cdots \sqcup D_r \), here each \( D_i \) is non-singular semialgebraic variety with \( \partial D_i = \emptyset \). Let \( D_i^\vee \subset \mathcal{P}(\mathcal{H}) \) be the dual semialgebraic variety. The dual semialgebraic variety of \( D_i \) is the set of hypersurfaces in \( \mathcal{P}(\mathcal{H}) \) which tangent to \( D_i \) at a certain point on \( D_i \). It can be regarded as a semialgebraic variety in \( \mathcal{P}(\mathcal{H}) \). If \( r := \text{dim} \mathcal{P} - \text{dim} D_i - 1 > 0 \), then \( \text{Zar}(D_i^\vee) \) (Zariski closure) has \( \mathbb{P}^n_\mathbb{R} \)-ruling structure. \( D_i^\vee \) spans an irreducible semialgebraic variety \( \mathcal{F}(D_i) \subset \mathcal{H} \). If \( F_i := \text{Zar}(\mathcal{F}(D_i)) \cap \mathcal{P} \) satisfies \( \text{dim} F_i = \text{dim} \mathcal{P} - 1 \), we say \( F_i \) is a face component of \( \mathcal{P} \). The defining equation of \( F_i \) is denoted by \( \text{disc}_{D_i} \). If \( \text{dim} D_i = \text{dim} X \), we say \( \text{disc}_{D_i} \) is a main discriminant. If \( \text{dim} X = 2 \) and \( \text{dim} D_i = 1 \), we say \( \text{disc}_{D_i} \) is an edge discriminant.

For example, if \( (A, \mathcal{H}) = (\mathbb{P}^n_\mathbb{R}, \mathcal{H}_{n+1,d}) \), then \( X \cong \mathbb{P}^n_\mathbb{R} \) and \( \Delta(X) = \{ X \} \), because \( \mathcal{H}_{n+1,d} \) is very ample. Since \( X \) is a Veronese variety, \( \text{disc}_X \) has too many terms to print. So, we give up (I2) and (I3) now. But we give some results for (I1) in this article.

If \( (A, \mathcal{H}) = (\mathbb{P}^n_+, \mathcal{H}_{n+1,d}) \), then \( X \) is isomorphic to \( n \)-simplex and \( \Delta(X) \) agree with its simplicial decomposition. Thus, some discriminants appear.

The set of cyclic polynomials \( \mathcal{H}_{n+1,d} := (\mathcal{H}_{n+1,d})^{c_{n+1}} \) is not very ample and \( X \) is an image of \( A/\mathcal{C}_{n+1} \). This \( X \) may have singularities. The critical decomposition is an analogue of simplicial decomposition or CW-decomposition which treat singularities carefully.

The set of symmetric polynomials \( \mathcal{H}_{n+1,d} := (\mathcal{H}_{n+1,d})^{s_{n+1}} \) is not also very ample and \( X \) is an image of \( A/\mathcal{S}_{n+1} \). Its boundary is defined by the discriminant \( \text{Disc}_{n+1} \) of an one-variable polynomial of degree \( n + 1 \).

Discriminants can be computed using the above ‘duality’. But the signature of a discriminant is not always constant on \( \mathcal{P} \). So, to obtain (I3), we need to add some inequalities to divide \( \mathcal{P} \) into some basic semialgebraic sets. These inequalities are called separators. Discriminants are unique up to a multiplication by a constant, but separators are not.

0.2. Main results in this article.

The style of this article is somewhat similaer to a lecture note. So, we introduce what are discussed in each section, and where are main theorems. If unfamiliar symbols appear, please see ‘List of symbols’ at the last of this article.
In §1, we study general theory. Theorem 1.16 will be one of new results. We prove that the edge discriminant of \( P_{3,3}^{-} := P_{3,3} \) agrees with \( \text{Disc}_d(p_0, p_1, \ldots, p_{d-1}, p_0) \).

Theorem 1.23 is an important theorem. This theorem show that extremal inequalities are determined by equality conditions including all infinitely near zeros.

§2 may not provide new results, but we write this section to understand how the notion of discriminants and separators work. Some results of §4 are determined by equality conditions including all infinitely near zeros. Theorem 1.23 is an important theorem. This theorem show that extremal inequalities are determined by equality conditions including all infinitely near zeros.

In §3, we determine all the extremal elements of \( P_{3,3}^{+} := P_{3,3}^{+} \). The main theorem is Theorem 3.18. If an extremal element \( f \) of \( P_{3,3}^{+} \) is irreducible, then \( V_C(f) \) is a rational curve which has a node in \( \text{Int}(P^3) \) (the interior of \( P^3 \)). If \( V_{\mathbb{R}}(f) \subset P^3_{\mathbb{R}} \) contacts to the boundary \( \partial P^3_{\mathbb{R}} \) at \( 2, 3 \) or \( 4 \) points (Theorem 3.9). Such a \( f \) is represented by using \( f_{pqr} \) in Definition 3.14 or \( g_{pq} \) in Definition 3.12. We explain what \( f_{pqr} \) and \( g_{pq} \) are.

AM-GM PSD form \( x^3 + y^3 + z^3 - 3xyz \) lies on \( \partial P^3_{3,3} \), but it is not extremal. \( x^2y + y^2z + z^2x - 3xyz \) and the Schur’s inequality of degree 3 are classical extremal elements of \( P_{3,3}^{+} \).

We had shown in [1] that the following \( f_s(x, y, z) \) is an extremal element of \( P_{3,3}^{+} \) \( (s \geq 0) \).

\[
f_s(x, y, z) := s^2(x^3 + x^3 + z^3) - (2s^3 - 1)(x^2y + y^2z + z^2x) + (s^2 - 2s)(xy^2 + yz^2 + xz^2) - 3(s^4 - 2s^3 + s^2 - 2s + 1)xyz.
\]

This is also extremal in \( P_{3,3}^{+} \). \( f_s(x^2, y^2, z^2) \) is studied in some articles including [9], as important irreducible extremal elements of \( P_{3,6} \). \( f_s \) is characterized by equality conditions \( f_s(1, 1, 1) = f_s(s, 1, 0) = f_s(0, s, 1) = f_s(1, 0, s) = 0 \). It means that if \( 0 \neq f \in P_{3,3}^{+} \) satisfies the above equality conditions, then \( f \) is a positive multiple of \( f_s \). Similarly, our new \( f_{pqr} \) is characterized by \( f_{pqr}(1, 1, 1) = f_{pqr}(0, p, 1) = f_{pqr}(1, 0, q) = f_{pqr}(r, 1, 0) = 0 \), except some degenerate \( (p, q, r) \). Thus \( f_s \) is the case \( p = q = r = s \) in \( f_{pqr} \) (see Remark 3.19). On the other hand, \( g_{pq} \) is a new family of \( E(P_{3,3}^{+}) \). \( g_{pq} \) is characterized by \( g_{pq}(1, 1, 1) = g_{pq}(0, 1, 0) = g_{pq}(0, 0, 1) = 0 \), if \( p > 0 \) and \( q > 0 \).

In §4, we treat \( P_{3,3}^{+} := P_{3,3}^{+} \) and \( P_{3,3}^{0} := P_{3,3}^{0} \). Theorem 4.8 provides (I3) for \( P_{3,3}^{+} \). All extremal elements of \( P_{3,3}^{0} \) are written in Corollary 4.11. All discriminants of \( P_{3,3}^{+} \) are given in Corollary 4.12.

Theorem 4.20 provides (I1) for \( P_{3,3}^{+} \). All discriminants of \( P_{3,3}^{+} \) are given in Proposition 4.21(3). (I3) is not solved. \( \text{disc}(C^b) \) in Proposition 4.21 is too complicated to compute (I3).

\( P_{3,3}^{+} \) and \( P_{3,3}^{0} \) do not have main discriminant, but we prove that \( P_{3,3}^{0} \) has the main discriminant in Theorem 4.25.

In §5, we study \( P_{4,4}^{0} \) and \( P_{4,4}^{0} \). (I1)—(I4) for \( P_{4,4}^{0} \) are given in Theorem 5.4, and these for \( P_{4,4}^{0} \) are given in Theorem 5.7. We present the (I3) part of Theorem 5.4 and 5.7 here in a different style.

**Theorem 0.2.** Let \( \sigma_1 := a_0 + a_1 + a_2 + a_3 \), \( \sigma_2 := \sum_{i<j} a_i a_j \), \( \sigma_3 := \sum_{i<j<k} a_i a_j a_k \), and \( \sigma_4 := a_0 a_2 a_3 \). Consider a family of quartic symmetric polynomials

\[
f(a_0, a_1, a_2, a_3) = \sigma_1^4 + p_1 \sigma_1^2 \sigma_2 + p_2 \sigma_2^2 + p_3 \sigma_1 \sigma_3 - (256 + 96p_1 + 36p_2 + 16p_3)\sigma_4 \in \mathcal{H}_{4,4}^{0}
\]

\( (p_1, p_2, p_3) \in \mathbb{R} \). Then

1. \( f(a_0, a_1, a_2, a_3) \geq 0 \) for all \( a_0, \ldots, a_3 \in \mathbb{R} \) if and only if \( 16 + 6p_1 + 2p_2 + p_3 \geq 0 \) and \( 9p_1^2 \leq 128 + 24p_1 + 36p_2 + 12p_3 \).

2. \( f(a_0, a_1, a_2, a_3) \geq 0 \) for all \( a_0 \geq 0, \ldots, a_3 \geq 0 \) if and only if “(ii)” and “(iii)” or “(iv)” hold.

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(i) \( p_1 \leq -8 \) and \( p_1^2 \leq 4p_2 \).
(ii) \( p_1 \geq -8 \) and \( 4p_1 + p_2 + 16 \geq 0 \).
(iii) \( p_1 \leq -14/3 \) and \( 9p_1^2 \leq 128 + 24p_1 + 36p_2 + 12p_3 \).
(iv) \( p_1 \geq -14/3 \) and \( 27 + 9p_1 + 3p_2 + p_3 \geq 0 \).

Next, we provide (I1).

**Theorem 0.3.** All the extremal elements of \( \mathcal{P}_{4,4}^{a_0} \) are positive multiples of the following polynomials:

\[
g_t(a_0, a_1, a_2, a_3) := 3\sigma_1^4 - 2(t + 7)\sigma_1^2\sigma_2 + (t + 3)^2\sigma_2^2 - 2(t^2 - 9)\sigma_1\sigma_3 - 4(t + 3)^2\sigma_4 \quad (t \in \mathbb{R}),
\]

\[
g_{\infty}(a_0, a_1, a_2, a_3) := \sigma_2^2 - 2\sigma_1\sigma_3 - 4\sigma_4,
\]

\[
p(a_0, a_1, a_2, a_3) := \sigma_2^2 - 3\sigma_1\sigma_3 + 12\sigma_4.
\]

Conversely, these are extremal elements of \( \mathcal{P}_{4,4}^{a_0} \).

\( g_t \) is characterized by the equality conditions \( g_t(t, 1, 1, 1) = g_t(-1, -1, 1, 1) = 0 \).

\( g_{\infty} \) is characterized by the equality conditions \( g_{\infty}(1, 0, 0, 0) = g_{\infty}(-1, -1, 1, 1) = 0 \).

\( p \) is characterized by the equality conditions \( p(s, 1, 1, 1) = 0 \) for all \( s \in \mathbb{R} \).

Here, we say \( f \) is characterized by the equality conditions \( f(x_1) = \cdots = f(x_r) = 0 \) if

\[
\mathbb{R}_+ \cdot f := \{ g \in \mathcal{P} \mid g(x_1) = \cdots = g(x_r) = 0 \}.
\]

Note that \( f \in \mathcal{P} \) is characterized by certain equality conditions, then \( f \) must be extremal. About the converse, please see [A4].

**Theorem 0.4.** All the extremal elements of \( \mathcal{P}_{4,4}^{a_0^+} \) are positive multiples of the following polynomials:

\[
f_{t}^{ab}(a_0, a_1, a_2, a_3) := 3\sigma_1^4 - 2(t + 7)\sigma_1^2\sigma_2 + (t + 3)^2\sigma_2^2 + 8(t + 1)\sigma_4^2
\]

\[
+ (t^2 - 6t + 21)\sigma_1\sigma_3 - 16(t^2 + 3)\sigma_4 \quad (0 \leq t \leq 5),
\]

\[
f_{t}^{b}(a_0, a_1, a_2, a_3) := 9\sigma_1^4 - 6(t + 7)\sigma_1^2\sigma_2 + (t + 7)^2\sigma_2^2
\]

\[
+ 12(t - 1)\sigma_1\sigma_3 - 12(t - 1)(3t + 13)\sigma_4 \quad (t \geq 5),
\]

\[
p(a_0, a_1, a_2, a_3) := \sigma_2^2 - 3\sigma_1\sigma_3 + 12\sigma_4.
\]

\[
q_{1}(a_0, a_1, a_2, a_3) := \sigma_1^2\sigma_2 - 4\sigma_2^2 + 3\sigma_1\sigma_3 - \sum_{i<j}a_ia_j(a_i - a_j)^2,
\]

\[
q_{2}(a_0, a_1, a_2, a_3) := \sigma_1\sigma_3 - 16\sigma_4.
\]

Conversely, these are extremal elements of \( \mathcal{P}_{4,4}^{a_0^+} \).

\( f_{t}^{ab} \) is characterized by the equality conditions \( f_{t}^{ab}(t, 1, 1, 1) = f_{t}^{ab}(0, 0, 1, 1) = 0 \).

\( f_{t}^{b} \) is characterized by the equality conditions \( f_{t}^{b}(t, 1, 1, 1) = f_{t}^{b}(0, 0, u, 1) = 0 \), where \( u \in \mathbb{R}_+ \) is a root of \( 3u^2 - (t + 1)u + 3 = 0 \).

\( q_{1} \) is characterized by the equality conditions \( q_{1}(1, 1, 1, 0) = q_{1}(1, 1, 0, 0) = q_{1}(1, 0, 0, 0) = 0 \).

\( q_{2} \) is characterized by the equality conditions \( q_{2}(s, 1, 0, 0) = 0 \) for all \( s \geq 0 \).

We should explain what is the discriminants of \( \mathcal{P} = \mathcal{P}(A, \mathcal{H}) \). Let \( s_0, s_1, \ldots, s_N \) be a basis of the vector space \( \mathcal{H} \), and let \( \Phi_{\mathcal{H}}: A \to \mathbb{P}_N \) be the rational map defined by \( \Phi_{\mathcal{H}}(a) = (s_0(a): \cdots: s_N(a)) \). \( X := \Phi_{\mathcal{H}}(A) \) is called the characteristic variety. Let \( \Delta(X) = \)
\{D_1, \ldots, D_r\} be the critical decomposition of \(X\) (see Definition 1.2). Each \(D \in \Delta(X)\) is a smooth semialgebraic variety, and \(D\) has its dual variety \(D^\vee\). Let \(\text{disc}(D)\) be the defining equation of the Zariski closure of \(D^\vee\) in \(\mathcal{H}\), and let \(\partial \mathcal{H}(\text{disc}(D))\) be the zero locus of \(\text{disc}(D)\) in \(\mathcal{H}\). If \(\dim (\partial \mathcal{H}(\text{disc}(D)) \cap \partial \mathcal{P}) = \dim \mathcal{P} - 1\), we say \(\text{disc}(D)\) is the discriminant of \(\mathcal{P}\). Assume that a subset \(B \subseteq A\) satisfies \(\Phi_{\mathcal{H}}(B) = D\). Then, for each \(f \in \partial \mathcal{H}(\text{disc}(D)) \cap \partial \mathcal{P}\), there exists a point \(a \in B\) such that \(\Delta(a) = 0\). In this case, we shall say that \(\text{disc}(D)\) is a discriminant corresponding to \(B\).

**Theorem 0.5.** Let denote the elements of \(\mathcal{P}_{n,d}^{0+}\) as

\[
f(a_0, a_1, a_2, a_3) = p_0 \sigma_1^2 + p_1 \sigma_1^2 \sigma_2 + p_2 \sigma_2^2 + p_3 \sigma_1 \sigma_3 - (256p_0 + 96p_1 + 36p_2 + 16p_3)\sigma_4,
\]

and use \((p_0, \ldots, p_3)\) as a coordinate system of \(\mathcal{P}_{n,d}^{0+}\).

(1) \(\mathcal{P}_{4,4}^{0+}\) has the following two discriminants:

\[
d_1 := 128p_0 + 24p_1 + 36p_2 + 12p_3 - 9p_1^2, \quad d_2 := 16p_0 + 6p_1 + 2p_2 + p_3.
\]

\(d_1\) corresponds to \(\{(t, 1, 1, 1) \in \mathbb{R}^4 \mid t \in \mathbb{R}, t \neq -3, 1\}\), and \(d_2\) corresponds to a point \((1, 1, -1, -1)\).

(2) \(\mathcal{P}_{4,4}^{0+}\) has the following five discriminants:

\[
d_1 := 128p_0 + 24p_1 + 36p_2 + 12p_3 - 9p_1^2, \quad d_2 := 4p_2 - p_1, \quad d_3 := 27p_0 + 9p_1 + 3p_2 + p_3, \quad d_4 := 16p_0 + 4p_1 + p_2 + 16, \quad d_5 := p_0.
\]

\(d_3\) corresponds to \(\{(0, 0, t, 1) \in \mathbb{R}^4 \mid 0 < t < 1\}\), \(d_4, d_5, d_6\) corresponds to points \((1, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 0)\) respectively.

(14) is an easy corollary of these results.

**Theorem 0.6.** (1) If \(f \in \mathcal{P}_{n,d}^{0}\) satisfies \(f(-1, -1, 1, 1) \geq 0\) and \(f(t, 1, 1, 1) \geq 0\) for all \(t \in \mathbb{R}\), then \(f(a, b, c, d) \geq 0\) for all \(a, b, c, d \in \mathbb{R}\).

(2) If \(f \in \mathcal{P}_{n,d}^{0}\) satisfies \(f(t, 1, 1, 1) \geq 0\) and \(f(0, 0, t, 1) \geq 0\) for all \(t \geq 0\), then \(f(a, b, c, d) \geq 0\) for all \(a, b, c, d \in \mathbb{R}_+\).

For general \(f \in \mathcal{P}_{n,d}^{0}\), Riemer, Timofte, Harris proved that \(f \in \mathcal{P}_{n,d}^{0}\) if \(f(x) \geq 0\) for all \(x \in \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \#\{x_1, \ldots, x_n\} \leq r_0\}\). Moreover, \(f \in \mathcal{P}_{n,d}^{0+}\) if \(f(x) \geq 0\) for all \(x \in \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \#\{x_1, \ldots, x_n\} - \{0\} \leq r_0\}\). (see Corollary 1.3 of [27], Corollary 2.1 of [28].)

In the case \(n = d = 4\), the above test condition are \(f(t, t, 1, 1) \geq 0\) and \(f(t, 1, 1, 1) \geq 0\) \((\forall t \in \mathbb{R})\) in the case \(f \in \mathcal{P}_{4,4}^{0+}\), and \(f(t, 1, 1, 1) \geq 0\) and \(f(0, 0, 0, 0) \geq 0\) \((\forall t \in \mathbb{R}_+)\) in the case \(f \in \mathcal{P}_{4,4}^{0+}\). We can find that the number test conditions are decreased in the case of \(f \in \mathcal{P}_{n,d}^{0+}\).

In §6, we study \(\mathcal{P}_{4,3}^{0+}\). (I2), (I3) and (I4) for \(\mathcal{P}_{4,4}^{0+}\) are given in Theorem 6.1. (II) is provided in Theorem 6.11. These theorems are very complicated to state here. Please read Theorem 6.1 and in Theorem 6.11.

In §7, we discuss about semialgebraic varieties. We didn’t explain about semialgebraic varieties so precisely in [1]. So, we study some basic properties of semialgebraic varieties here. Theorem 7.11 will be useful for study of real algebraic varieties. Corollary 7.14 may be known, but the author can not find it.
We shall explain a short history of study of PSD cones. Originally, \( P_{n,2d} \) is called a PSD cone, and \( \Sigma_{n,2d} \) is called a SOS cone. Hilbert proved, \( P_{n,3d} = \Sigma_{n,2d} \) if and only if \( n \leq 2 \) or \( 2d = 2 \) or \( (n, 2d) = (3, 4) \) ([19]). History of studies till 1991 are written in the section 6.6 of [6]. So we don’t explain them again. Choi and Lam found some extremal forms of \( P_{n,2d} \) which don’t belong to \( \Sigma_{n,2d} \) in [7]. In [26], Reznick studied not only the condition for \( f \in P_{n,2d} \) is included in \( \Sigma_{n,2d} \), but also the condition that \( f \in P_{n,2d} \) is extremal. In Theorem 1.23 of this article, we give another type of condition that \( f \in P(A, \mathcal{H}) \) is extremal.

An element \( f \in H_{n,2d} \) is called even, if each term of \( f \) is of the form \( x_1^{2e_1} \cdots x_n^{2e_n} \). Choi, Lam and Reznick studied cones of even symmetric forms \( P_{n,2d}^{es} \) in [9]. They studied the condition for \( P_{n,2d}^{es} = \Sigma_{n,2d}^{es} \). Note that \( P_{n,2d}^{es} \cong P_{n,d}^{es} \). In the case \( (n, d) = (3, 3) \) and \( (4, 3) \), Theorem 3.10 and 3.18 in this article will give more precise information. Harris proved \( P_{3,8}^{es} = \Sigma_{3,8}^{es} \) in [18]. Theorem 4.8 in this article determines the structure of \( P_{3,8}^{es} \cong P_{3,4}^{es} \). The relations \( P_{n,2d}^{es} \) and \( \Sigma_{n,2d}^{es} \) are studied by Goel, Kuhlmann and Reznick in [15] and [16]. A related study can be found in [5]. In this article, we don’t treat SOS cones \( \Sigma_{n,2d} \) but studies of \( E(P_{n,2d}) \) or \( E(P_{n,d}^+) \) give some useful results for \( E(\Sigma_{n,2d}) \).

About discriminants of \( P(A, \mathcal{H}) \), Nie shown some interesting results in [24]. He treated the case that \( A \) is an affine real algebraic quasi-variety. In this article, we only treat the cases that \( A \) is a compact semialgebraic quasi-variety. But they have very close relation. Compare with Nie’s results with Algorithm 1.11 in this article. [4] also provides many nice ideas. About discriminants of \( P_{3,6}, \Sigma_{3,6}, P_{4,4} \) and \( \Sigma_{4,4} \), very interesting results are obtained in [2]. In general, some discriminants of \( P(A, \mathcal{H}) \) have too many terms if \( \text{dim} \mathcal{H} \) is large. After §4 of this article, we only treat the cases \( \text{dim} \mathcal{H} \leq 6 \). In some of such cases, it is possible to determine discriminants.

### Section 1. General theories.

#### 1.1. Known results.

We explain some notations and some results in §1 and §2 in [1]. The precise definition of a semialgebraic quasi-variety will be discussed in §7. As a temporary, we may understand that a semialgebraic quasi-variety \( (A, \mathcal{R}_A) \) is a locally ringed space with semialgebraic set \( A \) and a sheaf of rings \( \mathcal{R}_A \) which represent real holomorphic functions on open subsets of \( A \). \( \mathcal{R}_A \) is similar as §3.2 in [6]. Using \( \mathcal{R}_A \), we define singular points of \( A \), and regular maps between semialgebraic quasi-varieties, and so on.

Let \( A \) be \( \mathbb{P}^n_\mathbb{R} \) or \( \mathbb{P}^n_+ \) or a compact semialgebraic quasi-variety, where

\[
\mathbb{P}^n_+ := \{(x_0, \ldots, x_n) \in \mathbb{P}^n_\mathbb{R} \mid x_i x_j \geq 0 \text{ for all } 0 \leq i < j \leq n\}.
\]

Note that after Proposition 1.8, we need to assume that \( A \) is compact. So, we cannot use \( A = \mathbb{R}^n \) or \( \mathbb{R}^n_+ \) in our theory.

For an arbitral invertible \( \mathcal{R}_A \)-module \( \mathcal{L} \), a finite dimensional vector subspace \( \mathcal{H} \) of \( \mathcal{L}(A) \) is called a signed linear system on \( A \) if \( \text{sign}(f(P)) \in \{\pm 1, 0\} \) is defined suitably for all \( f \in \mathcal{H} \), and \( P \in A \) (the strict definition is given at Definition 7.15). For example,

\[
\mathcal{H}_{n+1,d} := \{f(x_0, \ldots, x_n) \mid f \text{ is a homogeneous polynomial of degree } d\} \cup \{0\}
\]

is a signed linear system on \( \mathbb{P}^n_\mathbb{R} \). For \( f \in \mathcal{H}_{n+1,d} \) and \( P \in \mathbb{P}^n_+ \), we cannot define the value \( f(P) \) but can define \( \text{sign}(f(P)) \). If \( d \) is even, \( \mathcal{H}_{n+1,d} \) is also a signed linear system on \( \mathbb{P}^n_\mathbb{R} \).

\[
P(A, \mathcal{H}) := \{f \in \mathcal{H} \mid f(a) \geq 0 \text{ for all } a \in A\}
\]
is called the \textit{PSD cone} on $A$ in $\mathcal{H}$. For $f \in \mathcal{H}$, we denote $V(f) = V_A(f) := \{ a \in A \mid f(a) = 0 \}$.

$$\text{Bs} \mathcal{H} := \bigcap_{f \in \mathcal{H}} V_A(f) = \{ x \in A \mid f(x) = 0 \text{ for all } f \in \mathcal{H} \}$$

is called to be the \textit{base locus} of $\mathcal{H}$. Assume that $U := A - \text{Bs} \mathcal{H} \neq \emptyset$, $N := \dim_{\mathbb{R}} \mathcal{H} - 1 \geq 1$. Take a base \{ $s_0, s_1, \ldots, s_N$ \} $\subset \mathcal{H}$ as a $\mathbb{R}$-vector space. Let $\Phi_{\mathcal{H}} : U \rightarrow \mathbb{P}(\mathcal{H}^\vee)$ be the regular map defined by $\Phi(x) = (s_0(x); \cdots; s_N(x))$ for $x \in U$. The Euclidean closure of $\Phi_{\mathcal{H}}(U)$ in $\mathbb{P}(\mathcal{H}^\vee)$ is denoted by $X(A, \mathcal{H})$, and is called the \textit{characteristic variety} of $\mathcal{P}(A, \mathcal{H})$.

\textbf{Proposition 1.1.} Let $X := X(A, \mathcal{H})$, and let $Y$ be the convex closure of $X$ in $\mathbb{P}(\mathcal{H}^\vee)$. Then

$$\mathcal{P}(A, \mathcal{H}) = \mathcal{P}(X, \mathcal{H}_{N+1,1}) = \mathcal{P}(Y, \mathcal{H}_{N+1,1}),$$

where $\mathcal{H}_{N+1,1}$ is the set of linear polynomials on $\mathbb{P}(\mathcal{H}^\vee)$.

\textbf{Proof.} $\mathcal{P}(A, \mathcal{H}) = \mathcal{P}(X, \mathcal{H}_{N+1,1})$ is proved at Proposition 1.13 in [1]. $\mathcal{P}(X, \mathcal{H}_{N+1,1}) = \mathcal{P}(Y, \mathcal{H}_{N+1,1})$ is clear since every element of $\mathcal{H}_{N+1,1}$ is linear. \hfill \Box

Assume that a semialgebraic set $B$ is a subset of a complete real algebraic quasi-variety $V$. The minimal algebraic subset which contain $B$ is called the Zariski closure of $B$ and is denoted by $\text{Zar}_V(B)$. We denote the Euclidean closure of $B$ in $V$ by $\text{Cls}_V(B)$ or $\overline{B}$. Assume that $\text{Zar}_V(B) = V$. The interior of $B$ is defined by $\text{Int}(B) := V - \text{Cls}_V(V - B)$. The boundary of $B$ is defined by $\partial B := B - \text{Int}(B)$. Do not confuse with $\partial_{\text{V}}B := \text{Cls}_V(B) - \text{Int}(B)$. Note that $\text{Int}(B)$ and $\partial B$ does not depend on the choice of $V$. But $\text{Cls}_V(B)$ and $\partial_{\text{V}}B$ depend on $V$.

\textbf{Definition 1.2.} (Critical decomposition. See Definition 1.5 of [1]) Let $A$ be a reduced semialgebraic quasi-variety with $\dim A = n$. We shall define $\Delta^i(A)$ ($i = 0, \ldots, n$) by induction on $n$. If $\dim A = 0$, then $A = \{ P_1, \ldots, P_m \}$ where $P_i$ are points. In this case we put $\Delta^0(A) = \{ P_1, \ldots, P_m \}$, and put $\Delta^i(A) = \emptyset$ for $i \neq 0$.

Assume that $n = \dim A \geq 1$. Let $Z_1, \ldots, Z_r$ be all the irreducible components of $A$ with $\dim Z_i = n$. Put $A_i := \text{Int}(Z_i - \text{Sing}(A))$, and $\Delta^n(A) := \{ A_1, \ldots, A_r \}$. Note that $Z_i \cap Z_j \cap \text{Int}(A) \subset \text{Sing}(A)$ for $i \neq j$.

Let $Y_1, \ldots, Y_k$ be all the irreducible components of $A$ with $\dim Y_j \leq n - 1$, and let $B_j := Y_j - \{ A_1 \cup \cdots \cup A_r \}$. Put

$$B := \text{Sing}(A) \cup \partial A \cup B_1 \cup \cdots \cup B_k.$$

Then, we can regard $B$ to be a semialgebraic quasi-subvariety of $A$ with the reduced structure. Note that $\dim B < \dim A$. Thus we put $\Delta^i(A) := \Delta^i(B)$ for $i \neq n$.

We denote $\Delta(A) := \Delta^0(A) \cup \Delta^1(A) \cup \cdots \cup \Delta^n(A)$, and is called a \textit{critical decomposition} of $A$. Each element $D \in \Delta(A)$ is called a \textit{critical set} of $A$. Note that $D$ is a non-singular semialgebraic variety with $\partial D = \emptyset$.

\textbf{Definition 1.3.} (Dual variety. See Definition 1.17 of [1]) Let $\mathbb{P} = \mathbb{P}^N_{\mathbb{R}}$ and $\mathbb{P}^\vee$ be the set of all the hyperplanes in $\mathbb{P}$. Assume that $D \subset \mathbb{P}$ is a non-singular semialgebraic variety with $\partial D = \emptyset$ (i.e. $\Delta(D) = \{ D \}$). For $x \in D$, let $T_{D,x} := T_{\text{Zar}(D),x} \subset \mathbb{P}$ be the tangent space of $\text{Zar}(D)$ at $x$. Then,

$$D^\vee := \{ H \in \mathbb{P}^\vee \mid H \supset T_{D,x} \text{ for a certain } x \in D \}$$
is called the dual variety of $D$. Since $D$ is irreducible and non-singular, $D^\vee$ is irreducible. Thus $D^\vee$ is a semialgebraic variety. Note that $D^\vee$ may have singularities.

**Theorem 1.4.** (Theorem 1.18 of [1]) Let $X \subset \mathbb{P} = \mathbb{P}^N$ be a closed semialgebraic quasi-variety, $\mathcal{H} := \mathcal{H}_{N+1,1}$, $\mathcal{P} := \mathcal{P}(X, \mathcal{H})$, and $\pi : (\mathcal{H} - \{0\}) \to \mathbb{P}(\mathcal{H})$ be the natural surjection. Put $\mathbb{P}(\mathcal{P}) := \pi(\mathcal{P} - \{0\}) \subset \mathbb{P}(\mathcal{H})$. Then,

$$\partial \mathbb{P}(\mathcal{P}) \subset \bigcup_{D \in \Delta(X)} D^\vee.$$

**Definition 1.5.** For $D \in \Delta(X)$, we denote

$$\mathcal{F}(D) := \text{Cls}_{\mathcal{H}_{N+1,1}}(\pi^{-1}(D^\vee) \cap \partial \mathcal{P}).$$

If $\mathcal{F}(D) \in \Delta^{N-1}(\partial \mathcal{P})$, then $\mathcal{F}(D)$ is called a face component of $\mathcal{P}$ or of $\partial \mathcal{P}$, and an irreducible defining equation of the Zariski closure $\text{Zar}(\mathcal{F}(D))$ is called a discriminant of $\mathcal{P}$, and denoted $\text{disc}_D$ or $\text{disc}(D)$.

Especially, if $D \in \Delta^\dim(X)$ and $\mathcal{F}(D)$ is a face component, then $\mathcal{F}(D)$ is called a main component of $\mathcal{P}$, and $\text{disc}(D)$ is called a main discriminant of $\mathcal{P}$. In the case $D$ is a point, $\text{disc}(D)$ is linear and $\text{disc}(D)$ usually indicates a trivial condition. So, we sometimes ignore it.

**Definition 1.6.** For $I \subset A$, we put

$$\mathcal{H}_I := \{ f \in \mathcal{H} \mid f(a) = 0 \text{ for all } a \in I \}, \quad \mathcal{P}_I := \mathcal{P} \cap \mathcal{H}_I = \mathcal{P}(A, \mathcal{H}_a).$$

Especially, when $I = \{a\}$, we denote $\mathcal{P}_a := \mathcal{P}_I$. $\mathcal{P}_a$ is called the local cone of $\mathcal{P}$ at $a$.

If $0 \neq f \in \mathcal{P}$ is said to be extremal in $\mathcal{P}$, if $g$ and $h \in \mathcal{P}$ satisfy $f = g + h$ then $g$ and $h$ are multiples of $f$. For a PSD cone $\mathcal{P}$, we denote that

$$\mathcal{E}(\mathcal{P}) := \{ f \in \mathcal{E} \mid f \text{ is extremal in } \mathcal{E} \}. $$

$f \in \mathcal{E}(\mathcal{P})$ is said to be exposed if there exists a hyperplane $H \subset \mathcal{H}$ such that $H \cap \mathcal{P} = \mathbb{R}_+ \cdot f$, where $\mathbb{R}_+ := \{ a \in \mathbb{R} \mid a \geq 0 \}$.

For $f, g \in \mathcal{E}(\mathcal{P})$, we often identify them if there exists a constant $c > 0$ such that $f = cg$. We often omit the sentence ‘(unique) up to a multiplication by a (positive) constant.’ For example, we sometimes say ‘all the extremal elements of $\mathcal{P}$ are $e_1, \ldots, e_1$’, or ‘all the discriminants of $\mathcal{P}$ are $d_1, \ldots, d_k$’.

Let $Y$ be the convex closure of $X := X(A, \mathcal{H})$. Theoretically, we can determine $\mathcal{P}(X, \mathcal{H}_{N+1,1})$ or $\mathcal{P}(Y, \mathcal{H}_{N+1,1})$ by the following algorithm.

**Algorithm 1.7.** Let $X \subset \mathbb{P}^N$ and $\mathcal{H} := \mathcal{H}_{N+1,1}$ be same with Theorem 1.4.

Step 1. Let $X = X(A, \mathcal{H})$ or its convex closure. Determine $X$ and $\Delta(X)$ using algebraic geometry.

Step 2. For every $D \in \Delta(X)$, determine the extremal elements of $\mathcal{F}(D)$, and calculate the dual variety $\text{Zar}(D^\vee) = \text{Zar}(D)^\vee$, according to Algorithm 1.11. Usually, we need a computer here.

Step 3. $S := \bigcup_{D \in \Delta(X)} \text{Zar}(D^\vee)$ cuts $\mathcal{H}$ into blocks $B_1, \ldots, B_k$ such that $\partial B_i \subset S$ and $\text{Int}(B_i) \cap S = \emptyset$. Find out convex cone $\mathcal{P}$ which is a union of some $\overline{B}_i$’s. If there exists
\( f \in B \) such that \( f(a) < 0 \) for a certain \( a \in A \), then \( \text{Int}(B) \cap \mathcal{P} = \emptyset \). Contrary, if there exists \( 0 \neq f \in B \) such that \( f \in \mathcal{P} \), then \( \overline{B} \subset \mathcal{P} \).

Step 4. Obtained \( \mathcal{P} \) may not be a basic semialgebraic set. In such a case, find out a nice decomposition of \( \mathcal{P} \) into basic semialgebraic subsets. To divide \( \mathcal{P} \), we need some new polynomials which are not discriminants. Such polynomials are called separators. Note that discriminants are unique up to multiplication by non-zero constant, but there may be many possibility of the choice of separators.

When \( \dim \mathcal{P} \) or \( \deg \mathcal{F}(D) \) (\( \exists D \in \Delta(X) \)) is large, Step 4 is usually difficult. In such a case, we should be satisfied to complete Step 2.

**Proposition 1.8.** (Boundary Theorem. See Proposition 1.16 of [1]) Let \( A \) be a compact semialgebraic quasi-variety, and \( \mathcal{H} \) be a signed linear system on \( A \). Assume that \( \mathcal{P} := \mathcal{P}(A, \mathcal{H}) \subset \mathcal{H} \) is non-degenerate, and \( \dim \mathcal{P} \geq 2 \). Let \( f \in \mathcal{P} \).

1. If \( f(a) = 0 \) for a certain \( a \in A - \text{Bs} \mathcal{H} \), then \( f \notin \partial \mathcal{P} \).
2. If \( f \in \partial \mathcal{P} \), then there exists \( a \in A \) such that \( f(a) = 0 \).

**Proof.** Since the proof of (2) in [1] is too rough, we present precise proof of (2) here.

(2) Let \( \{s_0, \ldots, s_N\} \) be a base of \( \mathcal{H} \) such that \( s_0, \ldots, s_N \in \mathcal{P} \), and define \( \Phi_{\mathcal{H}} : A \to X \subset \mathbb{P}_R^N \) by \( s_0, \ldots, s_N \). We may assume that \( A = X \). Put

\[
W_i := \{(X_0 ; \ldots ; X_N) \in \mathbb{P}_R^N | X_0^2 + \cdots + X_N^2 \leq 3X_i^2 \}.
\]

Then \( W_0 \cup \cdots \cup W_N = \mathbb{P}_R^N \).

Assume that \( f \in \mathcal{P} \) satisfies \( f(a) > 0 \) for all \( a \in \overline{A} = X \). Take \( g \in \text{Int}(\mathcal{P}) \). We can regard \( f_i := f/X_i \) and \( g_i := g/X_i \) as holomorphic functions on \( W_i \). Since \( W_i \) is compact, there exists \( \varepsilon > 0 \) such that \( f_i(a) \pm \varepsilon g_i(a) > 0 \) for all \( a \in \overline{A} \cap W_i \). Put \( \varepsilon := \min\{\varepsilon_0, \ldots, \varepsilon_N\} \). Then \( f \pm \varepsilon g \in \mathcal{P} \). Thus \( f \notin \partial \mathcal{P} \). \( \square \)

**Proposition 1.9.** (Local Cone Theorem. See Proposition 1.26 of [1]) Let \( A \) be a compact semialgebraic quasi-variety, and \( 0 \neq \mathcal{H} \) be a signed linear system on \( A \). Then,

1. \( \partial \mathcal{P} \subset \bigcup_{a \in A} \mathcal{P}_a \).
2. If \( a \notin \text{Bs} \mathcal{H} \), then \( \mathcal{P}_a \subset \partial \mathcal{P} \). (If \( \mathcal{P}_a = 0 \), we regard \( 0 \subset \partial \mathcal{P} \).)
3. Assume that \( A \) is irreducible and \( A - \text{Bs} \mathcal{H} \neq \emptyset \). Then \( \partial \mathcal{P} = \text{Cl}_{\mathcal{H}} \left( \bigcup_{a \in A - \text{Bs} \mathcal{H}} \mathcal{P}_a \right) \).
4. Let \( 0 \neq f \in \mathcal{P}_a \). \( f \) is extremal in \( \mathcal{P} \) if and only if \( f \) is extremal in \( \mathcal{P}_a \).

The author should apologize for that Proposition 1.27 of [1] is not correct. Proposition 4.23 in this article presents a counter example. It should be modified as follow:

**Proposition 1.10.** (Face Component Theorem.) Let \( X \) be a closed semialgebraic subset of \( \mathbb{P}_R^N \) such that \( X \) is not included in any proper linear subspace of \( \mathbb{P}_R^N \). Assume that \( \mathcal{P} := \mathcal{P}(X, \mathcal{H}_{N+1,1}) \) is non-degenerate in \( \mathcal{H}_{N+1,1} \). Take \( x \in D \in \Delta^r(X) \). Then:

1. \( \dim \mathcal{P}_x \leq N - r \).
2. \( \mathcal{F}(D) = \text{Cl}_{\mathcal{H}} \left( \bigcup_{x \in D} \mathcal{P}_x \right) \).
Proof. For \( f \in \mathcal{H} := \mathcal{H}_1 \), let \( H_f \) be the hyperplane in \( \mathbb{P}(\mathcal{H}) \) defined by \( f = 0 \).

(1) Since \( \mathcal{P} \) is non-degenerate, \( \dim(U \cap \mathcal{P}) = N + 1 \) for any Euclidean open neighborhood \( U \) of \( x \). Let \( \mathcal{L} := \{ f \in \mathcal{H} \mid T_{D,x} \subset H_f \} \). Note that \( \dim T_{D,x} = \dim D = r \leq N + 1 \), since \( D \) is non-singular. The condition \( T_{D,x} \subset H_f \) means that \( f \) passes through independent \( r + 1 \) points. Thus, \( \dim \mathcal{L} = \dim \mathcal{H} - (r + 1) = N - r \). Since \( \mathcal{P}_x = \mathcal{P} \cap \mathcal{L} \), we have \( \dim \mathcal{P}_x \leq N - r \).

(2) \( \circ \) is clear. We prove \( \circ \). Take \( f \in D^V \subset \text{Int}(\mathcal{F}(D)) \). Then, \( f(x) = 0 \) for a certain \( x \in D \). That is, \( f \in \mathcal{P}_x \).

We explain an algorithm to calculate local cones and discriminants.

Algorithm 1.11. Let \( A \) be a compact reduced semialgebraic quasi-variety, \( \mathcal{H} \) be a signed linear system on \( A \), and let \( X = X(A, \mathcal{H}) \) or its convex closure.

(1) Let \( a \in \mathcal{A} \), and \( x := \Phi_{\mathcal{H}}(a) \in D \in \Delta^\nu(X) \). Let \( B := \Phi_{\mathcal{H}}^{-1}(D) \subset A \). Assume that \( \Phi_{\mathcal{H}} : B \to D \) is a finite surjective unramified morphism, and there exists a local coordinate system \((t_1, \ldots, t_r)\) at a certain neighborhood of \( a \) in \( B \).

Let \( \{s_0, \ldots, s_N\} \) be a base of \( \mathcal{H} \). Identify \( p_0s_0 + \cdots + p_Ns_N \in \mathcal{H} \) and \((p_0, \ldots, p_N) \in \mathbb{R}^{N+1} \). We take \((p_0: \cdots: p_N)\) as a homogeneous coordinate system of \( \mathbb{P}(\mathcal{H}) \). Then, \( T_{D,x} := \left\{ \left( \cdots : s_i(a) + \sum_{j=1}^r v_j \frac{\partial s_i}{\partial t_j}(a) : \cdots \right) \in \mathbb{P}^n_\mathbb{R} \mid (v_1, \ldots, v_r) \in \mathbb{R}^r \right\} \).

The defining equation of the hypersurface \( \text{Zar}(\mathcal{F}(D)) \) is denoted by \( \text{disc}(D) = \text{disc}(p_0, \ldots, p_N) \). If \( D^V \subset \partial \mathbb{P}(\mathcal{P}) \), \( \text{disc}(D) \) is called a discriminant of \( D \) or of \( \mathcal{P} \). Note that \( \text{disc}(D) \) can be obtained eliminating \( t_1, \ldots, t_r \) from the system of equations
\[
\sum_{i=0}^N p_is_i(a(t_1, \ldots, t_r)) = 0 \quad \text{and} \quad \sum_{i=0}^N p_i \frac{\partial s_i}{\partial t_j}(a(t_1, \ldots, t_r)) = 0 \quad (j = 1, \ldots, r).
\]
Here \( a(t_1, \ldots, t_r) \) is the function which represent the coordinate of \( a \). Consider (1.11.1) as a system of linear equations on \((p_0, \ldots, p_N)\). Then, its solution space is just \( \text{Zar}(\mathcal{P}_a) \) if \( \mathcal{P}_a \neq 0 \).

(2) Especially, consider the special case \( \dim D = N - 1 \). Then, \( \text{Zar}(D) \) is a hypersurface of \( \mathbb{P}^N_\mathbb{R} \) defined by a certain irreducible polynomial \( h(x_0, \ldots, x_N) \). Let \( h_i = \frac{\partial h}{\partial x_i} \). Then \( T_{D,x} = \{ (x_0 : \cdots : x_N) \in \mathbb{P}^N_\mathbb{R} \mid h_0(a)x_0 + \cdots + h_N(a)x_N = 0 \} \).

Thus, \( \text{disc}(D) = \text{disc}(h_0, \ldots, h_N) \) of \( \text{Zar}(D^V) \) can be obtained eliminating \( x_0, \ldots, x_N \) from the system of equations \( p_0x_0 + \cdots + p_Nx_N = 0 \) and \( p_i = h_i(x_0, \ldots, x_N) \) \((i = 0, \ldots, N)\).

(3) Assume that \( D \in \Delta^0(X) \), and \( D = (b_0 : \cdots : b_N) \). Then, \( \text{Zar}(D^V) \) is the hyper plane defined by \( b_0p_0 + \cdots + b_Np_N = 0 \). Thus, \( \text{disc}(D) = b_0p_0 + \cdots + b_Np_N \).

Remark 1.12. Consider the case \( A = \mathbb{P}^n_\mathbb{R} \) (or \( A = \mathbb{P}^n_+ \)), and \( \dim X = n \). Let \( D := \text{Int}(\text{Reg}(X)) \in \Delta^n(X) \). Then to solve the system of equation \( \sum_{i=0}^N p_i \frac{\partial s_i}{\partial t_j}(a) = 0 \) for all \( j = 0, \ldots, n \) is just the method to obtain extreme points of \( f = \sum_{i=0}^N p_is_i \) in EVA (Extreme Value Analysis). Thus the main discriminant \( \text{disc}(D) \) corresponds to the solution of EVA. Note that \( \dim \mathcal{F}(D) < N \) may happen. In such a case, simple EVA may not give a nice solution.
1.2. Some more general theorems.

Let $V$ and $W$ be non-singular real algebraic varieties with dim $V = n$, dim $W = m$, and $\varphi : V \to W$ be a regular map. Take a point $a \in V$ and put $b := \varphi(a)$. We can take open neighborhoods $a \in U_V \subset V$ and $b \in U_W \subset W$ such that $\varphi(U_V) \subset U_W$ and that $U_V, U_W$ have local coordinate systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ whose origins are $a, b$. $\varphi$ can be represented by functions $y_j = \varphi_j(x_1, \ldots, x_n)$ ($j = 1, \ldots, m$). Let $J_a := \left( \frac{\partial y_j}{\partial x_i} \right)_{(x_1, \ldots, x_n)=a}$ be the Jacobian matrix of $\varphi$ at $a$. Note that rank $J_a$ does not depend on the choice of $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$. We denote

$$\text{Sing}(\varphi) := \{ a \in V \mid \text{rank} \, J_a < \dim \varphi(V) \}.$$ 

**Proposition 1.13.** If $V$ is complete, then $\partial(\varphi(V)) \subset \varphi(\text{Sing}(\varphi))$.

**Proof.** Put $r := \dim \varphi(V)$, and assume that rank $J_a = r$. We may assume that

$$\det \left( \frac{\partial y_j}{\partial x_i} \right)_{1 \leq i \leq r, 1 \leq j \leq r} \neq 0$$

at $a$. Let $U' := \{ (x_1, \ldots, x_n) \in U_V \mid x_{r+1} = \cdots = x_n = 0 \}$. If $U_V$ is sufficiently small Euclidean open set, $\varphi|_{U'} : U' \to \varphi(U')$ is an isomorphism. Thus $b \not\in \partial(\varphi(V))$. \hfill $\square$

When $V$ has singularities, we put $\text{Sing}(\varphi) := \text{Sing}(\varphi|_{\text{Reg}(V)})$.

**Corollary 1.14.** Assume that $V$ is complete, but may has singularities. Let $A \subset V$ be a closed semialgebraic variety with Zar$V(A) = V$. Then,

$$\partial(\varphi(A)) \subset \varphi(\text{Sing}(\varphi) \cup \text{Sing}(A) \cup \partial A).$$

**Proposition 1.15.** Let $X^s_{3,d} := X(\mathbb{P}^2, \mathcal{H}^s_{3,d})$. If $d \geq 4$, then $X^+_{3,d} \cong \mathbb{P}^2 / \mathcal{S}_3$.

**Proof.** We denote the coordinate system of $\mathbb{P}^2_+$ by $(a : b : c)$, and put $S_{m,n} := a^m b^n + b^m c^n + c^m a^n$, $S_n := S_{n,0}$, and $U := abc$. $\Phi_{3,d} := \Phi_{3,d} : \mathbb{P}^2_+ \to X^+_{3,d}$ is decomposed as $\Phi_{3,d} : \mathbb{P}^2_+ \xrightarrow{\sigma} \mathbb{P}^2_+ / \mathcal{S}_3 \xrightarrow{\Phi_{3,d}} X^+_{3,d}$. By Proposition 2.13, 2.14 and §4.5 in [1], $\Psi_{3,4} : \mathbb{P}^2_+ / \mathcal{S}_3 \to X^+_{3,4}$ is an isomorphism. Since $B_3 S_1 \cap \mathbb{P}^2_+ = \emptyset$, the multiplication map $\times S_1 : \mathcal{H}^s_{3,d} \to \mathcal{H}^s_{s, d+1}$ induces an isomorphism $X^s_{3,d+1} \to X^s_{3,d}$. \hfill $\square$

In the cyclic case $Y^+_{n,d} := X(\mathbb{P}^{n-1}, \mathcal{C}_{n,d})$, we know that $Y^+_{n,d} \cong \mathbb{P}^{n-1} / \mathcal{C}_n$ if $d \geq n$, here $\mathcal{C}_n = \mathbb{Z}/n\mathbb{Z}$ (see Proposition 1.36 in [1]). When $n = 3$, let’s take a unique element $C^+_{n,d} := \{ \Phi^+_{3,d}(0 : s : 1) \mid s > 0 \} \in \Delta^1(Y_{3,d})$. We call $\text{disc}(C^+_{n,d})$ the edge discriminant of $\mathcal{P}^+_{3,d}$ (see Definition 2.7 in [1]). The following Theorem is a replacement of Proposition 2.10, Theorem 5.9 and Theorem 6.8 in [1].

We denote the discriminant of $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$ by $\text{Disc}_d(a_n, a_{n-1}, \ldots, a_1, a_0)$. 

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Theorem 1.16. Let’s denote the coordinate system of $\mathbb{P}^2_R$ by $(a; b; c)$, and put $S_{m, n} = S_{m, n}(a, b, c) := a^mb^n + b^nc^n + c^na^n$, $S_n := S_n(a, b, c) = S_{n, 0}(a, b, c)$, and $U := U(a, b, c) = abc$. Take the base of $\mathcal{H}_{3, d}^+$ so that $s_0 = S_d$, $s_1 = S_{d-1, 1}$, $s_2 = S_{d-2, 2}, \ldots$, $s_{d-1} = S_{1, d-1}, \ldots$. Here, if $i \geq d$, then $s_i$ is a multiple of $abc$. We represent $f \in \mathcal{H}_{3, d}^+$ as $f = \sum p_i s_i$. Then, the edge discriminant of $\mathcal{H}_{3, d}^+$ agrees with $\text{Disc}_d(p_0, p_1, \ldots, p_{d-1}, p_0)$.

**Proof.** Let $\mathcal{L}_{0, t}^+$ be the local cone of $\mathcal{H}_{3, d}^+$ at $(0; t; 1) \in \mathbb{P}^2_R$. Take $f \in \mathcal{L}_{0, t}^+ \subset \mathcal{H}(C_{n, d}^+)$. (We have $p_0 > 0$ and $t > 0$). Then $f(0, t, 1) = 0$. Since $f(0, x, 1) \geq 0$ for all $x > 0$, the equation $f(0, x, 1) = 0$ has a multiple root at $x = t$. Thus, the discriminant of $f$ is equal to 0. Since $s_i d_{d-1}(0, x, 1) = x^i (1 \leq i \leq d - 1)$, $S_d(0, x, 1) = x^d + 1$ and $U(0, x, 1) = 0$, we have $f(0, x, 1) = p_0 x^d + p_1 x^{d-1} + \cdots + p_d - 1 x + p_0$.

Since $\text{Disc}_d$ and $\text{Disc}_d^+$ agree, we have the conclusion. \(\square\)

Theorem 1.17. Consider the cases $\mathcal{P} := \mathcal{P}(A, \mathcal{H})$, $X := X(A, \mathcal{H}_{n, d})$, and $\Phi := \Phi \mathcal{H}: A \to X$. Let $\sigma: \mathbb{P}^n_R(1, 2, \ldots, n)$ be the natural surjection, and $\Psi: \mathbb{P}^n_R(1, 2, \ldots, n) \to \mathcal{H}$ be the rational map such that $\Psi \circ \sigma = \Psi$. Assume that $\Psi$ is a birational map. Let $D \in \Delta^r(X)$ with $r \geq \max\{2, [d/2]\}$. Then $\mathcal{F}(D)$ is not a face component of $\mathcal{P}$.

**Proof.** Let $r_0 := \max\{2, [d/2]\}$, and take $D \in \Delta^r(X)$ with $r_0 \leq r \leq n - 1$. Assume that $\mathcal{F}(D)$ is a face component of $\mathcal{P}$. But $\dim \mathcal{F}(D) = n - 1$.

(1) Consider the case $A = \mathbb{P}^n_R$. Let $\Omega := \{x_1, \ldots, x_n \in \mathbb{P}^n_R | \#\{x_1, \ldots, x_n\} \leq r_0\}$. $\Omega$ is included in a union of some $\mathbb{P}_R \subset \mathbb{P}^n_R$. Take general $f \in \mathcal{F}(D)$. There exists a semialgebraic subset $E \subset A$ such that $\Phi(E) = D$, and $a \in E$ such that $f(a) = 0$. Since $\mathcal{F}(D)$ is a face component, we may assume that the hyperplane $H_f \subset \mathbb{P}(\mathcal{H})$ corresponding to $f$ tangents to $X$ only at the unique point $\Phi(a)$. This means that if $b \in A - B_a \mathcal{H}$ satisfies $f(b) = 0$, then $\Phi(b) = \Phi(a)$. We can choose such $f$ and $a$.

By Corollary 1.3 of [27] or Corollary 2.1 of [28], there exists $b \in \Omega$ such that $f(b) = 0$. We denote this $b \in \mathcal{F}(D)$. $a$ can move a certain $r$-dimensional subset of $E$. But $\dim \Omega = r_0 - 1 < r$. Thus, there exists $\mathcal{F}(D)$ of $\mathcal{P}$ that is not a face component of $\mathcal{P}$.

(2) Consider the case $A = \mathbb{P}^n_R$. Let $\Omega' := \{x_1, \ldots, x_n \in \mathbb{P}^n_R | \#\{x_1, \ldots, x_n\} \leq r_0\}$. $\Omega'$ is also included in a union of some $(r_0 - 1)$-dimensional linear subspace of $\mathbb{P}^n_R$.

The following theorem will be well known.

**Definition 1.18.** Let $\mathcal{H}$ be a signed linear system on a semialgebraic variety $A$. A subset $\Omega \subset A$ is a test set for $(A, \mathcal{H})$, if $f(a) \geq 0$ for all $a \in \Omega$, then $f(a) \geq 0$ for all $a \in A$.

**Theorem 1.19.** Let $\mathcal{H}$ be a signed linear system on a compact semialgebraic variety $A$ with $\dim \mathcal{H} \geq 3$, and let $X := \text{Cls}(\Phi_{\mathcal{H}}(A))$ be the characteristic variety. Take a subset
Ω ⊆ A. If \( \text{Cls}(\Phi_{\mathcal{H}}(\Omega)) \) includes all the extremal points of the convex closure of \( X \), then \( \Omega \) is a test set for \( \mathcal{H} \).

**Example 1.20.** Consider the case \( A = \mathbb{P}^2 \), \( \mathcal{H} = \mathcal{H}_{5,3} \). Then
\[
\Omega := \{(1:1:1) \cup \{(0:t:1) \in \mathbb{P}^2 \mid t \geq 0\}
\]
is a test set for \( \mathcal{H}_{5,3} \) (see Theorem 3.1 of [1]). Thus, if \( f \in \mathcal{H}_{5,3} \) satisfies \( f(1, 1, 1) \geq 0 \) and \( f(0, t, 1) \geq 0 \) for all \( t \geq 0 \), then \( f(a, b, c) \geq 0 \) for all \( a, b, c \in \mathbb{R}_+ \).

1.3. Infinitely near zeros.

Max Noether introduced the notion of infinitely near points on algebraic surfaces. We clarify here the notion of infinitely near zeros of inequalities. We need blowing ups to treat infinitely near zeros. Note that Hironaka resolution theorem for real analytic space in [20] can be applied to semialgebraic varieties.

**Definition 1.18.** (Blowing up of a PSD cone at the base locus) Let \( A \) be a non-singular semi-algebraic quasi-variety which is compact with respect to Euclidean topology. Let \( \mathcal{P} \) be a PSD cone on \( A \) with \( \dim \mathcal{P} \geq 2 \). There exists an invertible \( \mathcal{R}_A \)-sheaf \( \mathcal{J} \) such that \( \mathcal{P} \subset H^0(A, \mathcal{J}) \). Assume that \( \text{Bs} \mathcal{P} \neq \emptyset \). Take the decomposition to irreducible components
\[
\text{Bs} \mathcal{P} = \bigcup_{j=1} B_j,
\]
where each \( B_j \) is an irreducible reduced closed semi-algebraic subvariety of \( A \).

1. Choose all the \( B_i \) such that \( \dim B_i = \dim A - 1 \). We rewrite them as \( D_1, \ldots, D_r \). Put
\[
m_j := \min \{ \text{mult}_{D_j} f \mid f \in \mathcal{P} \}.
\]
The set \( D_1 \cup \cdots \cup D_r \) or the formal Weil divisor \( D := m_1 D_1 + \cdots + m_r D_r \) is called the fixed part of \( \text{Bs} \mathcal{P} \). Take an affine open covering \( \mathcal{U} \) of \( A \) such that every open set \( U \subseteq \mathcal{U} \) has a coordinate system \((x_1, \ldots, x_n)\), and that \( \mathcal{J}|_U = \mathcal{R}|_U \cdot e_U \) by a certain \( e_U \in H^0(U, \mathcal{J}) \). Take an arbitral \( f \in \mathcal{P} \). We can write \( f = f_U(x) e_U \) on \( U \) where \( f_U \in H^0(U, \mathcal{R}_A) \) and \( x = (x_1, \ldots, x_n) \). Let \( h_{i,U} \in H^0(U, \mathcal{R}_A) \) be the irreducible defining equation of \( D_i \cap U \). If \( D_i \cap U = \emptyset \), we put \( h_{i,U} = 1 \). There exists \( g_U \in H^0(U, \mathcal{R}_A) \) such that \( f_U = h_{1,U}^{m_1} \cdots h_{r,U}^{m_r} g_U \). \( \{ h_{i,U} \mid U \in \mathcal{U} \} \) define an element \( h_i \in H^0(A, \mathcal{R}_A(-D_i)) \), and \( \{ g_U e_U \mid U \in \mathcal{U} \} \) define an element \( g \in H^0(A, \mathcal{J}) \) where \( \mathcal{J} := \mathcal{J} \otimes_{\mathcal{R}_A} \mathcal{R}_A(D) \). Note that \( g \) is regular near \( D_1 \cup \cdots \cup D_r \). We can write as \( f = h_1^{m_1} \cdots h_r^{m_r} g \). Put \( \mathbf{m} = (m_1, \ldots, m_r) \), and \( h_\mathbf{m} = h_1^{m_1} \cdots h_r^{m_r} \). We shall write as \( g = f/h_\mathbf{m} \).

\[
\mathcal{P}_0 := \{ f/h_\mathbf{m} \in H^0(A, \mathcal{J}) \mid f \in \mathcal{P} \}
\]
is also a PSD cone, and there exists an isomorphism \( \rho: \mathcal{P}_0 \rightarrow \mathcal{P} \) defined by \( \rho(g) = h_\mathbf{m} g \). \( \mathcal{P}_0 \) is called the resolution of the fixed part of \( \mathcal{P} \), and \( \rho: \mathcal{P}_0 \rightarrow \mathcal{P} \) is called the natural isomorphism. Note that \( \text{Bs} \mathcal{P}_0 \) does not have the fixed part.

2. Let \( \mathcal{P}_0 \) be the resolution of the fixed part of \( \mathcal{P} \), and \( \rho_0: \mathcal{P}_0 \rightarrow \mathcal{P} \) be the natural isomorphism. Let \( \psi: X \rightarrow A \) be a resolution of \( \text{Bs} \mathcal{P}_0 \), which is defined similarly as the case of complex algebraic varieties. (If \( \text{Bs} \mathcal{P}_0 = \emptyset \), then we put \( X = A \) and \( \psi = \text{id}_A \).) \( \psi: X \rightarrow A \) exists by the resolution theorem by Hironaka. Using \( \psi^*: \mathcal{J} \rightarrow \psi^* \mathcal{J} \), we can define
\[
\psi^* \mathcal{P}_0 := \{ \psi^*(f) \in H^0(X, \psi^* \mathcal{J}) \mid f \in \mathcal{P}_0 \}.
\]
It is easy to see that \( \psi^* \mathcal{P}_0 \) is a PSD cone on \( X \). Let \( \mathcal{Q} \) be the resolution of the fixed part of \( \psi^* \mathcal{P}_0 \), and \( \rho_1: \mathcal{Q} \rightarrow \psi^* \mathcal{P}_0 \) be the natural isomorphism. Then \( \text{Bs} \mathcal{Q} = \emptyset \). The isomorphism \( \rho := \rho_0 \circ (\psi^*)^{-1} \circ \rho_1: \mathcal{Q} \rightarrow \mathcal{P} \) is called a resolution of \( \text{Bs} \mathcal{P} \) with \( \psi: X \rightarrow A \).
(3) Take $P \in A$ and the local cone $P_P$. Assume that $\dim P_P \geq 2$. Let $A_0 := A$, $P_0 := P$ and $L_0 := P_P$.

Assume that $A_{i-1}$, $P_{i-1}$ and $L_{i-1}$ are defined, and $\dim L_{i-1} \geq 2$. Let $\rho_i : Q_i \to L_{i-1}$ be a resolution of $Bl_{L_{i-1}}$ with $\psi_i : A_i \to A_{i-1}$. Let $\bar{\psi}_i := \psi_1 \circ \cdots \circ \psi_i : A_i \to A$ and $\bar{\rho}_i := \rho_1 \circ \cdots \circ \rho_i : Q_i \to P$. For $d \in \mathbb{N}$, let

$$B_{i,d} := \{ Q \in A_i \mid \psi_i(Q) = P_{i-1} \text{ and } \dim(Q_i)Q \geq d \},$$

where $(Q_i)_Q$ is the local cone of $Q_i$ at $Q$. Since $\dim Q_i \geq 2$, we have $B_{i,1} \neq \emptyset$. Take a point $P_i \in B_{i,1}$, and put $L_i := (Q_i)_P$. $P := \bar{\rho}_i(L_i) \subset P$ is called an infinitesimal local cone of $P$ of the $i$-th order at $P_i \in A_i$, or at $P \in A$.

If $P_i \in B_{i,2}$, we can repeat the above process.

(4) Take $0 \neq f \in P$, and choose $P \in V(f) \cap A$. We do the process of (3) with the additional conditions as the following. Let $f_i := \bar{\rho}_i^{-1}(f)$, and we choose $P_i \in A_i$ so that $f_i(P_i) = 0$. Then $f_i \in L_i$. Replace $B_{i,d}$ in (3) by

$$B'_{i,d} := \{ Q \in A_i \mid f_i(Q) = 0, \psi_i(Q) = P_{i-1} \text{ and } \dim(Q_i)Q \geq d \}.$$

If $B'_{i,1} \neq \emptyset$, then $P_i \in B'_{i,1}$ is called a zero of $f$ infinitely near to $P$ of the $i$-th order, or simply infinitely near zero of $f$.

If $B'_{i,1} \neq \emptyset$ and $B'_{i,2} = \emptyset$, then we put $l := i$, $A_{i+1} := A_i$, $\psi_{i+1} := \text{id} : A_{i+1} \to A_i$, and take $P_i \in B'_{i,1}$ and put $Q_{i+1} := (Q_i)_P$. Here we stop to repeat the process.

If $B'_{i,1} = \emptyset$, then we put $l := i - 1$ and stop to repeat the process.

If $B'_{i,2} \neq \emptyset$, then take a point $P_i \in B'_{i,2}$ and repeat the process.

Obtained sequence $P_i, P_{i-1}, \ldots, P_0 = P$ with $A_{i+1} \psi_{i+1} A_i \to \cdots \to A_1 \psi_1 A_0 = A$ is called a sequence of zeros of $f$ infinitely near to $P$. Put $P' := \bar{\rho}_{i+1}(Q_{i+1}) \subset P$. $l$ is called the length of $P'$ and we denote $l = \text{length } P'$. If $l \geq 1$, $P'$ is called a minimal infinitesimal local cone of $P$ at a point $P$.

**Example 1.22.** Take a point $P = (a_0; a_1; a_2) \in \mathbb{P}_+^2$ with $a_0 \neq 0$, and consider the local cone $P := (\mathbb{P}_+^2)_P$ $(d \geq 3)$. Then $Bl P = \{ P \}$. Let $\psi_C : X_C \to \mathbb{P}_+^2$ and $\psi : X \to \mathbb{P}_+$ be the blowing ups at $P$, and put $E_C := \psi_C^{-1}(P), E := \psi^{-1}(P) = E_C \cap X$. Let $\rho : Q \to P$ be the resolution of $Bl P$. Put $m_P := \min \{ \text{mult}_E f \mid f \in \psi^{-1}P\}$.

Take $f \in P$, and let $D := \psi_D f - m_P E$. $D$ is a Weil divisor defined by $\rho^{-1}(f) = 0$ in $X_C$. Let $L_f$ is a minimal infinitesimal local cone such that $f \in L_f \subset P$, if $f$ has zeros which is infinitely near to $P$. Otherwise, we put $L_f = P$. We denote the homogeneous coordinate system of $\mathbb{P}_+^2$ by $(x_0; x_1; x_2)$. On $U_0 := \mathbb{P}_-^2 - \mathbb{V}(x_0)$, we take a local coordinate system $x = x_1/x_0 - a_1/a_0, y = x_2/x_0 - a_2/a_0$.

Take $g_1 \in H_3, g_2 \in P^e_3, d_2 = 2$ such that $g_1(0) = 0$, $g_2(P) \neq 0$, and $V \cap \text{Int}(\mathbb{P}_+^2) \neq \emptyset$. We shall observe length $L_f$ and what kind of reducible elements are included in $L_f$.

(1) Consider the case $P = (1; 1; 1) \in \mathbb{P}_+^2$. Assume that $f \in P$ has an acnode at $P$. Then $m_P = 2$. Then $V_X(\rho^{-1}(f)) \cap E \cap X = \emptyset$. Thus $\text{length } L_f = 0$. Note that $V_X(\rho^{-1}(g_1^2 g_2)) \cap E \cap X = \emptyset$. Thus $g_1^2 g_2 \in L_f$.

(1') In (1), if the leading term of $f|_{U_0}$ is equal to $x^2 + y^{2m}$, then length $L_f = m - 1$. Thus $g_1^2 g_2 \notin L_f$ if $m \geq 2$.

(2) Let $P = (1; 1; 0) \in \mathbb{P}_+^2$. Take $f \in P$. Assume that $m_P = 1$ and $I_P(f, x_2) = 2$. Let $L \subset X_C$ be the strict transform of $V_+(x_2)$ by $\psi_C$, and put $Q := E \cap L \in X$. Then
exists a sequence of infinitely near zeros $\rho \geq f \in \mathcal{P}_Q$. Therefore, length $\mathcal{L}_f = 1$, and the sequence of zeros of $f$ infinitely near to $P$ is $\{Q, P\}$.

Note that $V_X(\rho^{-1}(f^{1/2}_3g_2)) \supset E$. Thus $g_2^3 \in \mathcal{L}_f$. On the other hand, if $g_3 \in \mathcal{P}^+_{3,d-1}$, then $x_2g_3 \in \mathcal{L}_f$.

(2') In (2), if the leading term of $f|_{U_0}$ is equal to $y - x^{2m}$ with $m \geq 2$, i.e. if $I_P(f, x_2) = 2m$, then length $\mathcal{L}_f = 2m - 1$. $g_2^3 \notin \mathcal{L}_f$, but $x_2g_3 \in \mathcal{L}_f$.

(3) Let $P = (1:0:0) \in \mathbb{P}^2_+$. Assume that $m_P = 1$, $I_P(f, x_1) = 1$ and $I_P(f, x_2) = m \geq 2$.

Let $\psi_i := \psi, X_i := X, E_i := E, L_i \subset X_i$ be the strict transform of $V_+(x_2) \subset \mathbb{P}^2_+$, $Q_i := E_1 \cap L_1 \subset X_i$. Inductively, let $\psi_{i+1} := \psi_{i+1}(Q_i) \subset X_{i+1}, L_{i+1} \subset X_{i+1}$ be the strict transform of $L_i$, and $Q_{i+1} := E_{i+1} \cap L_{i+1}$. It is easy to see that length $\mathcal{L}_f = m - 1$, and the sequence of zeros of $f$ infinitely near to $P$ is $\{Q_{m-1}, \ldots, Q_1, P\}$. $g_2^3 \notin \mathcal{L}_f, x_1g_3 \notin \mathcal{L}_f$, but $x_2g_3 \in \mathcal{L}_f$.

Note that infinitesimal cones can be represented using higher order differentials instead of blowing ups.

**Theorem 1.23.** Let $A$ be a non-singular compact semialgebraic variety with dim $A \geq 2$, $\mathcal{P}$ be a PSD cone on $A$, and $f \in \mathcal{E}(\mathcal{P})$. Assume that dim $\mathcal{P} \geq 2$. Then, there exists points $P_1, \ldots, P_r \in A$, and local cones or infinitesimal cones $\mathcal{P}_1, \ldots, \mathcal{P}_r \subset \mathcal{P}$ which satisfy

1. $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r = \mathbb{R}_+ \cdot f$.
2. $\mathcal{P}_i$ is the local cone $\mathcal{P}_{P_i}$ or an infinitesimal local cone of $\mathcal{P}_i$ at $P_i \in A$, for $i = 1, \ldots, r$.

Note that $P_1, \ldots, P_r$ are not always distinct.

**Proof.** Let $\mathfrak{L}$ be the set of all the local cones or infinitesimal local cones $\mathcal{P}_i$ of $\mathcal{P}$ such that $f \in \mathcal{P}_i \not\subset \mathcal{P}_i$, and that there exists a point $P_i \in A$ which satisfies (2). Take $\mathcal{P}_1, \ldots, \mathcal{P}_r \in \mathfrak{L}$ so that $d := \dim(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r)$ is minimal. We shall drive a contradiction assuming $d \geq 2$.

In the case that $\mathcal{P}_j$ is a local cone of $\mathcal{P}$ at a point $P_j \in A$, we put $\chi_j := \{(P_j, A)\}$, and let $\rho_j: \mathcal{Q}_j \rightarrow \mathcal{P}_j$ be a resolution of $\mathcal{B}_s \mathcal{P}_j$ with a resolution $\psi_j: A_j \rightarrow A$.

Consider the case that $\mathcal{P}_j$ is an infinitesimal local cone of $\mathcal{P}$ at a point $P_j \in A$. There exists a sequence of infinitely near zeros $P_{1,j}, P_{1,j-1}, \ldots, P_{0,j} = P_j$ of $f$ with $A_{1+j,j} \xrightarrow{\psi_{1+j,j}} A_{1,j} \rightarrow \cdots \rightarrow A_{1,j-1} \xrightarrow{\psi_{1,j-1}} A_{1,j-2} \rightarrow \cdots \rightarrow A_{1,j} \xrightarrow{\psi_{1,j}} A_{0,j} = A$. Put $\psi_{i,j} := \psi_{1,j} \circ \cdots \circ \psi_{i,j} : A_{i,j} \rightarrow A$. We take the natural injection $\tilde{\mathcal{P}}_{i,j}: \mathcal{Q}_{i,j} \rightarrow \mathcal{P}$, and $f_{i,j} := \tilde{\mathcal{P}}_{i,j}^{-1}(f) \in \mathcal{Q}_{i,j}$ on $A_{i,j}$ as (4) of Definition 1.21.

Note that $\mathcal{P}_j = \mathcal{P}_{1+j,j}(\mathcal{Q}_{1+j,j})$. We put $\chi_j := \{(P_{i,j}, A_{i,j}) | \ i = 0, 1, \ldots, l_j\}, f_j := f_{l_j+1,j}, A_j := A_{l_j+1,j}, \mathcal{Q}_j := \mathcal{Q}_{l_j+1,j},$ and $\psi_j := \tilde{\mathcal{P}}_{l_j+1,j}: A_j \rightarrow A$. $\rho_j := \tilde{\mathcal{P}}_{l_j+1,j}: \mathcal{Q}_j \rightarrow \mathcal{P}_j$ is a morphism such that $\rho_j(f_j) = f$.

Let $X := A_1 \times A_2 \times \cdots \times A_r A_1$. There exists a surjective birational morphism $\varphi_j: X \rightarrow A_1$ such that $\varphi_j \circ \varphi_j = \varphi_j \circ \varphi_j$ for all $i, j \in \{1, \ldots, r\}$. Let $\varphi' := \varphi_j \circ \varphi_j: X \rightarrow A, \mathcal{P}' := (\varphi')^* \mathcal{P}_j, \mathcal{P}' := \mathcal{P}_j \cap \cdots \cap \mathcal{P}_r$, and $\varphi': \mathcal{P}' \rightarrow \mathcal{P}'$ be a resolution of $\mathcal{B}_s \mathcal{P}'$ with a surjective birational morphism $\psi': Y \rightarrow X$. Put $\psi := \varphi' \circ \psi': Y \rightarrow A$, and $f' := \rho^{-1}(\varphi'(f)) \in \mathcal{Q}'$. Since $\psi_2 f = f_j h_j$ by a certain holomorphic function $h_j$ on $A_1$, we have $\varphi'^* f = (\varphi_2^* f_j)(\psi_2^* h_j)$. Thus, there exists a holomorphic function $h_j$ on $Y$ such that $(\varphi_j \circ \psi') f_j = f' h_j$. Since dim $\mathcal{Q}' = d \geq 2$, $\mathcal{B}_s \mathcal{Q}' = \emptyset$, and $f' \in \mathcal{E}($ $\mathcal{Q}')$, there exists a point $Q \in Y$ such that $f'_j(Q) = 0$, and $\mathcal{Q}' \not\subset \mathcal{Q}'$. Moreover, there exists $g' \in \mathcal{Q}'$ such that $g'(Q) \neq 0$. Let $g := (\varphi'^{-1}((\rho'(g'))) \in \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$, and $P_{r+1} := \tilde{\mathcal{Q}}(Q) \in A$. 

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If $P_{r+1} \notin \{P_1, \ldots, P_r\}$, then $f(P_{r+1}) = 0$ and $g(P_{r+1}) \neq 0$. Thus $f \in \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_{r+1} \subsetneq \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$. This contradicts the minimality of $d$.

So, we may assume that $P_{r+1} = P_1$. We may assume that $P_1 = \cdots = P_k$ and $P_j \neq P_{r+1}$ for $k < j \leq r$. Put $\mathcal{X} := \mathcal{X}_1 \cup \cdots \cup \mathcal{X}_k$. For any $(P_{i,j}, A_{i,j}) \in \mathcal{X}$, there exists a natural surjective birational morphism $\varphi_{i,j}: Y \to A_{i,j}$ such that $\varphi_{i,j} \circ \varphi_{i,j} = \varphi_{i,j}: Y \to A$. There exists $(P_{i,j}, A_{i,j}) \in \mathcal{X}$ such that $\varphi_{i,j}(Q) = P_{i,j}$. Among all the such sets $\{(i, j)\}$, we choose a $(i_0, j_0)$ such that $j_0$ is the maximum. By the choice of $l_{j_0}$, we have $i_0 < l_{j_0}$.

Let $\tilde{A} := A_{1+i_0, j_0}, \tilde{\beta} := \varphi_{1+i_0, j_0}, \tilde{\varphi} := \varphi_{1+i_0, j_0}, \tilde{\varphi}_i := \varphi_{1+i_0, j_0}, \tilde{\varphi}_j := \varphi_{1+i_0, j_0}, \tilde{f} := \tilde{\varphi}^{-1}(f) \in \tilde{\varphi}_i, \tilde{g} := \tilde{\varphi}^{-1}(g) \in \tilde{\varphi}_j, \tilde{\varphi} := \varphi_{1+i_0, j_0}, \tilde{P} := \varphi_{1+i_0, j_0},$ and $\tilde{Q} := \varphi_{1+i_0, j_0}$. Then $\tilde{\varphi} \neq \tilde{Q}$, $\tilde{f}(\tilde{Q}) = 0$ and $\tilde{g}(\tilde{Q}) \neq 0$. Thus $\tilde{Q} \tilde{\varphi} \subseteq \tilde{\varphi}_i$ and $\tilde{Q} \tilde{\varphi} \neq \tilde{\varphi}_i$. Then $(\tilde{\varphi}(Q), A) \notin \mathcal{X}$. Thus $\mathcal{P}_{r+1} := \tilde{\varphi}(Q)$ is an infinitesimal local cone of $\mathcal{P}$ such that $f \in \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_{r+1} \subsetneq \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$. This contradicts the minimality of $d$.

In the case $\dim A = 1$, we can obtain similar theorem using

$$\mathcal{P}_a(m) := \{ f \in \mathcal{P} \mid \text{ord}_a f \geq m \}$$

instead of infinitesimal local cones.

**Theorem 1.24.** Let $A$ be a non-singular compact semialgebraic curve, and $\mathcal{H}$ be a signed linear system on $A$. Assume that $\mathcal{B}_s \mathcal{H} \neq A$ and $\dim \mathcal{H} \geq 2$. Then, $0 \neq f \in \mathcal{H}$ is an extremal element of $\mathcal{P}$, if and only if there exists $a_1, \ldots, a_r \in A$ and $m_1, \ldots, m_r \in \mathbb{N}$ such that

1. $\dim(\mathcal{P}_a(m_1) \cap \cdots \cap \mathcal{P}_a(m_r)) = 1$, and
2. $f \in \mathcal{P}_a(m_1) \cap \cdots \cap \mathcal{P}_a(m_r)$.

**Proof.** It is easy to see that (1) and (2) $\implies$ $f$ is extremal.

Assume that $f$ is an extremal element of $\mathcal{P}$. There exists $a_1 \in A$ such that $f(a_1) = 0$. Let $a_1, \ldots, a_r \in A$ be all the zeros of $f$. Put $m_i := \text{ord}_{a_i} f$, and $\mathcal{P}_0 := \mathcal{P}_a(m_1) \cap \cdots \cap \mathcal{P}_a(m_k)$. Assume that $\dim \mathcal{P}_0 \geq 2$. $f$ is extremal on $\mathcal{P}_0$. Take any $g \in \mathcal{H}_0 := \text{Zar}_\mathcal{H}(\mathcal{P}_0)$. Since $\text{ord}_{a_i} f \leq \text{ord}_{a_i} g$, $g/f$ is regular on $A$. Put $\mathcal{H}_1 := \{ g/f \mid g \in \mathcal{H}_0 \}$. Since $\mathcal{B}_s \mathcal{H}_1 = \emptyset$, $\dim \mathcal{H}_1 = \dim \mathcal{H}_0 \geq 2$, and $1 = f/f$ is an extremal element of $\mathcal{H}_1$, there exists $a \in A$ such that $1 = 1(a) = 0$. A contradiction.

Theorem 0.1 can be restated as the following, using the notion of infinitely near zeros.

**Theorem 1.25.** Assume that $f \in \mathcal{P}_{3,6}$ is not a square of a cubic polynomial. Then, $f \in \mathcal{E}(\mathcal{P}_{3,6})$ if and only if $f$ has just 10 zeros on $\mathbb{P}^2_\mathbb{R}$ including all the infinitely near zeros.

**Proof.** Assume that $f \in \mathcal{E}(\mathcal{P}_{3,6})$ is not a square of cubic polynomial. Then $f$ is a limit of a sequence $\{f_n\}$ of exposed extremal elements in $\mathcal{P}_{3,6}$ (see [9]). Since each $f_n$ has distinct 10 zeros, $f$ also has just 10 zeros including all the infinitely near zeros.

Assume that $f$ has 10 zeros $P_1, \ldots, P_{10}$ including infinitely near zeros. Then $f$ cannot be a product of a quadric and a quartic, since their intersection consists of 8 points. Thus $f$ is irreducible. Note that each $P_i$ is a node of $V(C)$. There exists a unique irreducible sextic curve $C \subset \mathbb{P}^2_\mathbb{C}$ such that $C$ has nodes at $P_1, \ldots, P_{10}$. Thus $C = V_C(f)$ and $f$ is extremal.

Using the ideas in [9] and [8], we also obtain the following theorem.
Theorem 1.26. Assume $d \geq 3$, and $f \in \mathcal{E}(\mathcal{P}_{3,2d})$ is irreducible. Let $N$ be the numbers of zeros of $f$ in $\mathbb{P}^2_\mathbb{R}$ including all the infinitely near zeros. Then
\[
\frac{(d+1)(d+2)}{2} \leq N \leq (2d-1)(d-1).
\]

Proof. Let $P_1, \ldots, P_N$ be all the zeros of $f$ on $\mathbb{P}^2_\mathbb{R}$ including infinitely near zeros. There exist local cones or infinitesimal local cones $L_1, \ldots, L_r \subset \mathcal{P}_{3,2d}$ such that $L_1 \cap \cdots \cap L_r = \mathbb{R}+ \cdot f$. We may assume that $r = N$ and $L_i$ corresponds to $P_i$.

If $N < (d+1)(d+2)/2$, then there exists $g \in \mathcal{H}_{3,d}$ such that $P_1, \ldots, P_N \in V_\mathbb{R}(g)$ in the sense of Noether. Then $g^2 \in L_1 \cap \cdots \cap L_r = \mathbb{R}+ \cdot f$. This implies $g^2 = cf$ ($\exists c \in \mathbb{R}+$), and $f$ is reducible. Thus $N \geq (d+1)(d+2)/2$.

Let $C := V_\mathbb{C}(f) \subset \mathbb{P}^2_\mathbb{C}$. Then $p_a(C) \geq \sum p \nu(P)(\nu(P) - 1)/2 + g(C') \geq N + g(C')$, where $p_a(C) = (2d-1)(2d-2)/2$ is the arithmetic (or virtual) genus, $g(C')$ is the genus of the normalization $C'$ of $C$, and $\nu(P_i)$ is the multiplicity of $C$ at $P_i$. Thus, we have $N \leq (2d-1)(d-1)$. \qed

Section 2. The case $n = 2$.

2.1. Setting of this section.

In this section, we consider a very classical and elementary problem:

Problem 2.1. Let $f(x) = x^d + \sum_{i=1}^{d} c_i x^{d-i}$. Find the condition for that $f(x) \geq 0$ for all $x \in \mathbb{R}+$ (resp. for all $x \in \mathbb{R}$ when $d$ is even).

Many solutions are well known, and there are many methods to solve the above problem. For example, EVA (Extreme Value Analysis) method, SOS (Sum Of Squares) method, and so on. Moreover, solutions are never unique, because there are many possibilities of the choice of separators. The results in this section are never new, but we treat the above problem to understand how PSD method will work, and as a preparation for more complicated theorems after §4. Please pay attention to semialgebraic discussion which PSD method will brings.

In this section, for $F(a_0, a_1) = \sum_{i=0}^{d} p_i a_0^n a_1^{n-i} \in \mathcal{H}_{2,d}$, we say $F$ is monic if $p_0 = 1$. We say $F$ is at infinity if $p_0 = 0$. For $V \subset \mathcal{H}_{2,d}$, we denote
\[
V := \{ F \in V \mid F \text{ is monic} \}, \quad V^\infty := \{ F \in V \mid F \text{ is at infinity} \}.
\]

Letting $x := a_1/a_0$ and $c_i := p_i/p_0$, $F \in \mathcal{H}_{2,d}$ with $p_0 > 0$ can be identified with $f(x) = \frac{1}{p_0} F(1, x) = x^d + \sum_{i=1}^{d} c_i x^{d-i}$.

We denote $\mathcal{P}_{2,d}^+ := \mathcal{P}(\mathbb{P}^1_\mathbb{R}, \mathcal{H}_{2,d}), \mathcal{P}_{2,d} := \mathcal{P}(\mathbb{P}_\mathbb{R}, \mathcal{H}_{2,d}), \Phi_{2,d} := \Phi_{\mathcal{H}_{2,d}}, X_{2,d}^+ := \Phi_{2,d}(\mathbb{P}^1_\mathbb{R})$, and $X_{2,d} := \Phi_{2,d}(\mathbb{P}_\mathbb{R})$. Simply, we often write $\mathcal{H} := \mathcal{H}_{2,d}, \Phi := \Phi_{2,d}, \mathcal{P} := \mathcal{P}_{2,d}$ or $\mathcal{P}_{2,d}$, $X := X_{2,d}^+$ or $X_{2,d}$.

We denote the local cone of $\mathcal{P}$ at $(1:p) \in A$ by $\mathcal{L}_p$, and the local cone at $(0:1)$ by $\mathcal{L}_\infty \subset \mathcal{H}_{2,d}^\infty$. Note that the discriminant has the following properties.
\[
\text{Disc}_d(p_0, p_1, \ldots, p_d) = \text{Disc}(p_0, p_1, \ldots, p_d),
\]
\[
\text{Disc}(p_0, -p_1, p_2, -p_3, \ldots, (-1)^d p_d) = \text{Disc}(p_0, p_1, \ldots, p_d).
\]
Remark 2.2. (1) Consider the case $A = \mathbb{P}^1_+$. 
Since $\Phi_{2,d}: \mathbb{P}^1_+ \to X^+_2,d$ is an isomorphism, we have $\Delta^1(X^+_2,d) = \{X^\circ\}$, $\Delta^0(X^+_2,d) = \{P_0, P_\infty\}$, where $X^\circ := \text{Int}(X^+_2,d)$, $P_0 := \Phi_{2,d}(1:0)$, $P_\infty := \Phi_{2,d}(0:1)$. Note that $\text{disc}(P_0) = p_d$, $\text{disc}(P_\infty) = p_0$, and $\text{disc}(X^\circ) = \text{Disc}_d$. Thus
$$\partial P^+_2,d \subset V_\mathcal{K}(p_d) \cup V_\mathcal{K}(p_0) \cup V_\mathcal{K}(\text{Disc}_d).$$

(2) Consider the case $d$ is even and $A = \mathbb{P}^1_\mathbb{R}$.
Since $\Phi_{2,d}: \mathbb{P}^1_\mathbb{R} \to X^2_2,d$ is an isomorphism, we have $\Delta^1(X^2_2,d) = \{X^2_2,d\}$, and $\Delta^0(X^2_2,d) = \emptyset$. Since $\text{disc}(X^2_2,d) = \text{Disc}_d$, we have
$$\partial P^+_2,d \subset V_\mathcal{K}(\text{Disc}_d).$$

In this case, what we should do is to find nice separators.

2.2. Structure of $P^+_{2,3}$. 
As a warming up, we shall solve the following problem, (I) by classical EVA method, and (II) by PSD method.

Problem 2.3. Let $f(x) = x^3 + ax^2 + bx + c$. Find the condition for that $f(x) \geq 0$ for all $x \geq 0$.

(I) EVA solution: A standard EVA solution will be as the following:
$$c = f(0) \geq 0 \text{ is required.}$$
So assume that $c \geq 0$. Let $x_0 := (\sqrt{a^2 - 3b} - a)/3$ be a solution of $f'(x) = 0$ which will not be smaller. Consider the cases (i) $x_0$ is imaginal, (ii) $x_0 < 0$, and (iii) "$x_0 \geq 0$ and $f(x_0) \geq 0"$, we have the following solution:

Solution 2.4. $f(x) \geq 0$ for all $x \geq 0$ if and only if one of (1), (2), (3) holds:
(1) $a \geq 0$, $b \geq 0$ and $c \geq 0$.
(2) $a^2 \leq 3b$ and $c \geq 0$.
(3) $a^2 > 3b$, $c \geq 0$ and $2a^3 - 9ab + 27c \geq 2(a^2 - 3b)^{3/2}$.

(II) PSD solution: We solve Problem 2.3 using theory of PSD cone.
$$c = f(0) \geq 0 \text{ is required.}$$
(i) If $c = 0$, considering a condition for that $x^2 + ax + b \geq 0$ for all $x \geq 0$, we have (1) "$a \geq 0$ and $b \geq 0", \text{ or (2) } a^2 - 4b \leq 0."

Fig. 2.1. Graph of $D_3(1,a',b',1) = 0$
(ii) Assume $c > 0$. By considering $a' := a/\sqrt[3]{c}$, $b' := b/\sqrt[3]{c}$, $c' := c/\sqrt[3]{c} = 1$, $x' := x/\sqrt[3]{c}$, we can reduce to the case $c = 1$. Then $\text{Disc}_3(1, a, b, 1) = \text{Disc}_{3}^{*}(a, b)$, where $\text{Disc}_{3}^{*}$ was defined in Theorem 3.1 in [1](see Fig. 2.1). Thus, by the same argument with the proof of Theorem 3.1 in [1], we have the following:

**Solution 2.5.** $f(x) = x^3 + ax^2 + bx + c \geq 0$ for all $x \geq 0$ if and only if one of (1), (2) or (3) holds:

1. $a \geq 0$, $b \geq 0$ and $c \geq 0$.
2. $c = 0$ and $a^2 - 4b \leq 0$.
3. $c > 0$ and $\text{Disc}_3(1, a, b, c) = a^2 b^2 - 4b^3 - 4a^3 c + 18abc - 27c^2 \leq 0$.

**Remark 2.6.** (1) In Solution 2.5, $\text{Disc}_3$ and $c$ are discriminants of $\mathcal{P}_{2,3}^{+}$. $a$ and $b$ are separators. Remember that there are many possibility of the choice of separators.

(2) Solution 2.4 is equivalent to the above. Put $\tilde{D}_3 := -\text{Disc}_3(1, a, b, c)$. Note that $(2a^3 - 9ab + 27c)^2 - 2(a^2 - 3b)^3 = 27\tilde{D}_3$.

If $\tilde{D}_3 \geq 0$, $a^2 \geq 3b$, $b \leq 0$ and $c \geq 0$, then $2c(2a^3 - 9ab + 27c) = \tilde{D}_3 + b^2(a^2 - 3b) + b^3 + 27c \geq 0$.

Moreover, if $a^2 - 3b < 0$, $a < 0$, $b > 0$, and $c \geq 0$, then $\tilde{D}_3 = (b^2 - 4ac)(3b - a^2) + 6(-a)bc + b^3 + 27c \geq 0$.

Thus, $\text{Disc}_3(a^3 - 9ab + 27c) \geq (2a^2 - 3b)^3/2$ or $a^2 \leq 3b^2$ is equivalent to $\tilde{D}_3 \geq 0$ under the condition $c \geq 0$ and $a < 0$ or $b < 0$.

(3) The local cone of $\mathcal{P}_{2,3}^{+}$ at $(1; p) \in A = \mathbb{P}_1^+$ is

$$L_p = \begin{cases} \mathbb{R}_+ \cdot x(x-p)^2 + \mathbb{R}_+ \cdot (x-p)^2 & \text{if } p > 0, \\
\mathbb{R}_+ \cdot x^3 + \mathbb{R}_+ \cdot x^2 + \mathbb{R}_+ \cdot x & \text{if } p = 0, \\
\mathbb{R}_+ \cdot x^2 + \mathbb{R}_+ \cdot x + \mathbb{R}_+ \cdot 1 & \text{if } p = \infty. \end{cases}$$

(4) All the extremal elements of $\mathcal{P}_{2,3}^{+}$ are positive multiple of the following polynomials:

$x(x-p)^2$ $(p \geq 0)$, $(x-p)^2$ $(p \geq 0)$, $x$, $1$. For $p > 0$, $x(x-p)^2 \in \text{Sing}(V(\text{Disc}_3))$.

(5) Extremal polynomials $x(x-p)^2$ lies on $V(a^2 - 4b)$, not on $V(a^2 - 3b)$.

2.3. Structure of $\mathcal{P}_{2,4}$

One of EVA solutions relating Proposition 2.7 and 2.8 is published in [25]. Please compare with the following solution, and pay attention to semiflagraic geometry of the solution.

Extremal elements of $\mathcal{P}_{2,4}^{+}$ can be determined easily. When $s \neq t$, $\dim(L_s \cap L_t) \leq 1$ and $f_{s,t} := (x-s)^2(x-t)^2 \in (L_s \cap L_t)$. Thus, $f_{s,t}$ is an extremal element of $\mathcal{P}_{2,4}^{+}$. Similarly, $f_{s,\infty} := (x-s)^2$, $f_{s,s} := (x-s)^4$ and $p_{\infty, \infty} := 1$ are also extremal elements. Conversely, any extremal element of $\mathcal{P}_{2,4}^{+}$ is a positive multiple of $f_{s,t}$ ($s, t \in \mathbb{P}_2^+$). Moreover $f_{s,t} \in \text{Sing}(V(\text{Disc}_4))$.

**Proposition 2.7.** Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$,

$$\text{sep}_1(a, b, c, d) := \left( a - \frac{c}{\sqrt[3]{d}} \right)^2 - 16(b + 2\sqrt{d}), \quad \text{sep}_2(a, b, c, d) := \left( a + \frac{c}{\sqrt[3]{d}} \right)^2 - 16(b - 2\sqrt{d}).$$

Then, $f(x) \geq 0$ for all $x \in \mathbb{R}$ if and only if one of the following (1)-(3) holds:

1. $c = 0$ and $a^2 - 4b \leq 0$.
2. $d > 0$, $-2\sqrt{d} \leq b < 6\sqrt{d}$, $\text{Disc}_4(1, a, b, c, d) \geq 0$ and $\text{sep}_1(a, b, c, d) \leq 0$.
3. $d > 0$, $b \geq 6\sqrt{d}$, $\text{Disc}_4(1, a, b, c, d) \geq 0$, $\text{sep}_1(a, b, c, d) \leq 0$ and $\text{sep}_2(a, b, c, d) \leq 0$.

**Proof.** $d = f(0) \geq 0$ is always required.
(I) Case $d = 0$.
Then, $c = f'(0) = 0$ is required. Since $f(x) = x^2((x - a/2)^2 + (b - a^2/4))$, we have (1).

(II) Case $d > 0$.
By considering $a' := a/\sqrt{\alpha}$, $b' := b/\sqrt{\alpha}$, $c' := c/\sqrt{\alpha^2}$, $d' := d/\sqrt{\alpha^2} = 1$, $x' := x/\sqrt{\alpha}$, we can reduce to the case $d = 1$.

Put $\varphi(p, q, r) := Disc_4(1, p, r, q, 1)$. We have already analyzed the graph of $\varphi$ in the proof of Theorem 0.3 of \cite{1}.

\begin{align*}
V_r := \{(p, q) \in \mathbb{R}^2 \mid x^4 + px^3 + r x^2 + qx + 1 \in \mathcal{H}_{2,4}\} \subset \mathcal{H}_{2,4},
\end{align*}

and $P_r := V_r \cap \mathcal{P}_{2,4}$. $P_r$ must be a convex set and $\partial P_r \subset V(Disc_4(1, p, r, q, 1))$. Since $x^4 - cx^3 + bx^2 - ax + 1 \in \mathcal{P}_{2,4}$ if and only if $x^4 + ax^3 + bx^2 + cx + 1 \in \mathcal{P}_{2,4}$, $P_r$ must be symmetric with respect to the line $a + c = 0$.

(II-1) If $r < -2$, there are no such symmetric convex set on $V_r$ (see Fig 2.6). Thus $P_r = \emptyset$. Note that $r < -2$ is equivalent to $b + 2\sqrt{\alpha} < 0$.

(II-2) If $r = -2$, then $P_r = \{(0, 0)\}$. This is included in (2).

(II-3) Case $-2 < r < 6$.
$V(Disc_4(1, p, r, q, 1))$ has just two simple nodes at the points corresponding to

\begin{align*}
Q_1 := x^4 + 2\sqrt{b + 2\sqrt{\alpha}x^3} + bx^2 - 2\sqrt{bd + 2d^{3/2}}x + d = \left(x^2 + \sqrt{b + 2\sqrt{\alpha}x - \sqrt{d}}\right)^2,
\end{align*}

\begin{align*}
Q_2 := x^4 - 2\sqrt{b + 2\sqrt{\alpha}x^3} + bx^2 + 2\sqrt{bd + 2d^{3/2}}x + d = \left(x^2 - \sqrt{b + 2\sqrt{\alpha}x - \sqrt{d}}\right)^2
\end{align*}

(see Fig 2.2). $Q_1$ and $Q_2$ agrees with extremal polynomials $f_{s,t}$, where $a = -2(s + t)$, $b = (s + t)^2 + 2st$, $c = -2st(s + t)$ and $d = s^2t^2$. Note that $-2 < r < 6$ is equivalent to $-2\sqrt{\alpha} < b < 6\sqrt{\alpha}$. If $st \geq 0$, then $0 \leq (s - t)^2 = (s + t)^2 - 4st = b - 6\sqrt{\alpha}$. Thus $st < 0$ if $-2 < r < 6$. Note that $f_{s,t} \in V(sep_1)$ when $st < 0$. Clearly, two lines

\begin{align*}
sep_1^+(a, b, c, d) = a - \frac{c}{\sqrt{d}} - 4\sqrt{b + 2\sqrt{\alpha}}, \quad sep_1^-(a, b, c, d) = a - \frac{c}{\sqrt{d}} + 4\sqrt{b + 2\sqrt{\alpha}}
\end{align*}

are nice separators, for $Q_1 \in V(sep_1^+)$ and $Q_2 \in V(sep_1^-)$. $sep_1 = sep_1^+ \cdot sep_1^-$ is the defining equation of these two lines. It is easy to see that

$$
P_r = \{(p, q) \mid Disc_4(1, p, r, q, 1) \geq 0, sep_1(p, r, q, 1) \leq 0\}.
$$
When \(2 \leq r < 6\), the points corresponding to

\[
Q_3 := x^4 - 2\sqrt{b - 2\sqrt{d}}x^3 + bx^2 - 2\sqrt{bd} - 2d^{3/2}x + d = \left(x^2 - \sqrt{b - 2\sqrt{d}x + \sqrt{d}}\right)^2,
\]

\[
Q_4 := x^4 + 2\sqrt{b - 2\sqrt{d}}x^3 + bx^2 + 2\sqrt{bd} - 2d^{3/2}x + d = \left(x^2 + \sqrt{b - 2\sqrt{d}x + \sqrt{d}}\right)^2
\]

are also isolated singularities of \(V(\text{Disc}_4)\). But these are inside of \(P_r\). Thus we have (2).

(II-4) Case \(r \geq 6\). See Fig 2.3. Similarly, we have (3).

\[
\begin{align*}
\text{Fig.2.3. } r > 6
\end{align*}
\]

2.4. Structure of \(\mathcal{F}^+_{2,4}\).

Proposition 2.8. Let \(f(x) = x^4 + ax^3 + bx^2 + cx + d\),

\[
\begin{align*}
\text{sep}_2^- (a, b, c, d) &:= a + \frac{c}{\sqrt{d}} + 4\sqrt{b - 2\sqrt{d}}, \\
\text{sep}_3^- (a, c, d) &:= a\sqrt{d} + c, \\
\text{sep}_4^- (b, c, d) &:= c + 2\sqrt{bd} + 2d^{3/2}, \\
\text{sep}_5^- (a, b, d) &:= a + 2\sqrt{b + 2\sqrt{d}}.
\end{align*}
\]

Then, \(f(x) \geq 0\) for all \(x \geq 0\) if and only one of the following (1)-(5) holds:

1. \(d = 0\), and one of (1), (2), (3) in Solution 2.5 holds.
2. \(d > 0\), \(\text{sep}_3^- (a, c, d) \geq 0\) and \(\text{Disc}_4 (1, a, b, c, d) \leq 0\).
3. \(d > 0\), \(b > -2\sqrt{d}\), \(\text{sep}_3^- (a, c, d) \geq 0\), \(\text{sep}_4^- (b, c, d) \geq 0\) and \(\text{sep}_5^- (a, b, d) \geq 0\).
4. \(d > 0\), \(-2\sqrt{d} < b \leq 6\sqrt{d}\), \(\text{sep}_3^- (a, c, d) < 0\) and \(\text{Disc}_4 (1, a, b, c, d) \geq 0\).
5. \(d > 0\), \(b > 6\sqrt{d}\), \(\text{sep}_3^- (a, c, d) < 0\), \(\text{sep}_2^- (a, b, c, d) \geq 0\) and \(\text{Disc}_4 (1, a, b, c, d) \geq 0\).
Proof. Almost same with the proof of Proposition 2.7 and Theorem 0.3 in [1]. See Fig. 2.4, 2.5 and 2.6.

Remark 2.9. As in Fig. 2.4, 2.5 and 2.6, \( V(\text{Disc}_4) \) divides \( \mathbb{R}^3 \) into 6 domains. In each domain, its elements have same type roots. For example, if

\[
g \in \{ f \in \bar{H}_{2,4} \mid \text{Disc}_4(f) < 0, \text{sep}_3(f) < 0 \}\]

then \( g(x) = 0 \) has 2 distinct positive roots and 2 imaginal roots. If

\[
g \in \{ f \in \bar{H}_{2,4} \mid \text{Disc}_4(f) > 0, b > 6\sqrt{d}, \text{sep}_2(f) < 0, a < 0, c < 0 \}\]

then \( g(x) = 0 \) has 4 distinct positive roots. Remember that we assume \( d = 1 \) (or \( d > 0 \)).
2.5. Structure of $\mathcal{P}^+_{2,5}$.

We can present a theorem like Proposition 2.8 for $d = 5$. But it is too dirty to write here. We shall explain this reason.

Fig 2.7 is the graph of $\text{Disc}_5(1, a, 12, -1, d, 1) = 0$. Denote this plane by $V_{12,-1} := V(b - 12, c + 1)$. $\mathcal{P}^+_{2,5} \cap V_{12,-1}$ is the convex domain enclosed by the tick line. sign($\text{Disc}_5$) is not constant on $\mathcal{P}^+_{2,5} \cap V_{12,-1}$. To describe $\mathcal{P}^+_{2,5} \cap V_{12,-1}$ as a union of basic semialgebraic sets, we need some separators $s(a, d)$ which pass through some of $P_1$, $P_2$ and $P_3$. But the coordinates of $P_i$ are elements of degree 5 over the field $\mathbb{Q}(b, c)$. Thus, if a separator $s(a, d)$ is a rational polynomial, then deg $s(a, d) \geq 5$ and $s(P_i) = 0$ for all $i = 1, 2, 3$. There are some standard separators $s$ as listed below. The zero locus $V(s)$ intersects with $\partial \mathcal{P}^+_{2,5} \cap V_{12,-1}$ at least three points. So, we need some new separators to separate them. There is no inevitability in the choice of these new separators. This fact makes a theorem very dirty.

Here, we present datum for Fig. 2.7.

$V(\text{Disc}_5(1, a, b, c, d, 1)) \cap V_{b,c}$ has a parameterization:

$$a(t, b, c, e) = \frac{-4t^5 - 2bt^3 - ct^2 + e}{3t^4}, \quad d(t, b, c, e) = \frac{t^5 - bt^3 - 2ct^2 - 4e}{3t}$$

where $e = 1$. This can be obtained from eliminating $p, q, r$ from four of five relations $x^5 + ax^4 + bx^3 + cx^2 + dx + e = (x - t)^2(x - p)(x - q)(x - r)$ for all $x \in \mathbb{R}$, $(a, b, c, d, e) \in \mathcal{F}(X^5)$ if and only if $t \in \mathbb{R}_+$ and $p, q, r \in (\mathbb{C} - \mathbb{R}_+) \cup \{0\}$. The followings are standard separators:

$S_1(a, b, c) = 64a^{10} - 16a^9c^2 + 8a^8b^2c - 896a^8b - a^7b^4 + 2b^6c^2 + 1408a^7c - 100a^6b^3c + 4592a^6b^2 - 344a^6c^3 + 12a^5b^3 - 782a^5b^2c^2 - 13664a^5bc + 5248a^5 + 408a^4b^4c - 10096a^4b^3 + 2988a^4bc^3 + 9168a^4c^2 - 48a^3b^6 + 320a^3b^3c^2 + 41608a^3b^2c + 4b^3c) - 41600a^3b - 2457a^3c^4 - 512a^2b^5c + 7808a^2b^4c^2 - 6048a^2b^3c^3 - 37520a^2b^2c^4 + 48000a^2c^2 + 64ab^7 + 2016ab^6c^2 - 3792ab^5b^2c + 82000a^2bc + 10692ac^4 + 1600ac^3 - 128b^6c + 256b^5 - 1728b^4c^3 + 43200b^3c^2 - 180000bc - 5832c^5 + 200000$,

$S_2(a, b, d) = 16a^7d^2 - 16a^6d^2 + 32a^6d - 200a^5bd^2 + 16a^5 + 192a^4b^3 - 368a^4bd + 27a^4d^4 + 832a^3b^2d^2 - 200a^3b - 1132a^3d^3 - 768a^2b^4 + 2560a^2b^2d - 216a^2bd^3 - 5550a^2d^2$.
- 1152a^5b^2d^2 + 4000ab^2 + 5040abd^3 - 7500ad + 1024b^5 - 6400b^3d + 432b^2d^4 + 9000bd^2 - 1728d^5 - 3125,

\[ S_3(a, c, d) = S_2(d, c, a), \]
\[ S_4(b, c, d) = S_1(d, c, b). \]

For example, \( S_1(a, 12, -1) = S_2(a, 12, d) = S_3(a, -1, d) = S_4(12, -1, d) = 0 \) has just four real roots \( (a, d) = P_1, P_2, P_3 \) and \( P_4 = (\pm 0.1946 \cdots, 35.92 \cdots) \). \( P_4 \) is an isolated zero of \( \text{Disc}_5 \) inside of \( \mathcal{P}_{2,5}^+ \).

**Problem 2.10.** Find a nice decomposition of \( \mathbb{P}^2_5/\mathcal{G}_{n+1}^+ \) and \( \mathbb{P}^2_7/\mathcal{G}_{n+1}^+ \) to a union of basic semialgebraic subsets, for \( n = 5, 6, \cdots \). This problem is very close to the study of \( V(\text{Disc}_n) \).

**Section 3. The case \( n = d = 3 \).**

Structure of \( \mathcal{P}^{c+}_{3,3} \) is determined in [1]. In this section, we denote the coordinate system of \( \mathbb{P}^2_5 \) or \( \mathbb{P}^2_7 \) by \( (x: y: z) \). We proved the following theorem in [1]:

**Theorem 3.0.** Let \( f(x, y, z) := p_0(x^3 + y^3 + z^3) + p_1(x^2y + y^2z + z^2x) + p_2(xy^2 + yz^2 + zx^2) + p_3xyz \in \mathcal{K}_{3,3} \). Then, \( f(x, y, z) \geq 0 \) for all \( x, y, z \in \mathbb{R}_+ \) if and only if one of (1) or (2) holds.

1. \( p_0 \geq 0, 3p_0 + 3p_1 + 3p_2 + p_3 \geq 0 \) and \( \text{Disc}_3(p_0, p_1, p_2, p_0) \leq 0 \).
2. \( p_0 \geq 0, p_1 \geq 0 \) and \( p_2 \geq 0 \) and \( 3p_0 + 3p_1 + 3p_2 + p_3 \geq 0 \).

The extremal elements of \( \mathcal{P}^{c+}_{3,3} \) is a positive multiple of

\[
\begin{align*}
&f_s(x, y, z) := s^2(x^3 + x^3 + z^3) - (2s^3 - 1)(x^2y + y^2z + z^2x) \\
&+ (s^4 - 2s)(xy^2 + yz^2 + zx^2) - 3(s^4 - 2s^3 + s^2 - 2s + 1)xyz \\
&s \geq 0 \\
&\text{or } f_{\infty}(x, y, z) := xy^2 + yz^2 + zx^2 - 3xyz.
\end{align*}
\]

All the discriminants of \( \mathcal{P}^{c+}_{3,3} \) are \( \text{Disc}_3(p_0, p_1, p_2, p_0), 3p_0 + 3p_1 + 3p_2 + p_3 \) and \( p_0 \). On the other hand, \( p_1 \) and \( p_2 \) in (2) are separators. So, the conditions \( p_1 \geq 0 \) and \( p_2 \geq 0 \) can be replaced by other inequalities.

In this section, we determine all the elements of \( \mathcal{E}(\mathcal{P}^{c+}_{3,3}) \).

**3.1. Structure of \( \mathcal{P}^{c+}_{3,3} \).**

\( \mathbb{P}^2_5 \) is homeomorphic to a closed triangle. Let \( L_x := \{ (0: t: 1) \in \mathbb{P}^2_5 \mid t > 0 \} \), \( L_y := \{ (1: 0: t) \in \mathbb{P}^2_5 \mid t > 0 \} \), \( L_z := \{ (t: 1: 0) \in \mathbb{P}^2_5 \mid t > 0 \} \), and \( P_x := (1: 0: 0), P_y := (0: 1: 0), P_z := (0: 0: 1) \in \mathbb{R}^2_5 \). For \( w \in \{ x, y, z \} \), we denote the Euclidean closure of \( L_w \) in \( \mathbb{P}^2_5 \) by \( \overline{L}_w \).

Then \( \Delta^0(\mathbb{P}^2_5) = \{ P_x, P_y, P_z \}, \Delta^1(\mathbb{P}^2_5) = \{ L_x, L_y, L_z \}, \) and \( \Delta^2(\mathbb{P}^2_5) = \{ \text{Int}(\mathbb{P}^2_5) \} \) is the critical decomposition of \( \mathbb{P}^2_5 \) (see Example 1.6 of [1]).

Let \( \varphi := \Phi_{\mathcal{G}_{3,3}} : \mathbb{P}^2_5 \rightarrow \mathbb{P}^2_7(\mathcal{G}_{3,3}) \) and \( Y := \text{Cl}(\varphi(\mathbb{P}^2_5)) \). Since \( \varphi: \mathbb{P}^2_5 \rightarrow Y \) is an isomorphism, \( \Delta^0(Y) = \{ \varphi(P_x), \varphi(P_y), \varphi(P_z) \}, \Delta^1(Y) = \{ \varphi(L_x), \varphi(L_y), \varphi(L_z) \}, \Delta^2(Y) = \{ \text{Int}(Y) \} \) is the critical decomposition of \( Y \). Take \( D \in \Delta(Y) := \Delta^0(Y) \cup \Delta^1(Y) \cup \Delta^2(Y) \), and its dual semialgebraic variety \( D^\vee \in \mathbb{P}_7(\mathcal{G}_{3,3}) \). Let \( \pi: \mathcal{G}_{3,3} \rightarrow \{ 0 \} \rightarrow \mathbb{P}_7(\mathcal{G}_{3,3}) \) be the natural surjection, and let \( \mathcal{F}(D) := \text{Cl}(\pi^{-1}(D^\vee) \cap \partial \mathcal{P}^{c+}_{3,3}) \). \( \mathcal{F}(D) \) is a semialgebraic variety with
\( \dim \mathcal{F}(D) = \dim \mathcal{P}^+_{3,3} - 1 \). Thus \( \mathcal{F}(D) \) is a face component of \( \mathcal{P}^+_{3,3} \), and \( \partial \mathcal{P}^+_{3,3} = \bigcup_{D \in \Delta^d(Y)} \mathcal{F}(D) \).

Thus, \( \partial \mathcal{P}^+_{3,3} \) has 7 irreducible components. If \( D \in \Delta^d(Y) \), then \( \text{Zar}(D^\vee) \) (the Zariski closure in \( \mathbb{P}(\mathcal{H}_{3,3}) \)) has \( \mathbb{P}^{8-d} \)-ruling structure. Thus, every \( f \in \mathcal{E}(\mathcal{P}^+_{3,3}) \) lies in an intersection of some \( \mathcal{F}(D) \).

It is hard to describe the main discriminant \( \text{disc}_{\text{Int}}(Y) \). But for \( D \in \Delta^1(Y) \cup \Delta^0(Y) \), we can calculate \( \text{disc}_D \). Let’s denote \( f \in \mathcal{H}_{3,3} \) as

\[
f(x, y, z) = p_1x^3 + p_2y^3 + p_4z^3 + p_4x^2y + p_5y^2z + p_6z^2x + p_7xy^2 + p_8yz^2 + p_9zx^2 + p_{10}xyz.
\]

For \( D \in \Delta^0(Y) \), \( \text{disc}_{\varphi}(p_1) = p_1 \), \( \text{disc}_{\varphi}(p_2) = p_2 \), and \( \text{disc}_{\varphi}(p_3) = p_3 \).

Let’s calculate an edge discriminant \( \text{disc}_D \) for \( D = \varphi(L_\ell) \in \Delta^1(Y) \). \( \varphi(L_\ell) \) is a twisted cubic curve in \( \mathbb{P}^3 : (x_1 : \cdots : x_{10}) \) defined by

\[
l = x_2x_3 = x_5x_8, \quad x_3x_5 = x_8^2, \quad x_2x_8 = x_5^2, \quad x_1 = x_4 = x_6 = x_7 = x_9 = x_{10} = 0.
\]

\( \text{disc}_D \) is obtained by eliminating \( t \) from \( f(0, t, 1) = p_2t^3 + p_3t^2 + p_8t + p_3 = 0 \) and \( \partial f(0, t, 1)/\partial t = 3p_2t^2 + 2p_3t + p_8 = 0 \).

Thus

\[
\text{disc}_{\varphi(L_\ell)} = 8p_2p_3^2 - 4p_3p_5p_8 + p_8^3.
\]

disc_{\varphi(L_\ell)} \) and disc_{\varphi(L_\ell)} are similar.

### 3.2. Reducible elements of \( \mathcal{E}(\mathcal{P}^+_{3,3}) \).

Let \( X := \{ cx \mid c > 0 \} \cup \{ cy \mid c > 0 \} \cup \{ cz \mid c > 0 \} \subset \mathcal{H}_{3,3} \).

**Proposition 3.1.** If \( f \in X \), \( g \in \mathcal{H}_{3,3} \) and \( fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1}) \), then \( g \in \mathcal{E}(\mathcal{P}^+_{3,d}) \).

Conversely, if \( f \in X \) and \( g \in \mathcal{E}(\mathcal{P}^+_{3,d+1}) \), then \( fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1}) \).

**Proof.** Assume that \( fg \in \mathcal{E}(\mathcal{P}^+_{3,d+1}) \), and \( g = h_1 + h_2 \) for certain \( h_1, h_2 \in \mathcal{P}^+_{3,d} \) \(-\{0\}\). Since \( fg \) is extremal, there exists \( c_1, c_2 \in \mathbb{R}_+ \) such that \( fh_1 = c_1 fg \) and \( fh_2 = c_2 fg \). Thus, \( h_1 = c_1 g \) and \( h_2 = c_2 g \). That is, \( g \) is extremal.

Conversely, we assume \( f \in X \), \( g \in \mathcal{E}(\mathcal{P}^+_{3,d}) \), and \( fg = h_1 + h_2 \) for certain \( h_1, h_2 \in \mathcal{P}^+_{3,d+1} \) \(-\{0\}\). Since \( V_+(f) \subset V_+(h_1 + h_2) \subset V_+(h_1) \cap V_+(h_2) \), \( h_1 \) and \( h_2 \) must be multiples of \( f \). Let \( h_1 = fg_1 \) and \( h_2 = fg_2 \) \((g_1, g_2 \in \mathcal{H}_{3,d}) \). Since \( h_1, h_2 \in \mathcal{P}^+_{3,d+1} \) \(-\{0\}\), we have \( g_1, g_2 \in \mathcal{P}^+_{3,d} \) \(-\{0\}\). Since \( g = g_1 + g_2 \) is extremal, there exists \( c_1, c_2 \in \mathbb{R}_+ \) such that \( g_1 = c_1 g \) and \( g_2 = c_2 g \). Thus, \( fg \) is extremal. \( \square \)

**Proposition 3.2.** If \( f \in \mathcal{E}(\mathcal{P}^+_{3,2}) \), then the following (1) or (2) occurs.

1. \( f = f_1 f_2 \) by certain \( f_1, f_2 \in X \).
2. There exists \( f_1 \in \mathcal{H}_{3,1} \) such that \( f = f_1^2 \) and \( V_+(f_1) \cap \text{Int}(\mathbb{P}^2_+) \neq \emptyset \).

**Proof.** Since \( f \) is extremal, there exists \( P \in \mathbb{P}^2_+ \) such that \( f(P) = 0 \) by Proposition 1.8(2).

(i) Assume that \( P \in \text{Int}(\mathbb{P}^2_+) \) and \( V_+(f) = \{ P \} \). Then \( V_+(f) \subset \mathbb{P}^2_+ \) is a union of two distinct lines that intersect at \( P \). In this case, there exists \( g_1, g_2 \in \mathcal{H}_{4,1} \) such that \( f = g_1^2 + g_2^2 \). Then, \( f \) is not extremal.

(ii) Assume that \( P \in \text{Int}(\mathbb{P}^2_+) \) and \( V_+(f) \supset \{ P \} \). Then, there exists \( Q \in \mathbb{P}^2_+ \) such that \( f(Q) = 0 \) and \( P \neq Q \). Since \( f \) is PDS, \( V_+(f) \) cannot be a real conic. Thus \( V_+(f) \) must be a line passing through \( P \) and \( Q \). Thus (2) occurs.

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(iii) Assume that \( V_\mathbb{R}(f) \cap \text{Int}(\mathbb{P}_+^2) = \emptyset \) and \( P \in \partial \mathbb{P}_+^2 \). We may assume \( P = (a:0:1) \) where \( a \geq 0 \).

It is easy to see that \( \dim(\mathbb{P}^2_{3,2}) \geq \dim \mathbb{P}^2_{3,2} - 2 = 4 \). Since \( f \in (\mathbb{P}^2_{3,2})^\rho \) is extremal, there exists \( Q \in \partial \mathbb{P}_+^2 \) such that \( f(Q) = 0 \).

If \( Q = (b:0:1) \) \((a \neq b, b > 0)\), then, \( f = cy^2 \ (\exists c > 0) \). If \( Q = (0:b:1) \) \((b > 0)\), then, \( f = cxy \ (\exists c > 0) \).

\[ \square \]

**Proposition 3.3.** Let \( f \in \mathcal{E}(\mathbb{P}_{3,3}^+). \) If \( f \) is reducible, then the following (1) or (2) occurs.

1. \( f = f_1f_2f_3 \) and \( f_1, f_2, f_3 \in \mathcal{X} \).
2. \( f = f_1f_2^2, f_1 \in \mathcal{X}, f_2 \in \mathcal{X}_3,1 \) and \( V_\mathbb{R}(f_2) \cap \text{Int}(\mathbb{P}_+^2) \neq \emptyset \).

Conversely, if \( f \in \mathcal{X}_{3,3} \) satisfies (1) or (2), then \( f \) is an extremal in \( \mathbb{P}_{3,3}^+ \).

**Proof.** (i) Since \( f \) is reducible, we can write as \( f = f_1g \), where \( f_1 = ax + by + cz \) \((a, b, c \in \mathbb{R})\) and \( g \in \mathbb{P}^2_{3,2} \). Since \( f_1 \in \mathbb{P}^2_{3,1} \), we have \( a > 0, b \geq 0 \) and \( c > 0 \). Assume that \( a > 0 \) and \( b > 0 \). Then \( f = axg + byg + czg \) and \( axg, byg, czg \in \mathbb{P}^3_{3,3} \). Thus \( f \) is not extremal. This implies \( f_1 \in \mathcal{X} \).

We may assume \( f_1 = x \). Since \( f \) is extremal in \( \mathbb{P}_{3,3}^+ \), \( g \) must be extremal in \( \mathbb{P}_{3,2}^+ \). Then, we have the conclusion by Proposition 3.2.

(ii) We prove the converse part. For \( c > 0 \), the forms \( cx^3, cx^2y, cxyz \) are also extremal in \( \mathbb{P}_{3,3}^+ \). Thus if \( f \) satisfies (1), then \( f \) is extremal.

Consider the case \( f \) satisfies (2). Assume that \( f = g_1 + g_2 \) where \( g_1, g_2 \in \mathbb{P}_{3,3}^+ \). Then \( V_+ (f_1) \subset V_+ (g_1) \cap V_+ (g_2) \). Thus, there exists \( h_1, h_2 \in \mathbb{P}^2_{3,2} \) such that \( g_1 = f_1h_1, g_2 = f_1h_2 \). Since \( V_+ (f_2) \subset V_+ (h_1) \cap V_+ (h_2) \), there exits \( c_1, c_2 \in \mathbb{R}_+ \) such that \( h_1 = c_1f_2^2, h_2 = c_2f_2^2 \).

Thus \( f \in \mathcal{E}(\mathbb{P}_{3,3}^+) \).

\[ \square \]

### 3.3. Irreducible elements of \( \mathcal{E}(\mathbb{P}_{3,3}^+) \).

**Corollary 3.4.** If \( f \in \mathcal{E}(\mathbb{P}_{3,3}^+) \) is irreducible in \( \mathbb{R}[x, y, z] \), then \( f \) is irreducible in \( \mathbb{C}[x, y, z] \).

**Lemma 3.5.** Assume that \( f \in \mathcal{E}(\mathbb{P}_{3,3}^+) \) is irreducible. Then, \( V_\mathbb{C}(f) \) is a rational curve on \( \mathbb{P}_+^2 \) whose unique singular point lies on \( \mathbb{P}_+^2 \).

**Proof.** We shall prove that \( \text{Sing}(V_\mathbb{C}(f)) \cap \mathbb{P}_+^2 \neq \emptyset \). Assume that \( \text{Sing}(V_\mathbb{C}(f)) \cap \mathbb{P}_+^2 = \emptyset \). Then \( f(P) > 0 \) for all \( P \in \text{Int}(\mathbb{P}_+^2) \). We denote the homogeneous coordinate system of \( \mathbb{P}_+^2 \) by \((x_0:x_1:x_2)\). Let

\[ D_i := \{(x_0:x_1:x_2) \in \mathbb{P}_+^2 \mid x_0^2 + x_1^2 + x_2^2 \leq 5x_i^2 \} \]

\((i = 0, 1, 2)\). Note that \( \mathbb{P}_+^2 = D_0 \cup D_1 \cup D_2 \). On \( D_0 \), we put \( x := x_1/x_0, y := x_2/x_0, \) and \( f_0(x, y) := f(1, x, y) \). Let

\[ D := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4, x \geq y \geq 0 \} \subset D_0. \]

(1) We shall prove that there exists \( c_0 > 0 \) such that \( f_0(x, y) \geq c_0xy \) for all \((x, y) \in D \).

Note that if \( f_0(a, 0) = 0 \) for a certain \( a > 0 \), then \( V_\mathbb{C}(f) \) tangents to the \( x \)-axis at \((a, 0)\). Since any cubic curve has no bitangent line, there exists at most one \( a > 0 \) such that \( f_0(a, 0) = 0 \).
(1-1) Assume that there $f_0(x, 0) > 0$ on $0 \leq x \leq 2$. Then there exists no $(x, y) \in D$ such that $f_0(x, y) = 0$. Thus $m := \min \{f_0(x, y) \mid (x, y) \in D\} > 0$. Put $c_0 := m/4$, then $f_0(x, y) \geq c_0xy$ for all $(x, y) \in D$.

(1-2) Assume that $f_0(a, 0) = 0$ for a certain $0 \leq a \leq 2$. We can denote as

$$f_0(x, y) = c_1y + c_2(x - a)^2 + 2c_3(x - a)y + c_4y^2 + g(x - a, y)$$

where $g(s, t) = c_5s^3 + c_6s^2t + c_7st^2 + c_8t^3$. Since $f_0(a, y) \geq 0$ for all $y \geq 0$, we have $c_1 \geq 0$. If $c_1 = 0$, then $(a, 0)$ is a singular point of $V_C(f)$. Thus $c_1 > 0$. Then, there exists an open neighbourhood $(a, 0) \in U_a \subset D$ such that $f_0(x, y) \geq (c_1/2)y$ for all $(x, y) \in U_a$. Then $f_0(x, y) \geq (c_1/2)xy$ for all $(x, y) \in U_a$. So, we put $m_1(a) := c_1/4$.

Let $V := \{a \in [0, 2] \mid f_0(a, 0) = 0\}$, $U := \bigcup_{a \in V} U_a$, and $m_2 := \min_{a \in V} m_1(a)$. Then $f_0(x, y) \geq m_2xy$ for all $(x, y) \in U$. Note that

$$m_3 := \min \{f_0(x, y) \mid (x, y) \in \text{Cls}(D - U)\} > 0.$$ 

So, put $c_0 := \min\{m_2, m_3/4\}$. Then $f_0(x, y) \geq c_0xy$ for all $(x, y) \in D$.

By (1), there exists $c > 0$ such that $F(x_0, x_1, x_2) \geq cx_0x_1x_2$ for all $(x_0; x_1; x_2) \in \mathbb{P}^2_3$. So, $f(x_0, x_1, x_2) - cx_0x_1x_2 \in \mathbb{P}^+_{3,3}$ and $f$ is not extremal. A contradiction. Thus $\text{Sing}(V_C(f)) \cap \mathbb{P}^2_+ \neq \emptyset$.

Since $V_C(f)$ is a cubic curve, $V_C(f)$ has just one singular point $P$ and $V_C(f)$ is a rational curve. $P \in \mathbb{P}^2_3$, since $\text{Sing}(V_C(f)) \cap \mathbb{P}^2_+ \neq \emptyset$.

Lemma 3.6. Assume that $f \in \mathcal{E}(\mathbb{P}^+_{3,3})$ is irreducible, and $P$ is the unique singular point of $V_C(f)$. Then $P \notin \{P_x, P_y, P_z\}$, where $P_x = (1:0:0)$ and so on.

Proof. We use the same notation with the proof of the previous lemma. Assume the $P = (0, 0) \in D$. Then $f_0(x, y) = g_2(x, y) + g_3(x, y)$ where $g_d(x, y)$ is a homogeneous polynomial of degree $d$. Then $f(x_0, x_1, x_2) = g_2(x_1, x_2)x_0 + g_3(x_1, x_2)$. Considering the cases $x_0 = 0$ and $x_0 \to +\infty$, we conclude that $g_2(x_1, x_2) \in \mathbb{P}^+_{2,2}$ and $g_3(x_1, x_2) \in \mathbb{P}^+_{2,3}$. It is easy to see that $g_2 \not\equiv 0$ and $g_3 \not\equiv 0$. Thus $f$ is not extremal.

Lemma 3.7. Assume that $f \in \mathcal{E}(\mathbb{P}^+_{3,3})$ is irreducible. Then $V_C(f)$ does not have a cusp.

Proof. Assume that $P$ is a cusp of $V_C(f)$. We already proved that $P \in \mathbb{P}^2_3$ and $P \notin \{P_x, P_y, P_z\}$. Since $V_{R}(f) \cap U \neq \{P\}$ for any Euclidian open neighborhood $P \in U \subset \mathbb{P}^2_R$, we have $P \notin \text{Int}(\mathbb{P}^2_+)$. So we may assume that $P = (1: a: 0)$ $(a > 0)$. Let $f_0(x, y) = f(1, x, y)$.

As is well known in algebraic geometry, after a suitable substitution $s = a_0x + a_1y + a_2$, $t = b_0x + b_1y + b_2$, the equation $f_0(x, y) = 0$ can be reformed as $s^2 + t^3 = 0$ (see [31]). Two lines $V_{R}(s)$ and $V_{R}(t)$ intersect at $P$. Since $V_{R}(s^2 + t^3) \cap \text{Int}(\mathbb{P}^2_+) = \emptyset$, we have $V_{R}(t) \cap \text{Int}(\mathbb{P}^2_+) = \emptyset$. Thus $s^2, t^3 \in \mathbb{P}^+_{3,3}$ and $f$ is not extremal.

So, if $f \in \mathcal{E}(\mathbb{P}^+_{3,3})$, then $V_{R}(f)$ has a node $P$ on $\mathbb{P}^2_3$. $P$ must be a cusp. Remember that the isolated point $P \in V_{R}(f)$ is called an acnode, if $P$ is a node of $V_C(f)$ and $V_C(f)$ have just two imaginal tangent lines at $P$.

Lemma 3.8. Assume that $f \in \mathcal{E}(\mathbb{P}^+_{3,3})$ is irreducible, and $P$ is the acnode of $V_C(f)$. Then $P \notin \partial \mathbb{P}^2_+$. 

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Proof. Use \((x : y : z)\) as a homogeneous coordinate system of \(\mathbb{P}^2_+\). We assume \(P = (1 : a : 0) \in L_z\) \((a > 0)\). Let \(L_P := (\mathcal{P}^+_3)_P\) be the local cone of \(\mathcal{P}^+_3\) at \(P\).

Since \(f \in \mathcal{E}(\mathcal{P}^+_3, \mathcal{P}_3)\), there exists local cones or infinitesimal local cones \(\mathcal{L}_1, \ldots, \mathcal{L}_r \subset \mathcal{P}^+_3\) such that \(\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f\). Let \(P_i \in \mathbb{P}^2_+\) be the point corresponding to \(\mathcal{L}_i\).

(1) We shall show that if \(P_i = P\), then \(\mathcal{L}_i\) must be a local cone \(\mathcal{L}_i = \mathcal{L}_P\).

Note that \(\text{Bs} L_P = \{P\}\), and \(L_P\) does not have a fixed part. Let \(\psi : X \to \mathbb{P}^2_+\) be the blowing up at \(P\), and \(Q\) be the resolution of the fixed part of \(\psi^* L_P\). Let \(E := \psi^{-1}(P)\) and \(h\) be the defining equation of \(E\). Then, the natural isomorphism \(\rho : \Omega \to \psi^* L_P\) is given by \(\rho(g) = g/h^2\). Thus \(\rho^{-1}(f) \in \Omega\) has no zero on \(E\). Therefore length \(L_P = 0\), and \(\mathcal{L}_i = \mathcal{L}_P\).

(2) We may assume \(P_1 = P\) and \(P_i \neq P\) for \(i \geq 2\). Moreover we may assume \(\{P_1, \ldots, P_r\} = V_+(f)\). Since \(V_C(f)\) tangles to \(L_z\) at \(P\), we have \(f(1, 0, 0) \neq 0\) or \(f(0, 1, 0) \neq 0\). We may assume \(f(1, 0, 0) \neq 0\).

If \(P_i \in L_x (i \geq 2)\), then \(V_C(f)\) tangles to \(L_x\) at \(P_i\). Thus \(\#(L_x \cap V_+(f)) \leq 1\), since \(f\) is cubic. Moreover we have length \(L_i \leq 1\) observing the blowing up at \(P_i\).

If \(L_y \cap V_+(f) = \{P_i\}\), then we put \(Q := P_i\). If \(L_y \cap V_+(f) = \emptyset\), then let \(Q\) be any point on \(L_y\). Note that \(V_+(f) \cap \{P, Q\} \subset L_x = V_+(x)\). Let \(f_1(x, y, z) \in \mathcal{H}_3, \mathcal{P}_3\) be the defining equation of the line passing through \(P\) and \(Q\). Then \(x^r f^2 \in \mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r\), since length \(L_i \leq 1\) for \(i \geq 2\). This contradicts to \(\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f\), since \(f\) is irreducible.

For \(f \in \mathcal{H}_3, \mathcal{P}_3\), let \(D_f\) be a Weil divisors on \(\mathbb{P}^2_+\) defined by \(f = 0\). For \(g \in \mathcal{H}_3, \mathcal{P}_3\) and a point \(P \in \mathbb{P}^2_+\), we denote the local intersection number of \(D_f\) and \(D_g\) at \(P\), by \(I_P(f, g)\). If \((x, y)\) is a local coordinate system at \(P\), then \(I_P(f, g) = \dim_{\mathbb{C}} \mathcal{C}[x, y]/(f, g)\).

**Theorem 3.9.** We use \((x : y : z)\) as a homogeneous coordinate system of \(\mathbb{P}^2_+\). Assume that \(f \in \mathcal{E}(\mathcal{P}^+_3, \mathcal{P}_3)\) is irreducible. Then \(V_C(f) \subset \mathbb{P}^2_+\) is a rational cubic curve which has an acnode at a certain \(P \in \text{Int}(\mathbb{P}^2_+)\). Moreover, one of the followings occurs.

1. There exists \(Q_x \in L_x, Q_y \in L_y\) and \(Q_z \in L_z\) such that \(V_x(f) = \{P, Q_x, Q_y, Q_z\}\).
2. There exists an element of \(w \in \{x, y, z\}\) such that \(V_+(f) \cap L_w = \emptyset\). Assume that \(w = z\). Then there exists \(Q_x \in L_x\) and \(Q_y \in L_y\) such that \(V_x(f) = \{P, Q_x, Q_y, P_z, P_y\}\), where \(P_z = (1:0:0)\) and \(P_y := (0:1:0)\).
3. The case taking some limits in (1) at least one of \(Q_x \to P_y, Q_y \to P_z, Q_z \to P_x\).
   
   For example, \(Q_z \to P_y\) implies \(I_{P_y}(f, x) = 2\).

4. The case taking limits in (1) at least one of \(Q_x \to P_z, Q_y \to P_x, Q_z \to P_y\).
   
   For example, \(Q_x \to P_y\) implies \(I_{P_y}(f, x) = 3\).

Proof. Take an irreducible \(f \in \mathcal{E}(\mathcal{P}^+_3, \mathcal{P}_3)\). We have proved that \(V_C(f) \subset \mathbb{P}^2_+\) is a rational cubic curve which has an acnode at \(P \in \text{Int}(\mathbb{P}^2_+)\).

There exists local cones or infinitesimal local cones \(\mathcal{L}_1, \ldots, \mathcal{L}_r \subset \mathcal{P}^+_3\) such that \(\mathcal{L}_1 \cap \cdots \cap \mathcal{L}_r = \mathbb{R}_+ \cdot f\). Let \(P_i \in \mathbb{P}^2_+\) be the point corresponding to \(\mathcal{L}_i\). We may assume that \(V_+(f) := \{P_1, \ldots, P_r\}\), \(P_1 = P\) and \(\mathcal{L}_1 = \mathcal{E}(\mathcal{P}^+_3, P)\). Moreover, if \(i \geq 2\), then \(P_i \neq P\) and length \(L_i \leq 1\). It is easy to see that \(\dim \mathcal{L}_1 = \dim \mathcal{P}^+_3 - 3 = 7\). Thus \(r \geq 2\).

(1) Consider the case \(V_+(f) \cap L_z = \emptyset\).

(I-1) We shall prove (i)“\(V_+(f) \cap L_z = \{P_x, P_y\}\) and \(I_{P_x}(f, z) = I_{P_y}(f, z) = 1\)” or (ii)“there exists \(Q \in \{P_x, P_y\}\) such that \(I_Q(f, z) \geq 2\).”
Assume that $I_Q(f, z) \leq 1$ for each $Q \in \{P_x, P_y\}$ and $\#(V_+(f) \cap L_z) \leq 1$. We may assume that $f(P_x) \neq 0$. Remember that $\#(V_+(f) \cap L_x) \leq 1$ and $\#(V_+(f) \cap L_y) \leq 1$. If $\#(V_+(f) \cap L_x) = 1$, we take $Q_x \in V_+(f) \cap L_x$. If $V_+(f) \cap L_x = \emptyset$, we take any $Q_x \in L_x$. We choose $Q_y \in L_y$ similarly. Then $V_+(f) \subset \{P, Q_x, Q_y, P_x, P_y\}$. Let $f_1(x, y, z) \in \mathcal{H}_3,1$ be the defining equation of the line passing through $P$ and $Q_z$. Then $xf_1^2 \in L_1 \cap \cdots \cap L_e = \mathbb{R}^+ \cdot f$. A contradiction. Thus $f(P_x) = f(P_y) = 0$.

(I-2) The case $V_+(f) \cap L_z = \emptyset$ and $I_Q(f, z) \geq 2$ for one of $Q \in \{P_x, P_y\}$ will be discussed at (IV). From (I-3) to the end of (II-1), we continue discussion assuming $I_{P_x}(f, w) \leq 1$ for all $w \in \{x, y, z\}$.

(I-3) We shall prove that $V_+(f) \cap L_x \neq \emptyset$ and $V_+(f) \cap L_y \neq \emptyset$.

Assume that $V_+(f) \cap L_y = \emptyset$. Then $V_+(f) \subset L_x \cup \{P, P_x\}$. Let $f_1(x, y, z) \in \mathcal{H}_3,1$ be the defining equation of the line passing through $P$ and $P_x$. Then $xf_1^2 \in L_1 \cap \cdots \cap L_e = \mathbb{R}^+ \cdot f$. A contradiction. Thus $V_+(f) \cap L_y \neq \emptyset$.

Let $V_+(f) \cap L_x = \{Q_x\}$, and $V_+(f) \cap L_y = \{Q_y\}$. Since the intersection number of $V_{C^2}(f)$ and $V_{C^2}(x)$ on $\mathbb{P}_{C^2}^2$ is equal to 3, we have $f(P_x) \neq 0$. Thus $V_+(f) = \{P, P_x, P_y, Q_x, Q_y\}$. This is the case (2).

(II) Consider the case $V_+(f) \cap L_x \neq \emptyset$, $V_+(f) \cap L_y \neq \emptyset$ and $V_+(f) \cap L_z \neq \emptyset$.

Since the cubic curve $V_{C^2}(f)$ does not have bitangent line, we have $V_+(f) \cap L_x = \{Q_x\}$, $V_+(f) \cap L_y = \{Q_y\}$, and $V_+(f) \cap L_z = \{Q_z\}$.

(II-1) We shall prove that $f(P_x) \neq 0$, $f(P_y) \neq 0$ and $f(P_z) \neq 0$.

Let $\pi: \mathbb{P}_{\mathbb{R}^2}^2 \to \mathbb{P}_{\mathbb{R}}^2$ be the surjective morphism defined by $\pi(x: y: z) = (x^2: y^2: z^2)$. Let $g := \pi^*f$. That is $g(x, y, z) = f(x^2, y^2, z^2)$. Note that $\#\pi^{-1}(P) = 4$, $\#\pi^{-1}(Q_w) = 2$, $\#\pi^{-1}(P_u) = 1$ for all $w \in \{x, y, z\}$. Thus $\#V_{\mathbb{R}}(g) = 4 + 3 \cdot 2 = 10$. Let $V_{10} := \pi^{-1}(\{P, Q_x, Q_y, Q_z\})$. We shall prove the $g$ is irreducible in $\mathbb{C}[x, y, z]$ in at (II-3). Let $C := V_{C^2}(g) \subset \mathbb{P}_{\mathbb{C}}^2$. Then

$$p_a(C) \geq \sum_{Q \in \text{Sing}(C)} \frac{\nu(Q)(\nu(Q) - 1)}{2} + g(C').$$

Since $V_{10} \subset \text{Sing}(C)$, we conclude that $V_{10} = \text{Sing}(C)$ and $g(C') = 0$. Thus, $V_{C^2}(g)$ is a rational curve which has 10 nodes at $V_{10}$. As is well known, there exists unique sextic curve which has 10 nodes at $V_{10}$. Thus $g \in \mathcal{E}(\mathbf{P}_{3,6})$.

Let $Q \in \pi^{-1}(P_x)$. Since $V_{6}(g) = V_{10}$, we have $g(Q) \neq 0$. Thus $f(P_x) \neq 0$. Similarly, we have $f(P_y) \neq 0$ and $f(P_z) \neq 0$.

(II-3) We shall prove that $g$ is reducible in $\mathbb{C}[x, y, z]$.

To begin with, we prove that $g$ is reducible in $\mathbb{R}[x, y, z]$. Assume that $g = g_1g_2$, where $g_1, g_2 \in \mathbb{R}[x, y, z]$. If $\deg g_1$ is odd, then $V_{10} = V_{\mathbb{R}}(g) \supset V_{\mathbb{R}}(g_1)$. This implies $V_{10}$ is not a set of isolated points. Thus we may assume $\deg g_1 = 2$ and $\deg g_2 = 4$. Then $g_2 \in \mathcal{E}(\mathbf{P}_{3,4}) \subset \Sigma_{3,4}$. But $g_2 = g_2^3 (\exists g_3 \in \mathcal{H}_{3,2})$ cannot construct $V_{10}$. Thus $g$ is irreducible in $\mathbb{R}[x, y, z]$.

Assume that $g$ is reducible in $\mathbb{C}[x, y, z]$. Then there exists $g_3 \in \mathbb{C}[x, y, z]_3 \subset \mathbb{R}[x, y, z]$ such that $g = g_3g_3$. It is easy to see that $g_3$ is irreducible in $\mathbb{C}[x, y, z]$.

Assume that $V_{C^2}(g_3)$ has a singular point $Q \in \mathbb{P}_{\mathbb{R}}^2$. Then $Q \in V_{10}$. But $V_{10}$ does not have such high multiplicity singularity. Thus $V_{C^2}(g_3)$ does not have singular points in $\mathbb{P}_{\mathbb{R}}^2$. Therefore $V_{10} \subset V_{C^2}(g_3) \cap V_{C^2}(g_2)$. This is impossible because the intersection number of $V_{C^2}(g_3)$ and $V_{C^2}(g_2)$ is nine. Thus, $g$ is irreducible in $\mathbb{C}[x, y, z]$.
(II) We shall observe $C = V_C(g)$ in the case (2), assuming that $V_+(f) = \{ P, Q_x, Q_y, P_x, P_y \}$, where $g(x, y, z) = f(x^2, y^2, z^2).

$\#\pi^{-1}(P) = 4$, $\#\pi^{-1}(Q_x) = 2$, $\#\pi^{-1}(Q_y) = 2$, $\#\pi^{-1}(P_x) = 1$, $\#\pi^{-1}(P_y) = 1$. Thus $\#V_+(g) = 10$. By the same argument as in (II-1), we conclude that $C$ is irreducible. 

Let $N(g)$ be the number of zeros of $g$ on $\mathbb{P}^2_{\mathbb{R}}$ counting infinitely near points exactly. $N(g) = 10$ in the cases (1) and (2). To treat the cases (3), (3') and (4), we shall observe how $N(g)$ changes by a deformation $Q_x \to P_y$ and so on. It is well known that $\lim N(g)$ never decreases. If $\lim N(g)$ is finite, we can prove that $\lim N(g) = 10$ and $C$ is irreducible, by the same way as (II-3). $N(g)$ is finite in any cases of (3), (3') and (4). Thus $C$ is irreducible and $\lim N(g) = 10$ for any $f$ in (3), (3'), (4).

(IV) We return the case (I-2). We may assume that $Q = P_x$. Note that $2 \leq I_{P_x}(f, z) \leq 3$. In (IV), we treat the cases $I_{P_x}(f, z) = 3$. Note that $V_+(f) \cap L_z = \emptyset$ and $f(P_y) \neq 0$.

(IV-1) Consider the case $V_+(f) \cap L_x = \emptyset$ and $I_{P_x}(f, z) = 3$.

Then $I_{P_x}(f, x) \geq 2$ by (I-1). $V_+(f) \cap L_y = \emptyset$ since $f$ is cubic. Assume that $I_{P_x}(f, x) = 2$. Let $f_1 \in \mathcal{H}_{3,1}$ be the defining equation of the line passing through $P$ and $P_x$. Then $z f_1^2 \in L_1 \cap \cdots \cap L_r = \mathbb{R} \cdot f$. A contradiction. Thus $I_{P_x}(f, x) = 3$. This is the case (4).

(IV-2) Consider the case $V_+(f) \cap L_x \neq \emptyset$ and $V_+(f) \cap L_y = \emptyset$.

Since $I_{P_x}(f, y) = 1$, we have $f(P_x) = 0$ by (I-1). This is the case (4).

(IV-3) Consider the case $V_+(f) \cap L_x \neq \emptyset$ and $V_+(f) \cap L_y = \emptyset$.

Then $V_+(f) \cap L_x = \{ Q_x \}$ and $V_+(f) \cap L_y = \{ Q_y \}$. Consider $\pi: \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$ as (II-1). Note that $C := V_C(g) \subset \mathbb{P}^2_{\mathbb{R}}$ has a node at $\pi^{-1}(P_x)$ which is analytically isomorphic to $y^2 + x^4 = 0$. Thus $N(g) = 11$. A contradiction.

(V) We treat the cases $I_{P_x}(f, z) = 2$. We may assume that $I_{P_x}(f, w) \leq 2$ for all $u, w \in \{ x, y, z \}$, by (IV). Note that $V_+(f) \cap L_z = \emptyset$.

(V-1) Consider the case $V_+(f) \cap L_y = \emptyset$ and $V_+(f) \cap L_x = \emptyset$.

Since $N(g) = 10$, we have $f(P_y) \neq 0$ and $f(P_x) \neq 0$. This is the case (3).

(V-2) Consider the case $V_+(f) \cap L_y = \emptyset$ and $V_+(f) \cap L_x = \emptyset$.

Since $I_{P_x}(f, y) = 1$, we have $f(P_x) = 0$ by (I-1). Since $V_+(f) \cap L_x = \emptyset$, we have $f(P_y) \neq 0$. Since $N(g) = 10$, we have $I_{P_x}(f, y) = 2$. This is the case (3).

(V-3) Consider the case $V_+(f) \cap L_y = \emptyset$ and $V_+(f) \cap L_x = \emptyset$.

Then $f(P_x) \neq 0$. Thus $I_{P_x}(f, x) \geq 2$ by (I-1). Since $N(g) = 10$, we have $I_{P_x}(f, x) = 2$. This is the case (3).

(V-4) Consider the case $V_+(f) \cap L_y = \emptyset$ and $V_+(f) \cap L_x = \emptyset$.

Since $I_{P_x}(f, y) = 1$, we have $f(P_x) = 0$ by (I-1). Note that $C := V_C(g) \subset \mathbb{P}^2_{\mathbb{R}}$ has a node at $\pi^{-1}(P_x)$ which is analytically isomorphic to $y^2 + x^4 = 0$. Since $N(g) = 10$, it must be $I_{P_x}(f, y) = 2$ and $I_{P_x}(f, x) = 2$. This is the case (3).

Theorem 3.10. Assume that $f \in \mathcal{E}(\mathbb{P}^3_{3,3})$. Let $g(x, y, z) = f(x^2, y^2, z^2)$. Then $g \in \mathcal{E}(\mathbb{P}^3_{3,6})$. Moreover, if $f$ is irreducible, then $V_C(g)$ is a rational curve and the number of zeros of $g$ on $\mathbb{P}^3_{\mathbb{R}}$ is equal to 10 counting infinitely near points exactly.

Proof. Take $f \in \mathcal{E}(\mathbb{P}^3_{3,3})$. If $f$ is irreducible, the theorem is proved in the proof of Theorem 3.9. If $f$ is reducible, $g \in \mathcal{E}(\mathbb{P}^3_{3,6})$ by Proposition 3.3. \qed
\textbf{Definition 3.11.} Let \( f \in \mathcal{E}(\mathcal{P}^+_3) \) be irreducible. We use the same notation as Theorem 3.9. If \( V_+(f) = \{P, Q_x, Q_y, Q_z\} \), then we say \( f \) is of type 1. If \( V_+(f) = \{P, Q_x, Q_y, P_x, P_y\} \), then we say \( f \) is of type 2. If (3) or (3') of Theorem 3.9 holds, then we say \( f \) is of type 1'. If (4) holds, then we say \( f \) is of type 2'.

We shall discuss whether such a \( f \in \mathcal{E}(\mathcal{P}^+_3) \) exists. We need to reduce complexity. Let \((1:a:b)\) be the coordinate of \( P \), where \( a > 0 \) and \( b > 0 \). By observing \( f'(x, y, z) := f(x, y/a, y/b) \), we may assume \( P = (1:1:1) \).

\textbf{3.4. Type 2 elements of} \( \mathcal{E}(\mathcal{P}^+_3) \).

Assume that \( f \in \mathcal{E}(\mathcal{P}^+_3) \) is irreducible, and \( V_+(f) = \{P, Q_x, Q_y, P_x, P_y\} \), here \( P = (1:1:1) \). We put \( Q_x = (0:1:p) \) and \( Q_y = (1:0:q) \) where \( p \geq 0 \) and \( q \geq 0 \). Let \( f_w := \partial f / \partial w \) for \( w \in \{x, y, z\} \). Solve the system of equations \( f(1,1,1) = 0, f_x(1,1,1) = 0, f_y(1,1,1) = 0, f(0,1,q) = 0, f_z(0,1,q) = 0, f(1,0,p) = 0, f_z(1,0,p) = 0, f(1,0,0) = 0, f(0,1,0) = 0 \). This is an elementary calculation of a 9 \( \times \) 10 matrix linear equation. This matrix is of rank 9 if \( p > 0 \) and \( q > 0 \). Using Mathematica, we know that the solution of the above equations is a multiple of the following \( g_{pq}(x, y, z) \).

\textbf{Definition 3.12.} We define \( g_{pq}(x, y, z) \) by:

\[
\begin{align*}
g_{pq}(x, y, z) &:= z^3 + p^2x^2z + q^2y^2z - 2pxz^2 - 2qyz^2 - (p^2 + q^2 - 4p - 4q + 3)xyz \nonumber \\
&\quad + (1 + p - q)(1 - p - q)x^2y + (1 - p + q)(1 - p - q)xy^2. \nonumber
\end{align*}
\]

\textbf{Theorem 3.13.} Assume that \( p \geq 0 \) and \( q \geq 0 \).

1. If \( p + q > 1 \), then \( g_{pq} \notin \mathcal{P}^+_3 \).
2. If \( p + q = 1 \), then \( g_{pq}(x, y, z) = (px + qy - z)^2z \).
3. If \( p + q < 1 \), then \( g_{pq} \) is of type 2.
4. If \( p + q < 1 \) and \( pq = 0 \), then \( g_{pq} \) is of type 2'.

\textit{Proof.} (1) Assume that \( p + q > 1 \). Let \( a := 1/2(p + q - 1) > 0 \). Since \( g_{pq}(x, x, 1) = (x - 1)^2(1 - 2(p + q - 1)x) \), we have \( g_{pq}(x, x, 1) < 0 \) if \( x > a \). Thus \( g_{pq} \notin \mathcal{P}^+_3 \).

(2) is easy to see.

(3) Assume that \( p + q < 1 \).

Let \( g(x, y, z) := (1+p-q)xy + (1-p+q)y - 2z \). Note that \( g(1, 1, 1) = 0 \). Fix \((x, y; z) \in \mathbb{P}^2_+ \).

Consider the case \( g(x, y, z) \geq 0 \). Then \( x \geq z \) or \( y \geq z \), since \( g(1, 1, 1) = 0 \).

If \( y \geq z \), then

\[
g_{pq}(x, y, z) = (1 - p - q)y(x - z)g(x, y, z) + (px + (1 - p)y - z)^2 z \geq 0. \quad (3.13.1)
\]

If \( x \geq z \), then

\[
g_{pq}(x, y, z) = (1 - p - q)x(y - z)g(x, y, z) + ((1 - q)x + qy - z)^2 z \geq 0. \quad (3.13.2)
\]

Consider the case \( g(x, y, z) < 0 \). Then \( x \leq z \) or \( y \leq z \), since \( g(1, 1, 1) = 0 \). If \( x \leq z \), then \( g_{pq}(x, y, z) \geq 0 \) by (3.13.1). If \( y \leq z \), then \( g_{pq}(x, y, z) \geq 0 \) by (3.13.2). Thus \( g_{pq}(x, y, z) \geq 0 \) for all \((x, y; z) \in \mathbb{P}^2_+ \), if \( p + q < 1 \).

Assume that \( x > 0 \), \( y > 0 \), \( z > 0 \) and \( g_{pq}(x, y, z) = 0 \). (3.13.1) and (3.13.2) implies that \( x = y = z \). It is easy to see that \( g_{pq} \) is irreducible. Thus \( g_{pq} \) is of type 2.

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(4) is easy to see. Note that
\[ g_{pq}(x, y, z) = (1 - p^2)x^2y + (1 - p)^2y^2 + p^2x^2z - 2pxz^2 - (1 - p)(3 - p)xyz + z^3, \]
\[ g_{00}(x, y, z) = x^2y + xy^2 + z^3 - 3xyz. \]

3.5. Type 1 elements of \( E(\mathcal{P}_{3,3}) \).

Assume that an irreducible \( f \in E(\mathcal{P}_{3,3}^+) \) is of type 1. Then \( V_+(f) = \{ P, Q_x, Q_y, Q_z \} \). We assume \( P = (1:1:1) \) as the previous case. Let \( Q_x = (0:p:1) \), \( Q_y = (1:0:q) \) and \( Q_z = (r:1:0) \). Solve the system if equations \( f(1,1,1) = 0, \ f_x(1,1,1) = 0, \ f_y(1,1,1) = 0, \ f(0,p,1) = 0, \ f_y(0,p,1) = 0, \ f(1,0,q) = 0, \ f_z(1,0,q) = 0, \ f(r,1,0) = 0, \ f_x(r,1,0) = 0 \). The solution of the above equations is a multiple of the following \( f_{pqr}(x, y, z) \).

**Definition 3.14.** We define \( f_{pqr}(x, y, z) \) as the following:
\[
\begin{align*}
  a_1(p, q) &:= pq - p + 1, \\
  a_2(p, q, r) &:= pq - p^2 + pqr - pq + 2pr + p - r + 1, \\
  c_1(p, q, r) &:= q^3a_1(p, q)a_1(r, p)a_2(p, q, r), \\
  c_2(p, q, r) &:= -a_1(p, q)(2p^2q^3r^3 - 2p^2q^2r^2 + 6pq^2r^3 - 2pq^3r^2 - 6pq^2r^3 + 3pq^2r^2 - 3pq^3r - 2pq^2r + 6pq^2 + 2pq^2 + 2p - 2), \\
  c_3(p, q, r) &:= r_1(p, q)(p^3q^3r - p^2q^2r^2 + 3pq^2r^3 - 3pq^3r^2 + 3pq^2r + q^2), \\
  c_4(p, q, r) &:= -c_1(p, q, r) - c_1(q, r, p) - c_1(r, p, q) - c_2(p, q, r) - c_2(q, r, p) - c_2(r, p, q) - c_3(p, q, r) - c_3(q, r, p) - c_3(r, p, q), \\
  f_{pqr}(x, y, z) &:= c_1(p, q, r) + c_1(q, r, p) + c_1(r, p, q) + c_2(p, q, r) + c_2(q, r, p) + c_2(r, p, q) + c_3(p, q, r) + c_3(q, r, p) + c_3(r, p, q) + c_4(p, q, r) + c_4(q, r, p) + c_4(r, p, q).
\end{align*}
\]
We shall study the condition of \( (p, q, r) \) for \( f_{pqr} \in \mathcal{P}_{3,3}^+ \).

**Lemma 3.15.** Assume that \( p > 0, q > 0 \) and \( r > 0 \). Then, \( f_{pqr}(1,0,0) > 0, \ f_{pqr}(0,1,0) > 0, \) and \( f_{pqr}(0,0,1) > 0 \), if and only if \( a_1(p, q) > 0, \ a_1(q, r) > 0 \) and \( a_1(r, p) > 0 \).

**Proof.** (1) Assume that \( a_1(p, q) > 0, \ a_1(q, r) > 0, \) and \( a_1(r, p) > 0 \).

Then \( a_2(p, q, r) = pr_1(p, q) + pa_1(q, r) + a_1(r, p) > 0 \). Thus,
\[
f_{pqr}(1,0,0) = c_1(p, q, r) = q^2a_1(p, q)a_1(r, p)a_2(p, q, r) > 0.
\]
Similarly, we have \( f_{pqr}(0,1,0) > 0 \) and \( f_{pqr}(0,0,1) > 0 \).

(2) Assume that \( f_{pqr}(1,0,0) > 0, \ f_{pqr}(0,1,0) > 0, \) and \( f_{pqr}(0,0,1) > 0 \). We shall derive a contradiction assuming \( a_1(p, q) < 0 \).

(2-1) we shall show that \( p > 1, 0 < q < 1, \ a_1(q, r) > 0 \) and \( a_1(r, p) > 0 \).

If \( p \leq 1 \), then \( a_1(p, q) = pq + (1 - p) \geq 0 \). If \( q \geq 1 \), then \( a_1(p, q) = p(q - 1) + 1 \geq 0 \).

Thus, if \( a_1(p, q) < 0 \), then \( p > 1 \) and \( 0 < q < 1 \). Then, \( a_1(q, r) = qr + (1 - q) > 0 \), and \( a_1(r, p) = r(p - 1) + 1 > 0 \).
Let

\[ b_1(p, q) := -p^2 q + p^2 - pq - 2p + 1 = (p - 1)^2 - p(p + 1)q, \]
\[ b_2(p, q) := p^2 q^2 - 2p^2 q - 2pq + p^2 - 2p + 1, \]
\[ r_0(p, q) := \frac{1 + p(1 - q)}{b_1(p, q)}, \]
\[ r_2(p, q) := \frac{p(1 - q)^2 - 1 - q}{q(pq + (p - 1))}. \]

(2-2) We shall show that \( b_1(p, q) > 0 \) and \( r_0(p, q) < r < r_2(p, q) \). Since

\[ 0 < f_{pqr}(1, 0, 0) = q^2 a_1(p, q) a_1(r, p) a_2(p, q, r), \]
\[ 0 < f_{pqr}(0, 1, 0) = r^2 a_1(q, r) a_1(p, q) a_2(q, r, p), \]
\[ 0 < f_{pqr}(0, 0, 1) = p^2 a_1(r, p) a_1(q, r) a_2(r, p, q), \]

we have \( a_2(p, q, r) < 0, a_2(q, r, p) < 0 \) and \( a_2(r, p, q) > 0 \). Note that \( a_2(p, q, 0) = 1 + p(1 - q) > 0 \). Since \( 0 > a_2(p, q, r) = b_1(p, q) r + a_2(p, q, 0) \), we have \( b_1(p, q) > a_2(p, q, 0)/r > 0 \). Thus, \( a_2(p, q, r) \) is monotonically decreasing on \( r \). The equation \( a_2(p, q, r) = 0 \) on \( r \) has just one root \( r = r_0(p, q) \). Since \( a_2(p, q, r) < 0 \), we have \( r > r_0(p, q) \).

Since \( 0 > a_2(q, r, p) = r q(pq + (p - 1)) + (-pq^2 + 2pq - p + q + 1) \) is monotonically increasing on \( r \), we have \( r < r_2(p, q) \). Since \( r_0(p, q) < r < r_2(p, q) \), we have

\[ 0 < r_2(p, q) - r_0(p, q) = \frac{-a_1(p, q) b_2(p, q)}{q(pq + (p - 1)) b_1(p, q)}. \]

Thus \( b_2(p, q) > 0 \).

Remember that \( 0 < a_2(r, p, q) = (p - 1)qr^2 - ((p - 1) - (p + 2)q)r + (1 - q) \). This is a quadric function on \( r \) with \( (p - 1)q > 0 \). Note that

\[ a_2(r_0(p, q), p, q) = \frac{2a_1(p, q) b_2(p, q)}{((p - 1)^2 - pq(p + 1))^2} < 0, \]
\[ a_2(r_2(p, q), p, q) = \frac{2a_1(p, q) b_2(p, q)}{(p - 1 + pq)^2} < 0. \]

Thus, if \( r_0(p, q) \leq r \leq r_2(p, q) \), then \( a_2(r, p, q) < 0 \). A contradiction. Thus we have \( a_1(p, q) > 0 \).

Similarly, we obtain \( a_1(q, r) > 0 \) and \( a_1(r, p) > 0 \).

\[ \square \]

**Theorem 3.16.** Assume that \( p > 0, q > 0 \) and \( r > 0 \). Then, \( f_{pqr} \) is of type 1 if and only if \( a_1(p, q) > 0, a_1(q, r) > 0 \) and \( a_1(r, p) > 0 \).

*Proof.* Assume that \( f_{pqr} \) is of type 1. Then \( f_{pqr}(1, 0, 0) > 0, f_{pqr}(0, 1, 0) > 0, f_{pqr}(0, 0, 1) > 0 \). Thus \( a_1(p, q) > 0, a_1(q, r) > 0, \) and \( a_1(r, p) > 0 \).

Conversely, assume that \( a_1(p, q) > 0, a_1(q, r) > 0, \) and \( a_1(r, p) > 0 \). Let \( \ell_t \subset P^2_\mathbb{R} \) be the line defined by \( y - z = t(x - z) \) where \( t \in \mathbb{R} \). The intersection point of \( V_{\mathbb{R}}(f_{pqr}) \) and \( \ell_t (\neq (1:1:1)) \) is given by \( P(t) := (x_{pqr}(t) : y_{pqr}(t) : z_{pqr}(t)) \), where

\[ x_{pqr}(t) := a_1(q, r)(t + (p - 1))^2\left(r^2 a_1(p, q) a_2(q, r, p) t - ((p^2 q^2 - 1) a_1(q, r) + 2 q r a_1(r, p) + 2 q p r^2 a_1(p, q))\right), \]
one of the following polynomials in (1)-(5):

$$y_{pqr}(t) := a_1(r,p)((1-q)t + q)^2 - ((p^2q^2r^2 + 1)a_1(r,p) + 2pm^2a_1(r,p) + 2pq^2a_1(r, p q, r)).$$

$$z_{pqr}(t) := (rt - 1)^2a_1(r,p)(a_1(r,q)q_a(q,r,p)t + q^2a_1(r,p)q_a(p, r, q)).$$

(1) Consider the case $t \leq 0$. Then $y_{pqr}(t) \leq 0$ and $z_{pqr}(t) \geq 0$. Thus $P(t) \notin \text{Int}(\mathbb{P}_2^+)$. (2) Consider the case $t > 0$. Let

$$t_1 := \frac{(p^2q^2r^2 + 1)a_1(q, r) + 2qra_1(r, p) + 2pq^2a_1(r, p)}{r^2a_1(p, q)a_2(q, r, p)},$$

$$t_2 := \frac{\frac{p^2q^2r^2 + 1}{a_1(q, r, p) + 2qra_1(p, q) + 2pq^2a_1(q, r)} + 2p^2q^2r^2a_1(r, p, q)}{a_1(p, q)a_2(p, q, r)}.$$

$z_{pqr}(t) \geq 0$ for all $t > 0$. If $t < t_1$, then $y_{pqr}(t) \leq 0$, and if $t \geq t_1$ then $y_{pqr}(t) \geq 0$. If $t < t_2$ then $y_{pqr}(t) \geq 0$, and if $t \geq t_2$ then $y_{pqr}(t) \leq 0$. Note that

$$t_1 - t_2 = r^2a_1(p, q)a_2(q, r, p)(a_1(r, p, q) + 2qra_1(p, q) + 2pq^2a_1(q, r, p)) > 0.$$

Thus, for every $t > 0$, at least one of $x_{pqr}(t)$, $y_{pqr}(t)$, $z_{pqr}(t)$ is non-negative, and at least one of $x_{pqr}(t)$, $y_{pqr}(t)$, $z_{pqr}(t)$ is non-positive. Thus $P(t) \notin \text{Int}(\mathbb{P}_2^+)$. Therefore $f_{pqr}$ is of type 1.

\[\square\]

**Remark 3.17.** (1) We consider the case $a_1(p, q) \to +0$. If $p = \frac{1}{1 - q}$, then

$$f_{pqr} = \frac{(q r - q - 1)^4}{(1 - q)^4} z(q x + (1 - q)y - z)\frac{a_1(q, r)4}{(1 - q)} \mathfrak{g}_{q, r}.4, (1 - q).$$

(2) Assume that $p \geq 0$, $q \geq 0$, $r \geq 0$, $a_1(p, q) > 0$, $a_1(q, r) > 0$, $a_1(r, p) > 0$ and $p q r = 0$. Then, $f_{pqr}$ is of type 1. Clearly, in the case (3) of Theorem 3.9, the conditions $Q_x \to P_z$, $Q_y \to P_x$ and $Q_z \to P_y$ corresponds to $p = 0$, $q = 0$ and $r = 0$ in $f_{pqr}$. (3) Note that

$$\mathfrak{f}_{P, r, q}(x, y, z) = \frac{1}{p q r} f_{p, q, r}(x, y, z).$$

### 3.6. Main Theorem of §3.

We summarize the obtained results in this section.

**Theorem 3.18.** Let $f(x, y, z) \in \mathcal{E}(\mathbb{P}^+_{3,3,3})$. Then, $f(x, y, z)$ is a positive multiple of one of the following polynomials in (1)-(5):

(1) $f_{pqr}(x, a y, b z)$ where $a > 0$, $b > 0$, $p \geq 0$, $q \geq 0$, $r \geq 0$, $pq - p + 1 > 0$, $qr - q + 1 > 0$ and $rp - r + 1 > 0$.

(2) $f_{pqr}(x, a z, b y)$ where $a > 0$, $b > 0$, $p \geq 0$, $q \geq 0$, $r \geq 0$, $pq - q + 1 > 0$, $qr - r + 1 > 0$, $rp - p + 1 > 0$ and $p q r = 0$.

(3) $g_{pqr}(x, a y, b z)$ or $g_{pqr}(y, a z, b x)$ or $g_{pqr}(z, a x, b y)$ where $a > 0$, $b > 0$, $p \geq 0$, $q \geq 0$ and $p q + q \leq 0$.

(4) $x(a x + b y + c z)^2$ or $y(a x + b y + c z)^2$ or $z(a x + b y + c z)^2$ where $a, b, c \in \mathbb{R}$, $(a, b, c) \neq (0, 0, 0)$ and $V_3(a x + b y + c z) \cap \text{Int}(\mathbb{P}^+_3) \neq \emptyset$. 

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(5) Monomials \(x^3, y^3, z^3, x^2y, y^2z, z^2x, xy^2, yz^2, zx^2\) or \(xyz\).

Conversely, all the polynomials in (1)-(5) are extremal in \(\mathcal{P}^+_{3,3}\).

Remark 3.19. Note that \(f_{ssss}(x, y, z) = (s^2 + s + 1)(s^2 - s + 1)^2 f_s(x, y, z)\). Thus \(f_{s}(x, y, z) \in \mathcal{E}(\mathcal{P}^+_{3,3})\) for \(s \geq 0\).

Section 4. The cases \(n = 3\) and \(4 \leq d \leq 6\).

4.1. Known results.

(4.4.1) \(\mathcal{P}_{3,4}\) is hard to study. Only (I1) was solved by Hilbert. Since \(X(\mathbb{P}^2_{\mathbb{R}}, \mathcal{H}_{3,4}) \cong \mathbb{R}^2\), \(\mathcal{P}_{3,4}\) has a unique discriminants. But it has too many terms to print. So, we give up (I2) and (I3). Hilbert proved \(P_{3,4} = \Sigma_{3,4}\). We shall give an alternative proof of this theorem. Compare with the proof of Proposition 6.3.4 in [6].

Theorem 4.1. (Hilbert) If \(f \in \mathcal{E}(\mathcal{P}_{3,4})\), then \(f\) is the square of a quadric polynomial.

Proof. Assume that \(f \in \mathcal{E}(\mathcal{P}_{3,4})\) is not a square of a quadric polynomial. It is easy to see that this implies that \(f\) is irreducible in \(\mathbb{R}[x, y, z]\).

If \(f\) is not explored, then there exists \(f_n \in \mathcal{E}(\mathcal{P}_{3,4})\) (\(n \in \mathbb{N}\)) such that \(\lim_{n \to \infty} f_n = f\) and all \(f_n\) are explored. If \(f\) is irreducible in \(\mathbb{R}[x, y, z]\), we can take \(f_n\) so that \(f_n\) are irreducible in \(\mathbb{R}[x, y, z]\). So we may assume that \(f\) is explored.

Consider \(V_\mathbb{R}(f)\). It is easy to see that if \(\dim_\mathbb{R} V_\mathbb{R}(f) = 1\) as a topological space, then \(f\) is not irreducible in \(\mathbb{R}[x, y, z]\), since \(f\) is PSD. If \(V_\mathbb{R}(f) = \emptyset\), then \(f\) can not be extremal. Thus \(V_\mathbb{R}(f)\) is a set of isolated points. Since \(f\) is explored, \(V_\mathbb{R}(f)\) does not contain infinitely near points. Since \(\dim_\mathbb{R} \mathcal{H}_{3,4} = 15\), \(V_\mathbb{R}(f)\) must contain at least \(5 = 15/3\) points.

If \(V_\mathbb{C}(f)\) is irreducible, this is impossible because any curves on \(\mathbb{P}^2_{\mathbb{C}}\) whose virtual genus is equal to 2, can have at most 4 singular points.

Thus \(f = gg\) where \(g \in \mathbb{C}[x, y, z] - \mathbb{R}[x, y, z]\) is a quadric. Note that \(V_\mathbb{R}(f) \subset V_\mathbb{C}(g) \cap V_\mathbb{C}(g)\). Thus \#\(V_\mathbb{R}(f)\) \leq 4. A contradiction.

Denote \(f \in \mathcal{E}(\mathcal{P}_{3,4})\) as \(f = g^2\). \(V_\mathbb{R}(g)\) must be a real conic curve or union of two real lines or a real double line (\(g = h^2\)). Converse is also true.

(4.1.2) \(\mathcal{P}^+_{3,4}\) is also hard to study. None of (I1), (I2) and (I3) are solved.

But we can give an example of \(f \in \mathcal{E}(\mathcal{P}^+_{3,4})\) which is irreducible in \(\mathbb{C}[a, b, c]\). For example, the equality conditions \(f(1, 1, 1) = f(2, 3, 1) = f(1, 2, 3) = f(0, 4, 3) = f(6, 0, 5) = f(0, 1, 0) = 0\) determine the extremal PSD form

\[
 f(x, y, z) := 591900050x^4 + 437205100x^3y - 766414561x^2y^2 + 217365672xy^3 - 165061670x^3z - 102695021x^2yz + 248518503xy^2z + 549666y^3z + 1531736792x^2z^2 + 118221267xyz^2 + 101630538y^2z^2 - 63674352x^3z^3 - 273946320y^2z^3 + 183282336z^4.
\]

Let \(g(x, y, z) := f(x^2, y^2, z^2)\). Then \(V_\mathbb{R}(g)\) has 18 acnodes and \(g \in \mathcal{E}(\mathcal{P}_{3,8})\).

(4.1.3) We also have an example of \(f \in \mathcal{E}(\mathcal{P}^+_{3,5})\) which is irreducible. For example, the following \(f(x, y, z) \in \mathcal{E}(\mathcal{P}^+_{3,5})\) has 6 acnode at \(V_6 := \{(4:1:1), (1:4:1), (1:1:4), (1/9:1:1), \ldots\}\).
\[
(1:1/9:1), (1:1:1/9) \}
\]. Moreover, \( f(1,0,0) = f(0,1,0) = 0 \).
\[
f(x, y, z) = 837x^3y - 465x^2y^2 - 645xy^2 + 837xy^3 + 1755x^2
\]
\[
- 17181x^3yz - 23876x^2y^2z - 17181xy^3z + 1755y^4z - 3486xz^3 + 19594xy^2z^2 - 3486y^3z^2 + 3287x^2z^3 - 11030xyz^3
\]
\[
+ 3287y^2z^3 - 1692xz^4 - 1692yz^4 + 648z^5.
\]
Let \( g(x, y, z) := f(x^2, y^2, z^2) \). Then \( V_R(g) \) has 26 acnodes at \( V_{26} := \pi^{-1}(V_6) \cup \{P_x, P_y\} \), where \( \pi: \mathbb{P}_R^2 \to \mathbb{P}_R^2 \) is defined by \( \pi(x:y:z) = (x^2: y^2: z^2) \). The dimension of the solution space of the system of equalities \( g(P) = g_x(P) = g_y(P) = g_z(P) = 0 \) for all \( P \in V_{26} \), is equal to 1. Thus \( g \in \mathcal{E}(\mathcal{P}_{3,10}) \).

**Proposition 4.2.** The above \( g \) is irreducible in \( \mathbb{C}[x, y, z] \).

**Proof.** (1) We prove that \( g \) is irreducible in \( \mathbb{R}[x, y, z] \). Assume that \( g = h_1h_2 \) where \( h_1 \in \mathcal{H}_{3,d}, h_2 \in \mathcal{H}_{3,e} \) with \( d + e = 10, d \leq e \). Since \( g \in \mathcal{E}(\mathcal{P}_{3,10}) \), we have \( h_1 \in \mathcal{E}(\mathcal{P}_{3,d}) \) and \( h_2 \in \mathcal{E}(\mathcal{P}_{3,e}) \). If \( d \) is odd, then \( \mathcal{P}_{3,d} = 0 \). Thus \( d \) is even. If \( d = 2 \), then \( h_1 = h_3^2 \). Then \( V_{26} \) must contain a line \( V_R(h_3) \). If \( d = 3 \), then \( h_1 = h_4^2 \). Since \( \text{Sing}(V_R(h_4)) \subset V_{26} \), we have \( V_R(h_4) = 0 \) or \( h_4 = h_3^2 + h_3^2 - 3h_5, h_6 \in \mathcal{H}_{3,1} \). It is easy to see that these are impossible.

(2) Thus, if \( g \) is reducible, there exists an imaginal \( h_7 \in \mathbb{C}[x, y, z] \) such that \( g = h_7h_7 \). Then \( P \in \text{Sing}(V_C(h_7)) \cap \mathbb{P}_R^2 \neq \emptyset \), then \( P \in \text{Sing}(V_C(h_7)) \cap \mathbb{P}_R^2 \). This is impossible, since \( P \in V_{26} \) is an acnode. Thus \( \text{Sing}(V_C(h_7)) \cap \mathbb{P}_R^2 = \emptyset \). This implies \( V_{26} \subset V_C(h_7) \cap V_C(h_{7}) \). But \( \#(V_C(h_7) \cap V_C(h_{7})) \leq 5^2 = 25 \).

(4.1.4) Cirstoje determined \( \mathcal{P}_{3,4}^0 \) in 2009 (see [11]).

From here to the end of this section, we denote the coordinate system of \( A \) by \((a_0: a_1: a_2) \) or \((a:b:c) \), and denote the coordinate system of \( \mathbb{P}(\mathcal{H}_C) \) by \((x_0: \cdots : x_N) \). We denote
\[
S_n := a^n + b^n + c^n, \quad S_{m,n} := a^m b^n + b^m c^n + c^m a^n, \quad T_{m,n} := S_{m,n} + S_{n,m}, \quad U := abc.
\]
We sometimes denote the above symbols with variables, e.g. \( U_S(x, y, z) = x^2 + y^2 + z^2 \).

**Theorem 4.3.** ([11]) Let \( f(a_0, a_1, a_2) := S_4 + p_1S_{3,1} + p_2S_{1,3} + p_3S_{2,2} - (1 + p_1 + p_2 + p_3)US_1 \in \mathcal{H}_{3,4}^0 \). Then, \( f(a_0, a_1, a_2) \geq 0 \) for all \( a_0, a_1, a_2 \in \mathbb{R} \) if and only if \( p_1^2 + p_1p_2 + p_2^2 \leq 3 + 3p_3 \).

\( \mathcal{P}_{3,4}^0 \) has a unique discriminant \( \text{disc}^0(p_1, p_2, p_3) := 3 + 3p_3 - p_1^2 + p_1p_2 + p_2^2 \), which is the main discriminant. Thus \( \mathcal{P}_{3,4}^0 \) is an elliptical ball. Extremal elements of \( \mathcal{P}_{3,4}^0 \) are given at Theorem 4.2 in [1]. Let
\[
\begin{align*}
g_{p,q}^X(a, b, c) := & S_4 + pS_{3,1} + qS_{1,3} \\
& + \left( \frac{p^2 + pq + q^2}{3} - 1 \right) S_{2,2} - \left( p + g + \frac{p^2 + pq + q^2}{3} \right) US_1, \\
p(s, t) := & \frac{-2S_{3,1}(s, t, 1) - S_{1,3}(s, t, 1) - US_1(s, t, 1)}{S_{2,2}(s, t, 1) - US_1(s, t, 1)}, \\
g_{p,t}^{A}(a, b, c) := & g_{p(s, t), p(t, s)}^{X}(a, b, c).
\end{align*}
\]
Then every extremal element of \( \mathcal{P}_{3,4}^0 \) is a positive multiple of \( g_{p,t}^{A} \). Conversely, \( g_{p,t}^{A} \) is extremal for all \( (s: t: 1) \in \mathbb{P}_R^2 \). This \( g_{p,t}^{A}(a, b, c) \) is characterized by the equality conditions \( g_{p,t}^{A}(s, t, 1) = \)}
$g_{s,t}(1,1,1) = 0$. (If $(1:s:t) = (1:1:1)$, we must treat it to be an infinitely near point to $(1:1:1)$.) Note that $g_{s,t}$ is also extremal in $\mathcal{P}^3_{3,4}$, but is not extremal in $\mathcal{P}_{3,4}$. If $(1:s:t) \in \mathbb{P}^2_+$, then $g_{s,t}$ is also extremal in $\mathcal{P}^3_{3,4}$.

(4.1.5) The structure of $\mathcal{P}^{d+}_{3,4} := \mathcal{P}(\mathbb{P}^2_+)$ is determined in [1] Theorem 4.4. Let

$$h_s := S_{3,1} + s^2 S_{1,3} - 2s S_{2,2} - (s - 1)^2 U S_1,$$

$$h_\infty := S_{1,3} - U S_1.$$  

All the extremal elements $\mathcal{P}^{d+}_{3,4}$ are positive multiples of $g_{s,t}$ ((s; t; 1) $\in \mathbb{P}^2_+$) and $h_s (s \in [0, +\infty])$. $h_s$ is characterized by the equality conditions $h_s(0, s, 1) = h_s(0, 0, 1) = h_s(1, 1, 1) = 0$.

(13) for $\mathcal{P}^{g+}_{4,3}$ is given by the following theorem. This is a corollary of Theorem 1.16 and [1] Theorem 4.4.

**Theorem 4.4.** Let $f(a_0, a_1, a_2) := S_1 + p_1 S_{1,3} + p_2 S_{1,3} + p_3 S_{2,2} - (1 + p_1 + p_2 + p_3) U S_1 \in \mathcal{H}_{3,4}$. Then, $f(a_0, a_1, a_2) \geq 0$ for all $a_0 \geq 0, a_1 \geq 0$ and $a_2 \geq 0$ if and only if one of the following 4 conditions holds.

1. $p_1 + p_2 \geq 0$ and $\text{disc}_1^{\ast}(p_1, p_2, p_3) \leq 0$.
2. $p_3 \geq -2, -2\sqrt{p_3 + 4} \leq p_1 + p_2 \leq 0$, $p_1 \geq -2\sqrt{p_3 + 2}, p_2 \geq -2\sqrt{p_3 + 2}$, and $\text{disc}_4^{\ast}(p_1, p_2, p_3) \geq 0$.
3. $p_3 \geq -2, p_1 \geq -2\sqrt{p_3 + 2}, p_2 \geq -2\sqrt{p_3 + 2}$, and $p_1 + p_2 \geq 0$.
4. $p_3 \geq 0$, and $\text{disc}_0^{\ast}(p_1, p_2, p_3) \geq 0$.

Let’s denote $f = p_0 S_4 + p_1 S_{3,1} + p_2 S_{2,2} - (p_0 + p_1 + p_2 + p_3) U S_1 \in \mathcal{H}_{3,4}$. Then, all the discriminants of $\mathcal{P}^{d+}_{3,4}$ are $\text{disc}_0^{\ast}, \text{disc}_4^{\ast}$, and $\text{disc}_0$. $\text{disc}_4^{\ast}$ is the main discriminant and $\text{disc}_4^{\ast}$ is the edge discriminant. Other functions $p_1 + 2\sqrt{p_3 + 2}, p_2 + 2\sqrt{p_3 + 2}, p_1 + p_2 + \sqrt{p_3 + 4}, p_1 + p_2, p_3 + 2$ are separators.

(4.1.6) The structure of $\mathcal{P}_{3,4}$ was also determined in [1] Theorem 4.11. The following theorem is (13) for $\mathcal{P}_{3,4}$.

**Theorem 4.5.** Take $f(a_0, a_1, a_2) := S_4 + p_1 T_{3,1} + p_2 S_{2,2} + (p_3 + 1 - 2p_1 - p_2) U S_1 \in \mathcal{H}_{3,4}$. Let $d_4(p_1, p_2, p_3)$ be the discriminant of the quartic equation $f(x, 1, 1) = 0$, and take its irreducible factor

$$\text{disc}_4(p_1, p_2, p_3) := d_4(p_1, p_2, p_3)/(16p_3),$$

$$= \text{Disc}_4(1, 2p_1, -(1 + 2p_1 - p_2 - p_3), -2(1 + p_1 + p_2 - p_3), (2 + 2p_1 + p_2))/(16p_3).$$

Then, $f(a_0, a_1, a_2) \geq 0$ for all $a_0, a_1, a_2 \in \mathbb{R}$, if and only if one of the (1), (2) and (3) holds.

1. $p_1 = 0$ and $p_2 \geq p_3^2 - 1$.
2. $0 < p_3 \leq 27, \text{disc}_4(p_1, p_2, p_3) \geq 0$ and $p_2 \geq p_3^2 - (p_3 + 2\sqrt{3p_3} + 1)$.
3. $27 < p_3, \text{disc}_4(p_1, p_2, p_3) \geq 0$ and $p_2 \geq (8 + p_3^2)/4$.

All the discriminants of $\mathcal{P}_{3,4}$ are $\text{disc}_4, p_2 - (8 + p_3^2)/4$ and $p_3$. But none of them is the main discriminant. Extremal elements of $\mathcal{P}_{3,4}$ are given at Proposition 4.13 in [1].

(4.1.7) The extremal elements of $\mathcal{P}_{3,5}$ and $\mathcal{P}_{3,4}^{\ast}$ are also determined in [1]. Please see Theorem 4.7 and 4.10 of [1].

(4.1.8) The structure of $\mathcal{P}_{3,5}^{d+}$ was also determined in [1]. Let

$$\text{disc}_{3}^{\ast}(p_0, p_1, p_2, p_3) := -\text{Disc}_3(p_0, 2p_0 + 2p_1, 3p_0 + 4p_1 + 2p_2 + p_3, 2p_0 + 2p_1 + 2p_2).$$

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The following theorem is (13) for $\mathcal{P}_{3,5}^{*+}$.

**Theorem 4.6.** Let $f(a_0, a_1, a_2) = p_0 S_5 + p_1 T_{4,1} + p_2 T_{3,2} + p_3 U S_2 - (p_0 + 2p_1 + 2p_2 + p_3 + 1)U S_{1,1} \in \mathcal{H}_{3,5}^{2*}$. Then, $f(a_0, a_1, a_2) \geq 0$ for all $a_0 \geq 0$, $a_1 \geq 0$ and $a_2 \geq 0$ if and only if $p_0 \geq 0$ and one of the following 4 conditions holds.

1. $p_0 + p_1 \geq 0, p_0 + p_1 + p_2 \geq 0$, and $p_0 + 2p_1 + p_3 > 0$.
2. $p_0 + p_1 \geq 0, p_0 + p_1 + p_2 \geq 0, p_0 + 2p_1p + p_3 \leq 0$, and $\text{disc}^{S+}_{3}(p_0, p_1, p_2, p_3) \geq 0$.
3. $-3p_0 \leq p_1 < p_0, p_0 + p_1 + p_2 \geq 0, \text{disc}^{S+}_{3}(p_0, p_1, p_2, p_3) \geq 0$ and $(p_2, p_3) \neq (-p_0 - p_1, -p_0 - 2p_1)$.
4. $3p_0 + p_1 \leq 0, 4p_0p_2 \geq 5p_0^2 + 2p_0p_1 + p_1^2$, and $\text{disc}^{S+}_{3}(p_0, p_1, p_2, p_3) \geq 0$.

All the extremal elements of $\mathcal{P}_{3,5}^{2*+}$ are written in Theorem 4.14(4) of this article. All the discriminants of $\mathcal{P}_{3,5}^{2*+}$ are $\text{disc}^{S+}_{3}$, $5p_0^2 + 2p_0p_1 + p_1^2 - 4p_0p_2$, $p_0 + p_1 + p_3$ and $p_0$. Other polynomials $p_0 + p_1, 3p_0 + p_1$ and $p_0 + 2p_1 + p_3$ are separators.

**Theorem 4.8.** Take $f(a, b, c) := \sum_{i=0}^{3} p_i s_i \in \mathcal{H}_{3,4}^{2*}$, here $p_0 := 1$. Let $d_4(p_1, p_2, p_3)$ be the discriminant of the quartic equation $f(x, 1, 1) = 0$, and take its irreducible factor $\text{disc}^{S+}_{3}(p_1, p_2, p_3) := d_4(p_1, p_2, p_3)/(16p_3)$. Then, $f(a_0, a_1, a_2) \geq 0$ for all $a_0, a_1, a_2 \in \mathbb{R}_+$ if and only if one of the (1)–(6) holds.

1. $p_3 = 0, p_1 \leq -1$ and $p_2 \geq p_1^2 - 1$.
2. $0 < p_3 \leq 3, -1 - p_3 \leq p_1$ and $p_2 \geq -2 - 2p_1$.
3. $0 < p_3 \leq 3, p_1 \leq -1 - p_3, \text{disc}^{S+}_{3}(p_1, p_2, p_3) \geq 0$ and $p_2 \geq p_1^2 - (p_3 + 2\sqrt{3p_3} + 1)$.
4. $3 < p_3, -4 \leq p_1$ and $p_2 \geq -2 - 2p_1$.
5. $3 < p_3, -2\sqrt{p_3/3} - 2 \leq p_1 \leq -4$ and $p_2 \geq (8 + p_1^2)/4$.
6. $3 < p_3, p_1 \leq -2\sqrt{p_3/3} - 2, \text{disc}^{S+}_{3}(p_1, p_2, p_3) \geq 0$ and $p_2 \geq (8 + p_1^2)/4$.

This theorem will be proved at the end of this subsection.
Let $X = X_{3,4}^{s,t} := X(\mathbb{P}_{3}^{2}, \mathcal{H}_{3,4}^{s,t})$, $\Phi := \Phi_{g_{3,4}^{s,t}} : \mathbb{P}_{3}^{2} \to X$. By Proposition 1.15, $\Psi_{s,t} : \mathbb{P}_{3}^{2} / \mathfrak{S}_{3} \to X$ is an isomorphism. Let

$$L_{b}^{t} := \{ (t: 1: 1) \in \mathbb{P}_{3}^{2} \mid 0 < t < 1 \text{ or } 1 < t < \infty \},$$

$$L_{b}^{t} := \{ (0: t: 1) \in \mathbb{P}_{3}^{2} \mid 0 < t < 1 \},$$

$C_{b} := \Phi(L_{b}^{t})$, $C_{0} := \Phi(L_{b}^{t})$, $P_{1} := \Phi(0: 0: 1) = (1: 0: 0: 0)$, $P_{2} := \Phi(0: 1: 1) = (2: 2: 1: 0)$ and $P_{3} := \Phi(0: 1: 1) = (0: 0: 0: 1)$. By Proposition 2.14 in [1], we have the following:

**Proposition 4.9.** $\Delta^{2}(X) = \{ X_{c} \}$, $\Delta^{1}(X) = \{ C_{b}, C_{0} \}$, $\Delta^{0}(X) = \{ P_{1}, P_{2}, P_{3} \}$.

By Algorithm 1.11(3), disc($P_{1}$) = $p_{0}$, disc($P_{2}$) = $2p_{0} + 2p_{1} + p_{2}$ and disc($P_{3}$) = $p_{3}$, where $(p_{0}, \cdots, p_{3})$ is a coordinate system of $\mathcal{H}_{3,4}^{s,t} \cong \mathbb{R}^{3}$. Thus $\mathcal{F}(P_{1})$ is at infinity, and $\mathcal{F}(P_{2}) = \mathcal{P}_{0}^{s,t}$. Thus, $\mathcal{F}(C_{b})$ and $\mathcal{F}(C_{0})$ are essential for $\partial \mathcal{P}_{3,4}^{s,t}$. Let

$$g_{t}(a, b, c) := g_{t}^{s,t}(a, b, c) = s_{0} - (t + 1)s_{1} + (t^{2} + 2t)s_{2} \in \mathcal{P}_{3,4}^{s,t},$$

$$e_{t}(a, b, c) := \left( S_{1,1} - \frac{1}{k}S_{1,1} \right)^{2} = s_{0} - \frac{2}{k}s_{1} + \frac{2k^{2} + 1}{k^{2}}s_{2} + 3 \left( \frac{1}{k} - 1 \right)^{2}s_{3} \in \mathcal{P}_{3,4}^{s,t},$$

where $s, t \in [0, \infty)$ and $k \in [0, 1]$. By Theorem 1.17, $\mathcal{F}(X_{c})$ is not a face component. Since $g_{t}(t, 1, 1) = 0$ and $e_{t}^{A}(t, 1, 1) = 0$, we have $g_{t}$, $e_{t}^{A}$ is in $\mathcal{F}(C_{b})$. Since $e_{t}^{A}(0, t, 1) = 0$ and $U_{S_{1}}(0, t, 1) = 0$, we have $e_{t}^{A}, U_{S_{1}} \in \mathcal{F}(C_{0})$.

**Proposition 4.10.** Let $L_{s,t}^{s,t}$ be the local cone of $\mathcal{P}_{3,4}^{s,t}$ at $(s, t) \in A = \mathbb{P}_{3}^{2}$.

1. If $0 < s \neq 1$, $0 < t \neq 1$ and $s \neq t$, then $L_{s,t}^{s,t} = \mathbb{R}_{+} \cdot e_{s,t}^{A} \neq 0$.

2. If $0 < t \neq 1$ then $L_{t}^{s,t} = \mathbb{R}_{+} \cdot g_{t} + \mathbb{R}_{+} \cdot e_{t}^{A}$.

3. If $0 < s \neq 1$ then $L_{0,s}^{s,t} = \mathbb{R}_{+} \cdot e_{0,t}^{A} + \mathbb{R}_{+} \cdot U_{S_{1}}$.

4. $L_{0,1}^{s,t} = \mathbb{R}_{+} \cdot (S_{1} + U_{S_{1}} - 2S_{2,2}) + \mathbb{R}_{+} \cdot e_{0,1}^{A} + \mathbb{R}_{+} \cdot (T_{3,1} - 2S_{2,2}) + \mathbb{R}_{+} \cdot U_{S_{1}}$.

**Proof.** (1) Assume that $0 < s \neq 1$, $t \neq 1$, and $s \neq t$. Then dim$L_{s,t}^{s,t} \leq 3 - 2 = 1$, by Proposition 1.10. On the other hand, $e_{s,t}^{A} \in L_{s,t}^{s,t}$. Thus, (1) holds.

(2) Assume that $0 < t \neq 1$. Then dim$L_{2,1}^{t} \leq 3 - 2 = 2$, by Proposition 1.10. Since $g_{t}^{A} \in L_{t}^{t} \cap L_{1}^{s,t} \cong \mathbb{R}_{+}$, $g_{t}^{A}$ is extremal. On the other hand, there exists $a' = (s': t') \in \mathbb{P}_{3}^{2}$ such that $e_{s',t'}^{A} = e_{s,t}^{A}$ and $s(a') \in \text{Int}(\mathbb{P}_{3}^{2} / \mathfrak{S}_{3})$. Thus $e_{s,t}^{A} \in L_{s,t}^{s,t} \cap L_{s,t}^{s,t} \cong \mathbb{R}_{+}$. So, $e_{s,t}^{A}$ is extremal, and we have (2).

(3) Assume that $0 < s \neq 1$. Then dim$L_{0,1}^{s} \leq 3 - 2 = 2$, by Proposition 1.10. Since $U_{S_{1}} \in L_{0,1}^{s,t} \cap L_{0,0}^{s,t} \cong \mathbb{R}_{+}$, and $e_{0,1}^{A} \in L_{0,1}^{s,t} \cap L_{s',t'}^{s,t} \cong \mathbb{R}_{+}$ for a certain $(s', t') \in \mathbb{P}_{3}^{2}$ as (2). Thus $U_{S_{1}}$ and $e_{0,1}^{A}$ are extremal, and we have (3).

(4) Note that $L_{0,1}^{s,t} = \mathcal{F}(P_{2}) \subset V(2p_{0} + 2p_{1} + p_{2})$. By Proposition 1.33 in [1], $\partial L_{0,1}^{s,t} \subset \mathcal{F}(P_{2}) \cap (V(p_{0}) \cup V(p_{2}) \cup \mathcal{F}(C_{b}) \cup \mathcal{F}(C_{0})). \mathcal{F}(P_{2}) \cap \mathcal{F}(C_{b}) = \lim_{t \to 0} L_{t}^{s,t} = \mathbb{R}_{+} \cdot g_{t} + \mathbb{R}_{+} \cdot e_{0,1}^{A} = \mathbb{R}_{+} \cdot (S_{1} + U_{S_{1}} - 2S_{2,2}) + \mathbb{R}_{+} \cdot e_{0,1}^{A}, \mathcal{F}(P_{2}) \cap \mathcal{F}(C_{0}) = \lim_{t \to 1} L_{t}^{s,t} = \mathbb{R}_{+} \cdot e_{0,1}^{A} + \mathbb{R}_{+} \cdot U_{S_{1}}$. By Theorem 0.3 in [1], we have $\mathcal{F}(P_{2}) \cap V(p_{3}) = \mathbb{R}_{+} \cdot g_{0} + \mathbb{R}_{+} \cdot (T_{3,1} - 2S_{2,2})$. Now, it is easy to see that $\mathcal{F}(P_{2}) \cap V(p_{0}) = \mathbb{R}_{+} \cdot U_{S_{1}} + \mathbb{R}_{+} \cdot (T_{3,1} - 2S_{2,2})$. Thus, we have (4).
Corollary 4.11. All the extremal elements of $\mathcal{P}_{3,4}^+$ are positive multiple of following polynomials: $e^X_k (0 < k \leq 1)$, $g_t (t \geq 0)$, $s_0 - 2s_2 = S_4 + US_1 - 2S_{2,2}$, $s_1 - 2s_2 = T_{3,1} - 2S_{2,2}$ and $s_3 = US_1$.

Corollary 4.12. Let's denote $f := \sum_{i=0}^3 p_i s_i \in \mathcal{H}_{3,4}^3$. Then, all the discriminants of $\mathcal{P}_{3,4}^+$ are positive multiples of $\text{disc}(C^b) = p_0^5 \text{disc}_4^s(p_1/p_0, p_2/p_0, p_3/p_0)$, $\text{disc}(C^0) = 4p_0p_2 - 8p_0^2 - p_1^2$, $\text{disc}(P_1) = p_0$, $\text{disc}(P_2) = 2p_0 + 2p_1 + p_2$ and $\text{disc}(P_3) = p_3$.

Proof. Since $\text{disc}^s_4$ is irreducible, and $g_t, e_{t,1}^A \in V(\text{disc}^s_4)$ for all $t \in \mathbb{R}$, we have $\text{disc}(C^b) = \text{disc}^s_4$. Since $US_1, e_{0,t}^A \in V(4p_2 - 8 - p_1^2)$ for all $t \in \mathbb{R}$, we have $\text{disc}(C^0) = 4p_2 - 8 - p_1^2$. \qed

Note that $e^X_k \in \mathcal{F}(C^0) \cap \mathcal{F}(C^b)$ if and only if $0 \leq k \leq 1/2$. When $1/2 < k \leq 1$, $e^X_k \in \mathcal{F}(C^b) - \mathcal{F}(C^0)$.

Proof of Theorem 4.8. We use the same symbols with the proof of Theorem 4.11 in [1]. There we denote $x := p_1, y := p_2$ and $z := p_3$. Fix a constant $v$, and let $H_v$ be the plane $z = v$ in $\mathcal{H}_4^3$. Let $P_v := \mathcal{P}_4^s \cap H_v$ and let $C_v$ be the curve defined by $\text{disc}_4^s(x, y, v) = 0$ on $H_v$. Moreover, let $C$ be the curve defined by $4y - 8 - x^2 = 0$ on $H_v$, and let $L$ be the line defined by $2 + 2x + y = 0$ on $H_v$. $C^b, C, L$ represent the zero loci of $\text{disc}(C^b), \text{disc}(C^0), \text{disc}(P_2)$ on $H_v$ respectively.

(1) If $v = 0$, then $\text{disc}^s_4(x, y, 0) = 3(y - x^2 + 1)(4x + y + 5)^3$. It is easy to see that (1) or (2) occurs in this case.

(II) Assume that $v > 0$.

Put $L(x \geq c) := \{(x, y) \in L \mid x \leq c\}$. If $x \geq 0$, the point $(x, -2x - 2) \in L$ corresponds to $(S_4 + US_1 - 2S_{2,2}) + x(T_{3,1} - 2S_{2,2}) + vUS_1 \in \partial \mathcal{P}$. Thus $L(x \geq 0) \subset \partial \mathcal{P}$.

Note that $L$ tangents to $C_v$ at $(x, y) = (-v - 1, 2v)$ with the multiplicity 2, and $L$ tangent to $C$ at $(-4, 6)$. When $0 \leq v \leq 3$, the point $(-v - 1, 2v)$ corresponds to $\frac{v + 1}{4}e_{1/2}^X + \frac{3 - v}{4}(S_4 + US_1 - 2S_{2,2}) + vUS_1 \in \partial \mathcal{P}$. Thus $L(x \geq -v - 1) \subset \partial \mathcal{P}$. When $v \geq 3$, the point $(-4, 6)$ corresponds to $e_{1/2}^X + (v - 3)US_1 \in \partial \mathcal{P}$. Thus $L(x \geq -4) \subset \partial \mathcal{P}$. This implies (2) and (4) occurs.

The curve $C_v$ has a node at

$$P_v : (x, y) = \left(-2 \sqrt[3]{\frac{v}{3} - 2}, \frac{v + 2\sqrt{3v} + 9}{3}\right).$$

$P_v$ corresponds to extremal polynomials $e^X_k$, where $k = \frac{1}{\sqrt{v/3} + 1}$, $v = 3(k - 1)^2/k^2$ ($0 \leq k \leq 1$). Moreover $P_v, Q_v \subset C \cap C_v$. 40
(III) Consider the case $0 < v \leq 3$.
Then, $\text{sep}(x, y, v) := y - x^2 + (v + 2\sqrt{3v} + 1)$ is a nice separator. As Fig. 4.1, (2) or (3) occurs.

(IV) Consider the case $v > 3$.
Put $C[P_v, -4] := \{ (x, y) \in C \mid -2\sqrt{v/3} - 2 \leq x \leq -4 \}$. Consider $f := \epsilon^X_k + (v - 3(1/k - 1)^2) U S_1 \in C$. $f$ corresponds to $(x, y) = (-2/k, 1/k^2 + 2) \in C$. Since $0 \leq k \leq 1/2$, we have $x \leq -4$. $v - 3(1/k - 1)^2 \geq 0$ is equivalent to $x = -2/k \geq -2(\sqrt{v/3} + 1)$. Thus $C[P_v, -4] \subset \partial \mathcal{P}$. As Fig. 4.2 and 4.3, (4), (5) or (6) occurs.
Remark 4.13. $\text{disc}^*_4(p_1, p_2, p_3)$ consists of 44 terms. When we choose $t_0 := S^1_1$, $t_1 := S^1_2 S_{1,1}$, $t_2 := S^1_3$, $t_3 := U S_1$ as a base of $\mathcal{H}_{3,4}$, and present $f = \sum_{i=0}^{3} q_i t_i$, $\text{disc}(C^b)$ become shorter. It consists of only 14 terms:
\[
d^*_4(q_0, q_1, q_2, q_3) = 27q_1^4q_2 - 216q_0q_1^2q_2^2 + 432q_0^2q_2q_3^2 + 36q_1^3q_2q_3 - 144q_0q_1q_2q_3^2 + 16q_1^2q_2q_3
\- 64q_0q_2q_3^2 + q_1^3q_2 - 36q_0q_1q_2q_3^2 + 8q_1q_2q_3^2 - 48q_0q_2q_3^2 + q_1q_3^3
\- 12q_0q_2q_3^2 - q_0q_3^4.
\]
Note that $\text{disc}^*_4(p, q, v) = d^*_4(1, -4 + p, 2 - 2p + q, 3 - 3p - 3q + v)$.

4.3. Structure of $\mathcal{P}^{s+}_{3,5}$.

To complete the Step 4 of Algorithm 1.7 for $\mathcal{P}^{s+}_{3,5}$, we have to study $V(\text{Disc}_5)$. This threefold is very complicated. So, here we shall give a study till Step 2. There are many types of extremal elements on $\mathcal{P}^{s+}_{3,5}$. We shall introduce them step by step, and give the main theorem at Theorem 4.20.

We choose $s_0 := S_5 - U S_{1,1}$, $s_1 := T_{4,1} - 2U S_{1,1}$, $s_2 := T_{3,2} - 2U S_{1,1}$, $s_3 := U S_2 - U S_{1,1}$, $s_4 := U S_{1,1}$ as a base of $\mathcal{H}_{3,5}$. Let $X := X^{s+}_{3,5}$, $\Phi := \Phi_{\mathcal{H}_{3,5}} : \mathbb{P}^2_s \to X$, $C^b := \Phi(L^b_{\ell t})$, $C^0 := \Phi(V^b_{\ell t})_+$, $P_1 := \Phi(0:0:1) = (1:0:0:0:0)$, $P_2 := \Phi(1:0:1) = (1:1:2:0:0)$, and $P_3 := \Phi(1:1:1) = (0:0:0:0:1)$. We denote the local cone of $\mathcal{P}^{s+}_{3,5}$ at $(s: t: 1) \in \mathbb{P}^2_s$ as $L^{s+}_{s,t}$. Note that $\mathcal{F}(P_1) = L^{s+}_{0,0} = \mathcal{P}_{3,5}^{s+}$, $\mathcal{F}(P_2) = L^{s+}_{1,0}$, $\mathcal{F}(P_3) = L^{s+}_{1,1} = \mathcal{P}_{3,5}^{s+0}$. By Theorem 1.4 and Theorem 1.17, we have
\[
\partial(\mathcal{P}^{s+}_{3,5}) = \mathcal{F}(C^b) \cup \mathcal{F}(C^0) \cup \mathcal{P}_{3,5}^{s+0} \cup \mathcal{L}_{0,0}^{s+} \cup \mathcal{F}(P_2).
\]
Take $f(a, b, c) = \sum_{i=0}^{4} p_i s_i(a, b, c)$. All the discriminants of $\mathcal{P}^{s+}_{3,5}$ are
\[
disc(P_1) = p_0, \quad disc(P_2) = p_0 + p_1 + p_2, \quad disc(P_3) = p_4,
\]
\[
disc(C^b) = 5p_0^2 + 2p_0p_1 + p_1^2 - 4p_0p_2, \quad \text{and disc}(C^b).
\]
disc$(C^b)$ is complicated, and will be given at Proposition 4.21.

The author should apologize for that Corollary 5.7 of [1] is not correct. It must be replaced by the following:

Theorem 4.14. (Corrected Corollary 5.7 of [1]) Let $L_{s,t}$ be the local cone of $\mathcal{P}_{3,5}^{s+0}$ at $(s: t: 1) \in A = \mathbb{P}^2_s$, and let
\[
\ell(t) := 2 - t^2 + t\sqrt{(t - 1)(t + 2)},
\]
\[
s_m(t) := (1/2)(\ell(t) - \sqrt{\ell(t)^2 - 4}),
\]
\[
f^A_t(a, b, c) := s_0 - (t + 1)s_1 + ts_2 + (t + 1)^2s_3,
\]
\[
f^B_t(a, b, c) := s_0 + (1 - 2\ell(t))s_1 + (t^3 + 2t^2 - 2 - 2(t^2 - 1)\ell(t))s_2
\- ((t + 1)^2(2t^2 - 4t + 1) + (t + 1)^2\ell(t))s_3,
\]
\[
f^C_t(a, b, c) := s_1 + (t^2 - 1)s_2 - 2(t + 1)s_3, \quad f^{C_\infty}_t(a, b, c) := s_2 - 2s_3.
\]

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(1) For all \( t \geq 0 \), \( f_t^C \) is an extremal element of \( P_{3,5}^{s_0^+} \), and
\[
L_{t,1}^{s_0^+} \cap L_{0,0}^{s_0^+} = \mathbb{R}_+ \cdot f_t^C \neq 0.
\]

(2) Let \( t \geq 2 \), and put \( s := s_m(t) \). Note that \( 0 < s \leq 1 \). Then \( f_t^B \) is an extremal element of \( P_{3,5}^{s_0^+} \), and
\[
L_{t,1}^{s_0^+} \cap L_{0,s}^{s_0^+} = \mathbb{R}_+ \cdot f_t^B \neq 0.
\]

(3) Let \( 0 \leq t \leq 2 \), Then \( f_t^A \) is an extremal element of \( P_{3,5}^{s_0^+} \), and
\[
L_{t,1}^{s_0^+} \cap L_{0,1}^{s_0^+} = \mathbb{R}_+ \cdot f_t^A \neq 0.
\]

(4) All the extremal elements of \( P_{3,5}^{s_0^+} \) are positive multiples of \( f_t^A \) \((0 \leq t \leq 2)\), \( f_t^B \) \((t \geq 2)\), \( f_t^C \) \((t \geq 0)\), \( s_2 - 2s_3 \) and \( s_3 \).

Proof. \( \dim L_{t,1}^{s_0^+} \leq 2 \) if \( t > 0 \) and \( t \neq 1 \), by Proposition 1.10(1). Thus \( \dim (L_{t,1}^{s_0^+} \cap L_{0,0}^{s_0^+}) \leq 1 \). So, if there exists \( 0 \neq f \in P_{3,5}^{s_0^+} \) such that \( f \in L_{t,1}^{s_0^+} \cap L_{0,0}^{s_0^+} \), then \( f \) is an extremal element of \( P_{3,5}^{s_0^+} \) and \( L_{t,1}^{s_0^+} \cap L_{0,0}^{s_0^+} = \mathbb{R}_+ \cdot f \) by Theorem 1.23.

(1) \( f \in L_{0,0}^{s_0^+} \) implies the coefficient of \( s_0 \) in \( f \) is equal to zero. Since, 
\[
f_t^C(u,1,1) = 2(u-1)^2(u-t),
\]
\[
f_t^C(0,1,1) = u(u+1)((u-1)^2 + t^2u),
\]
we have \( f_t^C(t,1,1) = f_t^C(0,0,1) = 0 \) and \( f_t^C \in P_{3,5}^{s_0^+} \) by Proposition 4.7.

(2) It is easy exercise to verify that \( s_m(t) \) varies \((0,1)\) when \( t \geq 2 \). Since
\[
f_t^B(u,1,1) = (u-t)^2(u-1)^2 \left( u + 2(t - \sqrt{(t-1)(t+3)}) \right),
\]
\[
f_t^B(0,u,1) = (u+1) \left( u^2 - (2-t^2 + t\sqrt{(t-1)(t+3)})u + 1 \right),
\]
we have \( f_t^B(t,1,1) = f_t^B(0,s_m(t),1) = 0 \) and \( f_t^B \in P_{3,5}^{s_0^+} \).

(3) follows from Proposition 4.7 and
\[
f_t^A(u,1,1) = u(u-1)^2(u-t),
\]
\[
f_t^A(0,u,1) = (u+1)(u-1)^2(u^2 - tu + 1).
\]

(4) All the extremal elements of \( L_{0,0}^{s_0^+} \) are \( f_t^C \) \((0 \geq 0)\) and \( f_\infty := s_2 \). \( L_{0,s}^{s_0^+} \cap L_{0,0}^{s_0^+} = \mathbb{R}_+ \cdot s_3 \). Thus we obtain (4). \( \square \)

Note that extremal elements of \( P_{3,5}^{s_0^+} \) are also extremal in \( P_{5}^{s^+} \). The above theorem implies \( f_t^A, f_t^B, f_t^C \) are characterized by equality conditions \( f_t^A(0,1,1) = f_t^A(1,1,1) = f_t^A(t,1,1) = 0 \), \( f_t^B(0,s_m(t),1) = f_t^B(1,1,1) = f_t^B(t,1,1) = 0 \), and \( f_t^C(0,0,1) = f_t^C(1,1,1) = f_t^C(t,1,1) = 0 \).

**Proposition 4.15.** For \( s > 0 \) and \( t \geq 2 \), put \( z := s - 2 + 1/s \). Let
\[
p_t^D(t,z) := -2z^3 - 3,
\]
\[
p_t^D(t,z) := z^2 + 2z + 2,
\]
\[
p_t^D(t,z) := -\frac{2t^3 + 4t^2 + 5t + 1}{t^2(t+2)} z^2 + \frac{2(4t^2 + 5t + 3)}{t^2} z.
\]
If \((t, z) \in \mathbb{R}_+^2\) satisfies \(t \geq 2\) and \(s_m(t) \leq s < 1\), then \(\mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+} = \mathbb{R}_+ \cdot f_{s,t}^D\). Thus \(f_{s,t}^D\) is an extremal element of \(\mathcal{P}_{3,5}^+\) characterized by \(f_{s,t}^D(1,1) = f_{s,t}^D(0,1) = 0\).

**Proof.** (i) Assume that \(t > 0\), \(t \neq 1\) and \(0 < s < 1\). Then \(\dim \mathcal{L}_{t,1}^{+,+} < \dim \mathcal{F}(C^b) = 4\) and \(\dim \mathcal{L}_{0,s}^{+,+} < \dim \mathcal{F}(C^0) = 4\). Since \(\dim \{(\mathcal{F}(C^b) \cap \mathcal{E})\} \leq 3\), and the codimension of \(\mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+}\) in \(\mathcal{F}(C^b) \cap \mathcal{E}\) is not less than 2, we have \(\dim (\mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+}) \leq 1\). Thus, if \(0 \neq f \in \mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+}\), then \(f\) is extremal in \(\mathcal{P}_{3,5}^+\), by Theorem 1.23.

(ii) We shall prove \(f_{s,t}^D \in \mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+}\), if \(t \geq 2\) and \(s_m(t) \leq s \leq 1\).

For a monic \(f \in \mathcal{H}_3^+,\) it is enough to show that \(f_{s,t}^D(u,1,1) \geq 0\) and \(f_{s,t}^D(0,u,1) \geq 0\) for all \(u \geq 0\). Using computer, we have

\[
f_{s,t}^D(0,u,1) = (u + 1)(u - s)^2(u - 1/s)^2 \geq 0.
\]

This also implies \(f_{s,t}^D(0,s,1) = f_{s,t}^D(0,1/s,1) = 0\).

On the other hand, \(f_{s,t}^D(u,1,1)\) is more complicated. Note that \(s_m(t) \leq s \leq 1\) is equivalent to \(0 \leq z \leq \ell(t) - 2\). Let

\[
f_{t,z}(a,b,c) := s_0 + \sum_{i=1}^4 p_i^D(t,z)s_i,
\]

\[
c_0(t,z) := t^2(t + 2),
\]

\[
c_1(t,z) := 2t^2(t + 2)(-2z + t - 3),
\]

\[
c_2(t,z) := -(5t + 1)z^2 - 2t^2(t - 7)z + t^2(t - 4)^2,
\]

\[
c_3(t,z) := 2(t + 2)z^2,
\]

\[
g(t,z,u) := c_0(t,z)u^3 + c_1(t,z)u^2 + c_2(t,z)u + c_3(t,z).
\]

Then \(f_{s,t}^D(u,1,1) = f_{t,z}(u,1,1) = \frac{(u - s)^2}{t^2(t + 2)} g\left(t, s + \frac{1}{s} - 2, u\right)\). Thus \(f_{s,t}^D(1,1,1) = 0\). \(f_{s,t}^D \in \mathcal{L}_{t,1}^{+,+} \cap \mathcal{L}_{0,s}^{+,+}\) is equivalent to \(g(t,z,u) \geq 0\) for \(z = s + 1/s - 2\) and for all \(u \geq 0\).

Note that \(c_0(t,z) > 0\) and \(c_3(t,z) > 0\).

(iii) We shall show that \(c_2(t,z) \geq 0\) if \(t \geq 2\) and \(0 < z < \ell(t) - 2\). \(c_2(t,z)\) is a concave quadratic function on \(z\), and \(c_2(t,0) = t^2(t - 4)^2 \geq 0\). Since

\[
c_2(t,\ell(t) - 2) = t^2(t + 2)(8t\sqrt{(t - 1)(t + 2)} - (8t^2 + 4t - 9)),
\]

\[
(8t\sqrt{(t - 1)(t + 2)})^2 - (8t^2 + 4t - 9)^2 = 9(8t - 9) > 0,
\]

we have \(c_2(t,z) \geq 0\).

(iv) Consider the case \(c_1(t,z) \geq 0\). Then \(g(t,z,u) \geq g(t,z,0) = c_3(t,z) \geq 0\). This is the case (1) of Solution 2.5. The case (2) of Solution 2.5 does not occur, since \(c_3(t,z) > 0\). So, it is enough to show \(\text{Disc}_3(c_0,c_1,c_2,c_3) \geq 0\) under the assumption \(c_1(t,z) < 0\).
(v) We assume $c_1(t, z) < 0$. Then $z > (t - 3)/2$. Using PC, we have
\[ \text{Disc}_3 \left( c_0(t, z), c_1(t, z), c_2(t, z), c_3(t, z) \right) = 4t^2(t + 2)b_1(t, z)^2b_2(t, z)b_3(t, z), \]
\[ b_1(t, z) := (2t + 1)z - t(t - 4), \]
\[ b_2(t, z) := -z^2 - 2t^2z + t^2(t - 2), \]
\[ b_3(t, z) := (t + 1)(5t^2 + 1)^2z^2 + 2t(t^4 - 13t^3 + 25t^2 + 27t - 4)z - t^2(t - 4)(t^2 - 3t - 1). \]

(v-1) We shall show $b_3(t, z) > 0$ for $t > 0$ and $\max\{0, (t - 3)/2\} \leq z \leq \ell(t) - 2$. If $t \geq (3 + \sqrt{13})/2$, then $t^2 - 3t - 1 \geq 0$ and $b_3(t, 0) \leq 0$. Since
\[ b_3(y + 2, (y + 1)/2) = (1/4)(13y^3 + 111y^4 + 502y^3 + 1342y^2 + 1881y + 1051), \]
$b_3((t - 3)/2, t) > 0$ for all $t \geq 2$. Thus, if $t \geq (3 + \sqrt{13})/2$ and $(t - 3)/2 \leq z \leq \ell(t) - 2$, then $b_3(t, z) \geq 0$. If $0 < t < (3 + \sqrt{13})/2$, as a quadratic function on $z$, $b_3(t, z)$ is strictly increasing on $z \geq 0$, and $b_3(t, z) \geq b_3(t, 0) \geq 0$.

(v-2) We shall show $\text{Disc}_3(c_0, c_1, c_2, c_3) \geq 0$ if $t \geq 2$ and $0 \leq z \leq \ell(t) - 2$. Assume that $t \geq 2$ and $0 \leq z \leq \ell(t) - 2$. Then $b_2(t, z) \geq 0$. Thus $\text{Disc}_3 \geq 0$. Note that $b_2(t, \ell(t) - 2) = 0$.

Thus $f^D_{s,t} \in L_{s,1}^{a_t} \cap L_{0,s}^{a_t}$, and $f^D_{s,t}$ is an extremal element of $P_{s,t}^{a_t}$.

\[ \square \]

Remark 4.16. (1) If $t \geq 2$ and $s = s_m(t)$, then $f^D_{s,t} = t^B$.
(2) Formally, we put $f^D_{s,\infty} := s_4$.
(3) $b_1(t, s + 1/s - 2) = 0$ implies $f^D_{s,t}(a, b, c) = S_1(S_2 - kS_{1,1})^2$, where $k = \frac{S_2(t, 1, 1)}{S_{1,1}(t, 1, 1)} = \frac{S_2(0, s, 1)}{S_{1,1}(0, s, 1)}$.

Proposition 4.17. Let
\[ s_G(t) := (t + 2)(7 - t), \quad s_H(t) := 9(t - 1)^2, \quad s_E(s, t) := \frac{s_G(t) - s}{(t + 2)(5t + 1)}, \]
\[ p^E_1(s, t) := \frac{s^2 - (t + 2)(5t^2 + t + 9)s + (t - 1)^2(t + 2)^2}{s(5t + 1)(t + 2)}, \]
\[ p^E_2(s, t) := \frac{1}{s(5t + 1)^3(t + 2)} \left( -t^2s^3 + (t - 1)(7t^3 - t^2 - 11t + 1)s^2 \right. \]
\[ + (t + 2)(17t^5 - 25t^4 + 199t^3 - 59t^2 + 76t + 8)s \]
\[ \left. + 9(t - 1)^4(t + 2)(t^2 - 12t - 1) \right), \]
\[ p^E_3(s, t) := \frac{1}{s(t + 2)^2(5t + 1)^3} \left( 2t^3 + 4t^2 + 5t + 1 \right) \]
\[ \left. - 2(t + 2)(7t^4 + 42t^3 + 37t^2 + 48t + 10)s^2 \right) \]
\[ + (t + 2)^2(9t^5 + 125t^4 + 682t^3 + 182t^2 + 523t + 125)s \]
\[ - 18(t - 1)^2(t + 2)^3(t^4 + 36t^3 + 34t^2 + 60t + 13) \right), \]
\[ p^E_4(s, t) := \frac{(t - 1)^3(6t^2 + 6t - 12 + s)^3}{s(t + 2)^2(5t + 1)^3}, \]

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\[ f_{s,t}^{E}(a, b, c) = s_0 + \sum_{i=1}^{4} p_i^{E}(s,t)s_i \quad \text{(if } s > 0), \]
\[ f_{0,0}^{E}(a, b, c) = (5t + 1)^2s_1 + (t - 1)^2(t^2 - 12t - 1)s_2 - 2(t^4 + 34t^2 + 60t + 13)s_3 + 24(t - 1)^4s_4. \]

If \( 0 \leq t \leq 7, t \neq 1 \) and \( 0 \leq s \leq \min\{s_G(t), s_H(t)\} \), then \( f_{s,t}^{E} \) is an extremal element of \( \mathcal{L}_{t,1}^{s+} \) characterized by \( f_{s,t}^{E}(t, 1, 1) = f_{s,t}^{E}(s_E(t), s_H(t), 1, 1) = 0. \) Moreover \( f_{0,7}^{E} = 1296 f_{0,0}^{E}. \)

**Proof.** (i) Assume that \( 0 \leq t \leq 7, t \neq 1 \) and \( 0 \leq s \leq \min\{s_G(t), s_H(t)\} \). We shall prove \( f_{s,t}^{E}(0, u, 1) \geq 0 \) for \( u \geq 0 \). Let
\[
h(s, t) := s^2 - (t + 2)(5t^2 - 4t + 6)s + (t + 2)^2s_H(t),
\]
\[
g(s, t, z) := s(t + 2)(5t + 1)^3z^2 + (5t + 1)^2h(s, t)z + t^2(s_G(t) - s)^2(s_H(t) - s).
\]

Then \( f_{s,t}^{E}(0, u, 1) = \frac{u^2(u + 1)g(s, t, u + 1/u - 2)}{s(t + 2)(5t + 1)^3}. \) To prove \( f_{s,t}^{E}(0, u, 1) \geq 0 \) for all \( u \geq 0 \), it is enough to show \( h(s, t) \geq 0. \) If \( 0 \leq t \leq 8/5 \), then \( h(s, t) = (s + 3(t - 1)(t + 1))^2 + st(t + 2)(8 - 5t) \geq 0. \)

If \( 8/5 < t \leq (10 + 2\sqrt{10})/5 \), then \( 5t^2 - 20t + 12 \leq 0. \) Thus
\[
h(s, t) = (s - 3(t - 1)(t + 1))^2 - s(t + 2)(5t^2 - 20t + 12) \geq 0.
\]

If \( t > (10 + 2\sqrt{10})/5 \), then
\[
\frac{(t + 2)(5t^2 - 14t + 6)}{2} - s_G(t) = \frac{(t + 2)(5t^2 - 12t - 8)}{2} > 0.
\]

Thus, \( h(s, t) \) is decreasing on \( 0 \leq s \leq s_G(t) \), and \( h(s, t) \geq h(s_G(t), t) = (t - 4)^2(t + 2)^2(5t + 1) \geq 0. \)

Thus we have \( f_{s,t}^{E}(0, u, 1) \geq 0 \) for \( u \geq 0 \).

(ii) Assume \( t \) and \( s \) are the same as (i). Using PC, we have
\[
f_{s,t}^{E}(u, 1, 1) = (u - t)^2(u - s_E(s, t))^2 \left( u + \frac{2(t + 2)(s_H(t) - s)}{s(5t + 1)} \right) \geq 0
\]
for all \( u \geq 0 \). Thus we have \( f_{s,t}^{E} \in \mathcal{L}_{t,1}^{s+} \cap \mathcal{L}_{s_E(s, t),1}^{s+}. \) Put \( s' := s_E(s, t) \).

(iii) Assume that \( 0 < t < 7, t \neq 1 \) and \( 0 < s < \min\{s_G(t), s_H(t)\} \). If \( \dim(\mathcal{L}_{t,1}^{s+} \cap \mathcal{L}_{s' - 1}^{s+}) \geq 2 \), then \( \dim(S_{a, b, c}(\mathcal{F}(C^n))) \geq 4. \) Thus, \( \dim(\mathcal{L}_{t,1}^{s+} \cap \mathcal{L}_{s', 1}^{s'}) = 1 \), and \( f_{s,t}^{E} \) is extremal.

Consider limits, then we conclude that \( f_{s,t}^{E} \) is extremal if \( 0 \leq t \leq 7, t \neq 1 \) and \( 0 < s < \min\{s_G(t), s_H(t)\} \). As the limit: \( s \to +0 \), we obtain \( f_{0,1}^{E} \), and it is also extremal. \[ \square \]

**Remark 4.18.** Let \( s_0 = 3(t - 1)^2(t + 2)/(2t + 1). \) Then
\[
f_{s_0,t}^{E}(a, b, c) = S_1(a, b, c) S_2(t, 1, 1, a, b, c)^2,
\]
\[
f_{s_G(t), t}^{E}(a, b, c) := s_0 + \frac{t^2 - 5t - 5}{7 - t} s_1 - \frac{t^2 - 6t + 2}{7 - t} s_2 - \frac{(t + 2)(t^2 - 3t - 2)}{7 - t} s_3 + \frac{(t - 1)^3}{7 - t} s_4,
\]
\[
f_{s_H(t), t}^{E}(a, b, c) := s_0 + \frac{t^2 + 5}{t + 2} s_1 + \frac{t^2 - t + 3}{t + 2} s_2 + \frac{t^4 - 6t^3 + 10t^2 + 13t + 13}{(t + 2)^2} s_3 + \frac{3(t - 1)^4}{(t + 2)^2} s_4.
\]
Proposition 4.19. Let
\[ f_t^F(a, b, c) := s_1 - s_2 - \frac{4t^2 + 5t + 3}{t + 2}s_3 + \frac{t^3 - 3t^2 + 3t - 1}{t + 2}s_4, \quad f_{\infty}^F(a, b, c) := s_4. \]

(1) If \( t \geq 7 \), then \( f_t^F \) is an extremal element of \( \mathcal{P}_{3,5}^+ \), and
\[ \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ \cap \mathcal{L}_{0,0}^+ = \mathbb{R}_+ \cdot f_t^F. \]

(2) If \( 5/2 \leq t < 7 \), then \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_{sG(t),t}^E \).

(3) If \( 0 \leq t < 5/2 \) and \( t \neq 1 \), then \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_{sH(t),t}^E \).

(4) If \( 0 < t \leq 2 \) and \( t \neq 1 \), then \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_t^A + \mathbb{R}_+ \cdot f_{sH(t),t}^E \).

(5) If \( 2 < t < 5/2 \), then \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_{sH(t),t}^E + \mathbb{R}_+ \cdot f_{1,t}^D \).

(6) If \( 5/2 \leq t < 7 \) and \( t \neq 4 \), \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_{sG(t),t}^E + \mathbb{R}_+ \cdot f_{1,t}^D \).

(7) \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,0}^+ = \mathbb{R}_+ \cdot f_{1,H}^E. \)

(8) If \( t \geq 7 \), then \( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ = \mathbb{R}_+ \cdot f_t^F + \mathbb{R}_+ \cdot f_{1,t}^D. \)

(9) \( f_t^F = f_{0,0}^E = f_{0,7}^E = 1296 = s_1 - s_2 - 26s_3 + 24s_4. \)

Proof. (1) If \( t \geq 7 \) and \( u \geq 0 \), then
\[ f_t^F(0, u, 1) := u(u + 1)(u - 1)^2 \geq 0, \]
\[ f_t^F(u, 1, 1) := u(u - t)^2 \left( 2u + \frac{t - 7}{t + 2} \right) \geq 0. \]

It is easy to see that \( \dim \left( \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ \cap \mathcal{L}_{0,0}^+ \right) \leq 1. \) Thus we have (1).

(4) Put \( L_t := \mathcal{L}_{t,1}^+ \cap \mathcal{L}_{0,1}^+ \). Then \( \dim L_t \leq 2 \). If \( 0 < t \leq 2 \), then \( f_t^A, f_t^H \in L_t \). Since \( f_t^A \) and \( f_t^H \) are extremal, we have \( L_t = \mathbb{R}_+ \cdot f_t^A + \mathbb{R}_+ \cdot f_t^H \).

(2), (3) is easy to see. (5)-(8) can be proved similarly as (4).

Theorem 4.20. All the extremal elements of \( \mathcal{P}_{3,5}^+ \) are positive multiples of \( f_t^A \) \((0 \leq t \leq 2)\), \( f_t^C(t \geq 0)\), \( f_{s,t}^D(t \geq 2, s_m(t) \leq s \leq 1)\), \( f_{s,t}^E(t \leq 7, 0 \leq s \leq \min\{s_G(t), s_H(t)\}\), \( f_t^F \) \((t \geq 7)\), \( s_2 - 2s_3, s_3 \) and \( s_4 \).

Proof. (1) We shall determine \( \partial \mathcal{L}_{t,1}^+ \) and extremal elements of \( \mathcal{L}_{t,1}^+ \) for \( t > 0, t \neq 1 \). Put \( S := \mathbb{R}_+ \cdot \{ f_{s,t}^E \in \mathcal{F}(C^b) \mid 0 \leq t \leq 7, t \neq 1 \) and \( 0 \leq s \leq \min\{s_G(t), s_H(t)\} \} \subset \text{Sing} \left( \mathcal{F}(C^b) \right) \).

(I-1) Case \( 0 < t \leq 2, t \neq 1 \). \( \mathcal{L}_{t,1}^+ \) is enclosed by the following 4 walls (Fig. 4.4):

(i) \( \mathbb{R}_+ \cdot f_t^C + \mathbb{R}_+ \cdot f_t^A = \mathcal{F}(P_3) \cap \mathcal{L}_{t,1}^+. \)

(ii) \( \mathbb{R}_+ \cdot f_t^A + \mathbb{R}_+ \cdot f_{sH(t),t}^E = \mathcal{F}(P_3) \cap \mathcal{L}_{t,1}^+. \)

(iii) \( \mathbb{R}_+ \cdot \{ f_{s,t}^E \mid 0 \leq s \leq s_H(t) \} = S \cap \mathcal{L}_{t,1}^+. \)

(iv) \( \mathbb{R}_+ \cdot f_0^D + \mathbb{R}_+ \cdot f_t^F = \mathcal{F}(P_3) \cap \mathcal{L}_{t,1}^+. \)

Thus all the extremal elements of \( \mathcal{L}_{t,1}^+ \) are \( f_t^A, f_t^C \) and \( f_{s,t}^E \) \((0 \leq s \leq s_H(t)) \). These are extremal elements of \( \mathcal{P}_{3,5}^+ \) by Proposition 1.9(4). Note that \( f_{sH(t),t}^E = f_{1,t}^H \).

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Thus all the extremal elements of $L_{s,t}$ (0 $< t \leq 2$) are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$ ($0 \leq s \leq s_H(t)$).

Thus all the extremal elements of $L_{s,t}$ (2 $< t \leq 5/2$) are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$ ($0 \leq s \leq s_H(t)$).

Thus all the extremal elements of $L_{s,t}$ (5/2 $< t < 7$) are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$.

Thus all the extremal elements of $L_{s,t}$ (t $\geq 7$) are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$.

Note that $Zar(F(P_1)) = H_{3,5}^\infty = \mathbb{R}^4$, $Zar(F(C^0) \cap F(P_1)) = V(p_1)$, $Zar(F(P_2) \cap F(P_1)) = V(p_1 + p_2) = 0$ and $Zar(F(P_3) \cap F(P_1)) = V(p_4)$.

(I-2) Case $2 < t \leq 5/2$. $L_{s,t}$ is enclosed by the following 5 walls (Fig.4.5):

(a) $\mathbb{R}_+ \cdot f^C_t + \mathbb{R}_+ \cdot f^D_{sm(t),t} = F(P_3) \cap L_{s,t}$.
(b) $\mathbb{R}_+ \cdot \{f^E_{s,t} \mid s_m(t) \leq s \leq 1\} = F(C^0) \cap L_{s,t}$.
(c) $\mathbb{R}_+ \cdot f^D_{H(t),t} + \mathbb{R}_+ \cdot f^E_{P_3} = F(P_2) \cap L_{s,t}$.
(d) $\mathbb{R}_+ \cdot \{f^E_{s,t} \in F(C^0) \mid 0 \leq s \leq s_H(t)\} = S \cap L_{s,t}$.
(e) $\mathbb{R}_+ \cdot f^E_{P_3} = F(P_1) \cap L_{s,t}.

Thus all the extremal elements of $L_{s,t}$ (0 $< t \leq 2$) are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$ ($0 \leq s \leq s_H(t)$). Note that $f^D_{P_3} = f^C_t$.

(I-3) Case 5/2 $< t < 7$ (Fig.4.6). Replace $s_H(t)$ by $s_G(t)$ in (I-2). Note that $f^E_{s_G(t),t} = f^E_{P_3}$.

(I-4) Case $t \geq 7$. $L_{s,t}$ is enclosed by the following 4 walls (Fig.4.7):

(a) $\mathbb{R}_+ \cdot f^C_t + \mathbb{R}_+ \cdot f^D_{sm(t),t} = F(P_3) \cap L_{s,t}$.
(b) $\mathbb{R}_+ \cdot \{f^D_{s,t} \mid s_m(t) \leq s \leq 1\} = F(C^0) \cap L_{s,t}$.
(c) $\mathbb{R}_+ \cdot f^D_{P_3} + \mathbb{R}_+ \cdot f^E_{s,t} = F(P_2) \cap L_{s,t}$.
(d) $\mathbb{R}_+ \cdot f^C_t \cap \mathbb{R}_+ \cdot f^E_{s,t} = F(P_1) \cap L_{s,t}$.

Thus all the extremal elements of $L_{s,t}$ are $f^C_t$, $f^D_{s,t}$ ($s_m(t) \leq s \leq 1$) and $f^E_{s,t}$.

(II) We shall determine all the extremal elements of $F(P_1) = L_{0,0}^+ = \mathbb{P}^{3,5}_{3,5}.$

(III) All the extremal elements on $F(C^0)$ are positive multiples of $f^C_t$ ($t \geq 0$), $f^D_{s,t}$ ($t \geq 7$) and $f^E_{s,t}$ ($0 \leq t \leq 7$) by (I).

$F(P_3) \cap F(P_1)$ is enclosed by the following 3 walls (Fig.4.8):

(a) $\mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot f^D_{P_3} = F(P_2) \cap F(P_1)$.
(b) $\mathbb{R}_+ \cdot \{f^C_t \mid t \geq 0\} = F(C^0) \cap F(P_3) \cap F(P_1)$.
(c) $\mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot f^D_{P_3} = F(P_2) \cap F(P_3) \cap F(P_1)$.
(iii) \( \mathbb{R}_+ \cdot (s_2 - 2s_3) + \mathbb{R}_+ \cdot s_3 = \mathcal{F}(C^0) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_1) \).

Note that \( f_0^C = s_1 - s_2 - 2s_3 \) and \( f_0^D = s_2 - 2s_3 \).

\[ \mathcal{F}(C^0) \quad \mathcal{F}(P_2) \quad \mathcal{F}(P_1) \]

\[ s_3 \quad f_0^C \quad f_0^D \]

\[ \mathbf{Fig.4.8. \mathcal{F}(P_2) \cap \mathcal{F}(P_1)} \]

(II-2) \( \mathcal{F}(P_2) \cap \mathcal{F}(P_1) \) is enclosed by the following 4 walls (Fig.4.9):

(i) \( \mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot f_0^C = \mathcal{F}(P_3) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \).

(ii) \( \mathbb{R}_+ \cdot \{ f_0^F \mid t \geq 7 \} \subset \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \).

(iii) \( \mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot s_3 = \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \).

(iv) \( \mathbb{R}_+ \cdot f_0^C + \mathbb{R}_+ \cdot f_0^D \subset \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \).

Only (iv) is not clear. We prove it. Note that \( f_0^C \cdot f_0^F \in \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \cap \mathcal{F}(P_3) \), and \( \dim (\mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2)) = 2 \). Using Proposition 4.21 below, we have

\[
\text{disc}_{C^0}(0, p_1, -p_1, p_3, p_4) = -p_2^2(2p_1 + p_3 + p_4)^2 \\
\times (192p_1^3 + 144p_1^2p_3 + 36p_1p_3^2 + 3p_3^3 + 11p_1p_4 - 62p_1p_3p_4 - p_3^2p_4 + 16p_1^2p_4).
\]

Thus \( \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \subset \mathcal{F}(P_3) \cap \mathcal{F}(P_1) \cap \mathcal{F}(P_2) \). This implies (iv).

(II-3) \( \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \) is enclosed by the following 3 walls (Fig.4.10):

(i) \( \mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot (s_2 - 2s_3) = \mathcal{F}(P_3) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \).

(ii) \( \mathbb{R}_+ \cdot (s_2 - 2s_3) + \mathbb{R}_+ \cdot s_3 \subset \mathcal{F}(C^0) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \).

(iii) \( \mathbb{R}_+ \cdot s_4 + \mathbb{R}_+ \cdot s_3 \subset \mathcal{F}(P_2) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_1) \).

Thus, all the extremal elements on \( \mathcal{F}(P_1) \) are positive multiples of \( f_0^C \) \( (t \geq 0), \ f_{0,t}^0 \) \( (0 \leq t \leq 7), \ f_0^F \) \( (t \geq 7), \ s_2 - 2s_3, s_3 \) and \( s_4 \).

(III) We shall determine all the extremal elements of \( \mathcal{F}(P_2) \). Note that \( \text{Zar}(\mathcal{F}(C^0) \cap \mathcal{F}(P_2)) = V(p_1 + 3) \).

(III-1) \( \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \) is enclosed by the following 3 walls (Fig.4.11):

(i) \( \mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot f_{1,2}^D = \mathcal{F}(P_3) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

(ii) \( \mathbb{R}_+ \cdot \{ f_{1,t}^D \mid t \geq 2 \} = \mathcal{F}(C^0) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

(iii) \( \mathbb{R}_+ \cdot s_4 + \mathbb{R}_+ \cdot s_3 = \mathcal{F}(P_1) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

Note that \( f_{1,\infty}^D = s_4 \).

\[ \mathcal{F}(C^0) \quad \mathcal{F}(P_3) \]

\[ f_{1,2}^D \quad f_4^D \quad s_3 \]

\[ \mathbf{Fig.4.11. \mathcal{F}(C^0) \cap \mathcal{F}(P_2)} \]

(III-2) \( \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \) is enclosed by the following 4 walls (Fig.4.12):

(i) \( \mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot f_{1,2}^D = \mathcal{F}(P_3) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

(ii) \( \mathbb{R}_+ \cdot \{ f_{1,t}^D \mid t \geq 2 \} = \mathcal{F}(C^0) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

(iii) \( \mathbb{R}_+ \cdot s_4 + \mathbb{R}_+ \cdot s_3 = \mathcal{F}(P_1) \cap \mathcal{F}(C^0) \cap \mathcal{F}(P_2) \).

Note that \( f_{1,\infty}^D = s_4 \).

\[ \mathcal{F}(C^0) \quad \mathcal{F}(P_3) \quad \mathcal{F}(P_2) \]

\[ f_{1,2}^D \quad f_4^D \quad s_3 \]

\[ \mathbf{Fig.4.12. \mathcal{F}(P_3) \cap \mathcal{F}(P_2)} \]
(III-2) $\mathcal{F}(P_3) \cap \mathcal{F}(P_2)$ is enclosed by the following 4 walls (Fig.4.12):

(i) $\mathbb{R}_+ \cdot s_3 + \mathbb{R}_+ \cdot t^2 = \mathcal{F}(C^0) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_2)$.

(ii) $\mathbb{R}_+ \cdot \{ t^2 \mid 2 \geq t \geq 0 \} \subset \mathcal{F}(C^b) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_2)$.

(iii) $\mathbb{R}_+ \cdot t^4_1 + \mathbb{R}_+ \cdot (s_2 - 2s_3) \subset \mathcal{F}(C^b) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_2)$.

(iv) $\mathbb{R}_+ \cdot t^4_1 + \mathbb{R}_+ \cdot s_3 = \mathcal{F}(P_1) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_2)$.

(i) follows from $\text{disc}_{C^b}(p_0, p_1, -p_0 - p_1) = (3p_0 + p_1)^2$. (iii) follows from $\text{disc}_{C^b}(p_0, p_1, -p_0 - p_1, p_3, 0) = 3(p_0 + 2p_1 + p_3)^2(4p_0 + 4p_1 + p_3)$. Note that $t^4_{1,2} = t^4_1 = s_0 - 3s_1 + 2s_2 + 9s_3$.

(III-3) $\mathcal{F}(P_2) \cap \mathcal{F}(P_2)$ is given in (II-1).

Structure of $\mathcal{F}(C^b) \cap \mathcal{F}(P_2)$ is complicated, but all the extremal elements of $\mathcal{F}(C^b) \cap \mathcal{F}(P_2)$ are positive multiples of $t^4_1$ ($0 \leq t \leq 2$), $t^E_{1,1}$ ($t \geq 2$), $t^E_{1,2}$ ($t \geq 7$), $s_2 - 2s_3$, $t^2 = t^2_{s,1,2}$, $t^2 = t^2_{s,1,2}$, $f^2_{s,1,2,3}$, $s_4$, and $s_2$ by (I).

(IV) The structure of $\mathcal{F}(C^0)$ is complicated. But $\partial(\mathcal{F}(C^0))$ is determined in the above proof. Thus, all the extremal elements of $\mathcal{F}(C^0)$ are positive multiples of $t^4_{1,2}$ ($t \geq 2, s_m(t) \leq s \leq 1$), $s_2 - 2s_3 = t^4_{s,1,2}$, $s_3$ and $s_4$.

Since an extremal element of $\mathfrak{P}_{3,5}^+$ is an extremal element of $\mathcal{F}(D)$ for a certain $D \in \Delta(X_{3,5}^+)$, we have the conclusion. \hfill \Box

**Proposition 4.21.** (1) $\text{disc}(C^b) = \text{disc}_{C^b}$. Here $\text{disc}_{C^b}$ is the irreducible factor with highest degree of the discriminant of the quintic equation $f(x, 1, 1) = 0$.

(2) If we present $f = S^5 + aS^2_1S^1_1 + bS^1_1S^2_1 + cS^2_1U + dS^1_1U$, then

\[ \text{disc}(C^b) = 729a^4b^3 - 5832a^2b^4 + 11664b^5 + 972a^3b^3c - 3888ab^4c + 432a^2b^4c - 1728b^5c + 27a^3b^2c^2 - 972ab^3c^2 + 216a^2b^3c^2 - 1296b^5c^2 + 27a^2b^2c^3 - 324b^3c^3 - 27b^3c^4 + 729a^4b^3d - 7776a^2b^3d - 216a^3b^3d + 19440b^4d + 864ab^4d + 486a^3b^2cd - 3780a^2b^2cd + 1026a^3b^2cd + 10800b^3cd - 3672ab^5cd + 576a^2b^3cd - 2304b^4cd + 666a^3bc^2d^2 - 3492ab^2c^2d + 544a^2b^2c^2d - 2752b^3c^2d + 16a^3c^3d - 630abc^3d + 164a^2bc^3d - 1200b^3c^3d + 16a^2c^4d - 228bc^4d - 16c^5d - 729a^5d^2 + 6075a^3bd^2 - 13500ab^5d^2 - 1998a^2b^5d^2 - 216a^3b^2d^2 + 7560b^3d^2 + 864ab^3d^2 + 16a^2b^3d^2 - 64b^4d^2 + 10534a^2cd^2 + 3555a^3bcd^2 + 60330bcd^2 + 27400ab^2d^2 + 136a^2b^2d^2 - 560b^3d^2 + 825a^2c^2d^2 - 300a^3c^2d^2 + 2250bc^2d^2 + 1991abc^2d^2 + 65a^2bc^2d^2 - 396b^2c^2d^2 + 340a^3cd^2 + 8a^2c^3d^2 - 97bc^2d^2 - 8c^4d^2 + 2025a^3d^3 + 216a^4d^3 - 9000abc^3d - 9542a^2bd^3 - 804b^3d^3 + 8ab^2d^3 + 16a^2b^2d^3 - 64b^3d^3 - 3750acd^3 - 345a^2cd^3 - 132a^3cd^3 - 400bcd^3 + 598abcd^3 + 8a^2bcd^3 - 48b^2cd^3 - 50c^2d^3 + 149ac^2d^3 + a^2c^2d^3 - 12bc^2d^3 - 3c^3d^3 + 3125d^4 + 375ad^4 - 12a^2d^4 - 6a^2d^4 + 45bd^4 + 72abd^4 + 225cd^4 + 18acd^4 - 27d^4, \]

\[ \text{disc}(C^0) = a^2 - 4b. \]

(3) If we present $f(a, b, c) = \sum_{i=0}^{4} p_i s_i(a, b, c)$, all the discriminants of $\mathfrak{P}_{3,5}^+$ are

\[ \text{disc}(P_1) = p_0, \quad \text{disc}(P_2) = p_0 + p_1 + p_2, \quad \text{disc}(P_3) = p_4, \quad \text{disc}(C^0) = 5p_0^2 + 2p_0p_1 + p_1^2 - 4p_0p_2, \quad \text{and \ disc}(C^b) \text{ as the above.} \]
Proof. All the extremal elements of $\mathcal{F}(C^b)$ are positive multiples of $f^A_t (0 \leq t \leq 2)$, $f^C_t (t \geq 0)$, $f^D_{s,t} (t \geq 2, s_m(t) \leq s \leq 1)$, $f^E_{s,t} (0 \leq t \leq 7, 0 \leq s \leq \min\{s_G(t), s_H(t)\})$, $f^F_t (t \geq 7)$, and $s_2 - 2s_3$. Using computer, it is easy to see all of these are on $V(\text{disc}(C^b))$. \hfill\Box

Remark 4.22. Put the above disc$(C^b) := g(a, b, c, d)$. Then
\[
\text{Disc}_5 (1, 2p_1, 2p_2 + p_3, -2(1 + 2p_1 + p_2 + p_3 - p_4), (-1 - 2p_2 + p_3 + p_4), 2(1 + p_1 + p_2)) = 16p_4 g(-5 + p_1, 5 - 3p_1 + p_2, 5 - p_1 - 2p_2 + p_3, -6 + 3p_1 - 3p_2 - 3p_3 + p_4).
\]

The following proposition gives a counter example of Proposition 1.27 of [1] (see Proposition 1.10).

Proposition 4.23. Let $\mathcal{L}_{s+t}$ be the local cone of $\mathcal{P}_{s+t}^+$ at $(s:t) \in A = \mathbb{P}_s^+$. If $0 < s \neq 1$, $0 < t \neq 1$ and $s \neq t$, then $\mathcal{L}_{s+t}^+ = \mathbb{R} \cdot S_1 \varepsilon_{s,t}^A$.

Proof. Assume $0 < s \neq 1$, $t \neq 1$, and $s \neq t$. dim $\mathcal{L}_{s+t}^+ = 4 - 2 = 2$, by Proposition 1.10. Since $\varepsilon_{s,t}^A(s,t) = 1$, we have $S_1 \cdot \varepsilon_{s,t}^A \in \mathcal{L}_{s,t}^+$.

Take $0 \neq f \in \mathcal{L}_{s,t}^+$. We regard $f$ as a function on $\mathbb{P}_s^2/\mathfrak{S}_3$, and present $f = g_1(p,q)r + g_2(p,q)s$ where $p := S_1$, $q := S_1,1$ and $r = U$. Since $\deg g_1 = 2$, we can write $g_1(p,q) = c_1p^2 + c_2q$. Put $p_0 := S_1(s,t,1)$, $q_0 := S_1,1(s,t,1)$ and $r_0 := U(s,t,1) = st$. Note that $(p_0, q_0, r_0) \in \text{Int}(\mathbb{P}_s^2/\mathfrak{S}_3)$, $p_0 > 0$ and $r_0 > 0$. Since $f(p_0, q_0, r_0) = 0$, $f(p, q, r) \geq 0$ and $f$ is linear on $r$, we have $g_1(p_0, q_0) = g_2(p_0, q_0) = 0$.

Assume that $g_1 \neq 0$. Since $g_2$ is quadric on $q$ and $c_1p_0^2 + c_2q_0 = 0$, we can write $g_2(p, q) = g_1(p, q)(p_3q_3(p, q))$ cannot be positive semidefinite. Thus, $g_1 = 0$.

We can write $g_2(p, q) = c_3(p - c_4p^2)^2 + c_5p^4$. Since $g_2(p_0, q_0) = 0$, we have $c_5 = 0$ and $q_0 - c_4p_0^2 = 0$. Thus $f = \alpha S_1 \varepsilon_{s,t}^A$ for a certain $\alpha > 0$. \hfill\Box

4.4. Structure of $\mathcal{P}_{3,6}^{0+}$.

About sextic symmetric homogeneous inequalities, [9] and [12] are interesting results. But there are no crucial results. Structures of $\mathcal{P}_{3,d}^{0+}$ and $\mathcal{P}_{3,6}^{0+}$ change whether $d \leq 5$ or $d \geq 6$. If $d \leq 5$, $\mathcal{P}_{3,d}^{0+}$ and $\mathcal{P}_{3,6}^{0+}$ do not have the main components. But $\mathcal{P}_{3,6}^{0+}$ has the main component. This fact makes the structure of $\mathcal{P}_{3,6}^{0+}$ very complicated.

Proposition 4.24. We regard $\mathcal{H}_{3,6}$ as a linear system on $\mathbb{P}_3^2/\mathfrak{S}_3$, and take $a_0 = (p_0, q_0, r_0) \in \text{Int}(\mathbb{P}_s^2/\mathfrak{S}_3)$. Let
\[
\begin{align*}
&f^{(1)}_{a_0}(p, q, r) := 9p_0(p_0^2 - 3q_0)r - (p_0^3 - 27r_0 + vq_0^3)pq + (p_0q_0 - 9r_0 + vq_0q_0)p^3, \\
&f^{(2)}_{a_0}(p, q, w) := p_0^2(q_0^2 - p_0^2q_0^2)(wq_0^2 - 3(w + v^2)q_0^2), \\
&f_{a_0}(p, q, w) := \left( f^{(1)}_{a_0}(p, q, r) \right)^2 + f^{(2)}_{a_0}(p, q, r),
\end{align*}
\]
where $(p, q, r) \in \mathbb{P}_3^2/\mathfrak{S}_3$ and $v, w \in \mathbb{R}$. If $v \geq \frac{2(p_0^2 - 27q_0)}{p_0^3 - 3q_0}$, there exists $w_0(v) \in \mathbb{R}$ such that $f_{a_0}(p, q, w) \in \mathcal{H}_{3,6}$ for all $w \geq w_0(v)$.
Proof. Since \( p = a + b + c > 0 \), we may assume \( p = 1 \). Then \( 0 \leq q \leq 1/3 \) and \( 0 \leq r \leq 1/27 \). Similarly, we may assume \( p_0 = 1 \), \( 0 < q_0 < 1/3 \) and \( 0 < r_0 < 1/27 \). Assume that \( v > 0 \) and \( w > 0 \). Then

\[
D_{v,w} := \{ (1: q: r) \in \mathbb{P}^2_+ / \mathfrak{S}_3 \mid w - 3(w + v^2)q \leq 0 \}
\]
is a non-empty closed semialgebraic set. For any \( v_0 \in \mathbb{R} \), \( \bigcap_{w > 0} D_{v_0,w} = \{ (1:1/3:1/27) \} \). If \((1: q: r) \in \mathbb{P}^2_+ / \mathfrak{S}_3 - D_{v,w} \), then \( f^{(2)}_{a_0,v,w}(1, q, r) \geq 0 \). Thus \( f^{(2)}_{a_0,v,w}(1, q, r) \geq 0 \). Since

\[
f_{a_0,v,w}(1, q, r) = q^3(1 - 3q^0)^2 z + \frac{1}{3}(w(1 - 3q_0)^2 + (1 - 3q_0)v((1 - 3q_0)v - 2(1 - 27r_0)))y + \text{higher order terms with respect to } y \text{ and } z,
\]
there exists a neighborhood \( U \subset \mathbb{P}^2_+ / \mathfrak{S}_3 \) of \((1: 1/3: 1/27)\) such that \( f_{a_0,v,w}(1, q, r) \geq 0 \) for all \((1: q: r) \in U\), if \( v \geq \frac{2(1 - 27q_0)}{1 - 3q_0} \). Note that the graph of \( \mathbb{P}^2_+ / \mathfrak{S}_3 \) is as Fig.4.13. Thus \( D_{v,w} \subset U \) for \( w \gg 0 \).

\[\text{Fig.4.13. Graph of } \mathbb{P}^2_+ / \mathfrak{S}_3\]

Theorem 4.25. \( \mathfrak{P}^{s_0+}_{3,6} \) has the main component.

Proof. Let \( \mathcal{L}^{s_0+}_{3,6} \) be the local cone of \( \mathfrak{P}^{s_0+}_{3,6} \) at the point \((1: s: t) \in \mathbb{P}^2_+ \). Let \( \sigma_1(a, b, c) := a + b + c, \sigma_2(a, b, c) := bc + ca + ab, \sigma_3(a, b, c) := abc \), and \( \sigma_4 : \mathbb{P}^2_+ \rightarrow \mathbb{P}^2_+ / \mathfrak{S}_3 \subset \mathbb{P}^2_+ \) be the map defined by \( \sigma := (\sigma_1, \sigma_2, \sigma_3) \). Assume that \( (1: s: t) \in \text{Int}(\mathbb{P}^2_+ / \mathfrak{S}_3) \). Note that \( \dim \mathfrak{P}^{s_0+}_{3,6} = 6 \) and \( \dim \mathcal{L}^{s_0+}_{3,6} \leq 5 - 2 = 3 \). Put \( a_0 = \sigma(1: s: t) \). Assume that \( v \geq \frac{2(p^2_0 - 27q_0)}{p^2_0 - 3q_0} \), 52
and \( w \geq w_0(v) \). Then \( f_{a_0,v,*} \in L_{s,t}^{0+} \). Thus \( \dim L_{s,t}^{0+} \leq 5 - 2 = 3 \). Let
\[
L_{a_0,s}^0 := \{(p, q, r) \in \mathbb{P}_+^2/\mathbb{G}_3 \mid f_{a_0,s}^{(1)}(p, q, r) = 0\},
L_{a_0}^0 := \{(p, q, r) \in \mathbb{P}_+^2/\mathbb{G}_3 \mid qop^2 - p_0^2q = 0\},
L_{v,w}^2 := \{(1: q: r) \in \mathbb{P}_+^2/\mathbb{G}_3 \mid w - 3(w + v^2)q \leq 0\}.
\]
Then \( L_{a_0,s}^0 \cap L_{a_0}^1 \) and \( L_{a_0,s}^0 \cap L_{v,w}^2 \) contains at most one point. Thus \( \# \{ (p: q: r) \in \mathbb{P}_+^2/\mathbb{G}_3 \mid f_{a_0,v,*}(p, q, r) = 0 \} \leq 2 \). Thus \( \dim (L_{s,t}^{0+} \cap L_{s',t}^{0+}) \leq 1 \). if \( \sigma(1: s: t) \neq \sigma(1: s': t') \in \text{Int}(\mathbb{P}_+^2/\mathbb{G}_3) \). This implies \( \dim \mathcal{F}(\text{Int}(X_{3,6}^{0+})) = 5 \). Thus \( \mathcal{P}_{3,6}^{0+} \) has the main component. \( \square \)

The author calculated the main discriminant of \( \mathcal{P}_{3,6}^{0+} \). But it is too long to present here. It will be provided in a file.

**Section 5. Quartic Inequalities of Four Variables.**

In this section, we shall study \( \mathcal{P}_{4,4}^{0} \) and \( \mathcal{P}_{4,4}^{0+} \). We write the homogeneous coordinate system of \( A = \mathbb{P}_R^4 \) or \( A = \mathbb{P}_+^4 \) by \( \{a_0: a_1: a_2: a_3\} \). We regard \( a_{4n+i} = a_i \) for \( n \in \mathbb{Z} \). We denote
\[
S_d := \sum_{i=0}^3 a_i^d, \quad T_{p, q} := \sum_{i=0}^3 a_i^p(a_i^q + a_{i+2}^q + a_{i+3}^q), \quad S_{p, p} := \sum_{0 \leq i < j \leq 3} a_i^p a_j^p,
T_{p, q, q} := \sum_{i=0}^3 a_i^q(a_i^{q+1}a_{i+2}^{q+1} + a_{i+1}^{q+1}a_{i+3}^{q+1} + a_{i+2}^{q+1}a_{i+3}^{q+1}), \quad U := a_0a_1a_2a_3.
\]

A polynomial \( f \in \mathcal{H}_{s,d}^n \) or \( \mathcal{H}_{n,d}^n \) is called monic, if the coefficient of \( S_d = a_0^d + \cdots + a_n^{d-1} \) is equal to 1. We say \( f \) is at infinity if the coefficient of \( S_d \) is equal to 0. For a subset \( V \subset \mathcal{H}_{n,d}^n \), we denote
\[
\tilde{V} := \{ f \in V \mid f \text{ is monic} \}.
\]

We denote as \( \mathbb{P}_R^n : (a_0: \cdots : a_n) \) when we treat \( \mathbb{P}_R^n \) with a homogeneous coordinate system \( \{a_0: \cdots : a_n\} \). Similarly we denote as \( \mathbb{R}^n : (x_1, \ldots, x_n) \) when we study \( \mathbb{R}^n \) with a coordinate system \( \{x_1, \ldots, x_n\} \).

**5.1. Structure of \( \mathbb{P}_R^3/\mathbb{G}_4 \).**

Let \( \{a_0: \cdots : a_n\} \) be the homogeneous coordinate system of \( \mathbb{P}_R^n \), and let \( \sigma_k = \sigma(a_0, \ldots, a_n) \) be the \( k \)-th symmetric function of \( a_0, \ldots, a_n \) \((0 \leq k \leq n + 1) \). The sequence of functions \( \{\sigma_1, \ldots, \sigma_{n+1}\} \) defines the regular map \( \sigma : \mathbb{P}_R^n \to \mathbb{R}(1, 2, \ldots, n+1) \), where \( \mathbb{P}_R(1, 2, \ldots, n+1) \) is the real weighted projective space which is defined as the real part of the complex weighted projective space \( \mathbb{P}_C(1, 2, \ldots, n+1) \). The image \( \sigma(\mathbb{P}_R^n) \) is isomorphic to \( \mathbb{P}_R^n/\mathbb{G}_{n+1} \) as semialgebraic varieties. Note that \( \mathbb{P}_C/\mathbb{G}_{n+1} \cong \mathbb{C}(1, 2, \ldots, n+1) \), but \( \mathbb{P}_R/\mathbb{G}_{n+1} \neq \mathbb{R}(1, 2, \ldots, n+1) \). In general, \( (PQ) \) represents an open line segment, \( [PQ] := (PQ) \cup \{P, Q\} \) represents a closed line segment, and \( PQ \) represents a line.

**Definition 5.1.** Assume that a finite group \( G \) acts on a semialgebraic variety \( A \). Let \( \sigma : A \to A/G \) be the natural surjection. A closed semialgebraic subset \( A_0 \subset A \) is called a fundamental domain of \( A/G \), if \( \sigma(A_0) = A/G \) and \( \sigma : \text{Int}(A_0) \to \sigma(\text{Int}(A_0)) \subset A/G \) is an isomorphism.
Example 5.2. (1) Let $A = \mathbb{P}^n_\mathbb{R}$ and $G = \mathbb{Z}/(n+1)\mathbb{Z}$. Then $(\mathbb{P}^n_\mathbb{R})^G = \{1\}$, and \(\text{Sing}(\mathbb{P}^n_\mathbb{R}/G) = \sigma((\mathbb{P}^n_\mathbb{R})^G) = \{\sigma(1)\}\), here $1 = (1:1:\cdots:1) \in A$. The following $A_c$ is a fundamental domain.

\[
A_c := \left\{ (a_0: \cdots : a_{n-1}:1) \in \mathbb{P}^n_\mathbb{R} \mid a_0 + a_1 + \cdots + a_{n-1} + 1 \geq 0, a_0 \leq 1, a_1 \leq 1, \ldots, a_{n-1} \leq 1 \right\}.
\]

(2) Let $A = \mathbb{P}^n_+ \text{ and } G = \mathbb{Z}/(n+1)\mathbb{Z}$. Then $(\mathbb{P}^n_+)^G = \{1\}$, and

\[
A_c^+ := \left\{ (a_0: \cdots : a_{n-1}:1) \in \mathbb{P}^n_+ \mid 0 \leq a_0 \leq 1, \ldots, 0 \leq a_{n-1} \leq 1 \right\}
\]

is a fundamental domain.

(3) Let $A = \mathbb{P}^n_\mathbb{R}$ and $G = \mathbb{S}_{n+1}$. Then

\[
A_c := \left\{ (a_0: \cdots : a_{n-1}:1) \in A_c \mid a_0 \leq a_1 \leq \cdots \leq a_{n-1} \right\}
\]

is a fundamental domain.

(4) Let $A = \mathbb{P}^n_+ \text{ and } G = \mathbb{S}_{n+1}$. Then

\[
A_c^+ := \left\{ (a_0: \cdots : a_{n-1}:1) \in \mathbb{P}^n_+ \mid 0 \leq a_0 \leq a_1 \leq \cdots \leq a_{n-1} \leq 1 \right\}
\]

is a fundamental domain.

Note that $\mathbb{P}^3_\mathbb{C}/\mathbb{S}_4 \cong \mathbb{P}_\mathbb{C}(1,2,3,4)$ has cyclic quotients singularities at $\hat{P}_0 := (0:1:0:0)$, $\hat{P}_0^* := (0:0:1:0)$ and $\hat{P}_0^* := (0:0:0:1)$.

Proposition 5.3. About the structures of $\mathbb{P}^3_\mathbb{R}/\mathbb{S}_4$ and $\mathbb{P}^3_+ /\mathbb{S}_4$, we have the followings:

(1) Let $\sigma: \mathbb{P}^3_\mathbb{R} \longrightarrow \mathbb{P}^3_\mathbb{R}/\mathbb{S}_4 \longrightarrow \mathbb{P}_\mathbb{R}(1,2,3,4)$ be the natural map. Then $\sigma^{-1}(\hat{P}_0^*) = \emptyset$, $\sigma^{-1}(\hat{P}_0^*) = \emptyset$, and $\sigma(1,0,0,1) = \hat{P}_0$.

(2) $\Delta^2(\mathbb{P}^3_\mathbb{R}/\mathbb{S}_4) = \{ \hat{D}_1 \}$, $\Delta^1(\mathbb{P}^3_\mathbb{R}/\mathbb{S}_4) = \{ \hat{C}_1, \hat{C}_2 \}$, and $\Delta^0(\mathbb{P}^3_\mathbb{R}/\mathbb{S}_4) = \{ \hat{P}_0, \hat{P}_1, \hat{P}_2 \}$, where $\hat{D}_1, \hat{C}_1$ and $\hat{P}_1$ are as follows:

\[
\begin{align*}
\hat{D}_1 & := \{ \sigma(s:t:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid s + t + 2 > 0, s < t \}, \\
\hat{C}_1 & := \{ \sigma(s:1:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid s \neq -3, 1 \}, \\
\hat{C}_2 & := \{ \sigma(s:1:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid -1 < s < 1 \}, \\
\hat{P}_1 & := \sigma(1:1:1:1) = (4:6:4:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4), \\
\hat{P}_2 & := \sigma(-1:-1:1:1) = (0:-2:0:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4).
\end{align*}
\]

(3) $\Delta^2(\mathbb{P}^3_+/\mathbb{S}_4) = \{ \hat{D}_1^+, \hat{D}_0 \}$, $\Delta^1(\mathbb{P}^3_+/\mathbb{S}_4) = \{ \hat{C}_1^+, \hat{C}_2^+, \hat{C}_3, \hat{C}_4 \}$, and $\Delta^0(\mathbb{P}^3_+/\mathbb{S}_4) = \{ \hat{P}_1, \hat{P}_3, \hat{P}_4, \hat{P}_5 \}$, where $\hat{D}_1^+, \hat{D}_0, \hat{C}_1^+, \hat{C}_2^+, \hat{C}_3$ and $\hat{P}_i$ are as follows:

\[
\begin{align*}
\hat{D}_1^+ & := \{ \sigma(s:t:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < t, s \neq 1, t \neq 1 \}, \\
\hat{D}_0 & := \{ \sigma(0:s:t:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < t < 1 \}, \\
\hat{C}_1^+ & := \{ \sigma(s:1:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < 1 \text{ or } s > 1 \}, \\
\hat{C}_2^+ & := \{ \sigma(s:s:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < 1 \}, \\
\hat{C}_3 & := \{ \sigma(0:s:1:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < 1 \text{ or } 1 < s \}, \\
\hat{C}_4 & := \{ \sigma(0:0:s:1) \in \mathbb{P}_\mathbb{R}(1,2,3,4) \mid 0 < s < 1 \}, \\
\hat{P}_3 & := \sigma(0:1:1:1) = (3:3:1:0) \in \mathbb{P}_\mathbb{R}(1,2,3,4), \\
\hat{P}_4 & := \sigma(0:0:1:1) = (2:1:0:0) \in \mathbb{P}_\mathbb{R}(1,2,3,4), \\
\hat{P}_5 & := \sigma(0:0:0:1) = (1:0:0:0) \in \mathbb{P}_\mathbb{R}(1,2,3,4).
\end{align*}
\]
(4) \( \text{disc}(\hat{D}_1) = \text{Disc}_4 \), and \( \hat{C}_1 \cup \hat{C}_2 \subset \text{Sing}(V(\text{Disc}_4)) \), here \( V(f) \) is the zero locus of \( f \) in \( \mathbb{P}_\mathbb{R}(1,2,3,4) \).

(5) \( \text{Cls} \hat{C}_1 \) is isomorphic to a cubic curve on \( \mathbb{P}_\mathbb{R}^2 \) with a cusp at \( \hat{P}_1 \).

(6) \( \hat{C}_2 = (\hat{P}_1 \hat{P}_2) \) is an open line segment with ends \( \hat{P}_1 \) and \( \hat{P}_2 \).

(7) \( \mathbb{P}_\mathbb{R}^2/\mathcal{S}_4 \) is the semialgebraic subset of \( \mathbb{P}_\mathbb{R}(1,2,3,4) \) defined by \( \text{Disc}_4(1, \sigma_1, \sigma_2, \sigma_3, \sigma_4) \geq 0 \), \( 8\sigma_2 \leq 3\sigma_1^2 \), and \( 64\sigma_4 - 16\sigma_2^2 + 16\sigma_1\sigma_2 - 16\sigma_1\sigma_3 - 3\sigma_1^4 \leq 0 \). Here, \( \sigma_i \) is the elementary symmetric polynomials of \( a_0, a_1, a_2, a_3 \) of degree \( i \).

**Proof.** Easy exercise. \( \square \)

**5.2. The PSD cone \( \mathcal{P}_{4,4}^{0} \).**

In this subsection, we shall study \( \mathcal{P}_{4,4}^{0} := \mathcal{P}(\mathbb{P}_\mathbb{R}^3, \mathcal{H}_{4,4}^{0}) \). The aim of this subsection is to prove the following theorem.

**Theorem 5.4.** (1) For

\[
 f = S_4 + pT_{3,1} + qS_{2,2} + rT_{2,1,1} - (4 + 12p + 6q + 12r)U \in \mathcal{H}_{4,4}^{0},
\]

\[
 f(a_0, a_1, a_2, a_3) \geq 0 \text{ for all } (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \text{ if and only if }
\]

\[
 p + r \geq 0 \quad \text{and} \quad -9q^2 + 12p + 12q + 12r + 8 \geq 0.
\]

(2) All the extremal elements of \( \mathcal{P}_{4,4}^{0} \) are positive multiples of \( g_t \) \((t \in \mathbb{P}_\mathbb{R}^1) \) or \( p := S_{2,2} + 6U - T_{2,1,1} \), where

\[
 g_t(a_0, a_1, a_2, a_3) := 3(S_4 - 4U) - 2(t + 1)(T_{3,1} - T_{2,1,1}) + (t^2 + 2t - 1)(S_{2,2} - 6U),
\]

\[
 g_\infty(a_0, a_1, a_2, a_3) := S_{2,2} - 6U.
\]

(3) All the discriminants of \( \mathcal{P}_{4,4}^{0} \) are \( \text{disc}_{C_1} = 9p^2 + 12p + 12q + 12r + 8 \) and \( \text{disc}_{p_2} = p + r \).

(4) \( \{ (t:1:1:1) \in \mathbb{P}_+^3 | t \geq 0 \} \cup \{ (-1:1:1:1) \} \) is a test set for \( \mathcal{P}_{4,4}^{0} \).

This theorem will be proved at the end of this subsection.

We choose \( s_0 := S_4 - 4U \), \( s_1 := T_{3,1} - 12U \), \( s_2 := S_{2,2} - 6U \), \( s_3 := T_{2,1,1} - 12U \) as a base of \( \mathcal{H}_{4,4}^{0} \), and define \( \Phi_{4,4}^{0} : \mathbb{P}_\mathbb{R}^3 \to \mathbb{P}_\mathbb{R}^3 \) by \( \Phi_{4,4}^{0}(a) = (s_0(a) : s_1(a) : s_2(a) : s_3(a)) \).

Since \( \text{Bs} \mathcal{H}_{4,4}^{0} = \{ (1:1:1:1) \} \), \( \Phi_{4,4}^{0} : \mathbb{P}_\mathbb{R}^3 \to \mathbb{P}_\mathbb{R}^3 \) is not regular. So, we don’t need to consider \( \Psi(\hat{P}_1) \). Let \( X_{4,4}^{0} := \Phi_{4,4}^{0}(\mathbb{P}_\mathbb{R}^3) = X(\mathbb{P}_\mathbb{R}^3, \mathcal{H}_{4,4}^{0}) \subset \mathbb{P}(\mathcal{H}_{4,4}^{0}) \), and let \( \Psi : \mathbb{P}_\mathbb{R}^3/\mathcal{S}_4 \to X_{4,4}^{0} \) be the rational map such that \( \Phi_{4,4}^{0} = \Psi \circ \sigma \).

Since \( \Psi : (\mathbb{P}_\mathbb{R}^3/\mathcal{S}_4 - \{ \hat{P}_1 \}) \to X_{4,4}^{0} \) is a birational morphism, every \( D \in \Delta(X_{4,4}^{0}) \) is obtained as \( D = \Psi(\hat{D}) \) for a certain \( \hat{D} \in \Delta(\mathbb{P}_\mathbb{R}^3/\mathcal{S}_4) \). Since \( (1:1:1:1) \in \text{Bs} \mathcal{H}_{4,4}^{0} \), there are no element in \( \Delta(X_{4,4}^{0}) \) corresponding to \( \hat{P}_1 \). Let \( D_1 := \Psi(\hat{D}_1), C_1 := \text{Cls}(\Psi(C_1)), C_2 := \Psi(\hat{C}_2), \) and \( P_i := \Psi(\hat{P}_i) \) for \( i = 0, 2, 3 \). We put

\[
 P_1 := (2:3:1:1) = \lim_{t \to 1} \Phi_{4,4}^{0}(t, 1, 1, 1).
\]

\[
 P_1 \notin \Phi(\mathbb{P}_\mathbb{R}^3 - \text{Bs} \mathcal{H}_{4,4}^{0}), \lim_{t \to 1} \Phi_{4,4}^{0}(t, 1, 1, 1) \neq P_1, \text{ and } \{ P_1 \} \notin \Delta(X_{4,4}^{0}).
\]

We denote the coordinate system of \( \mathbb{P}(\mathcal{H}_{4,4}^{0}) = \mathbb{P}_\mathbb{R}^3 \) by \( (x_0 : x_1 : x_2 : x_3) \), here \( \Phi_{4,4}^{0} \) is defined by \( x_i = s_i(a) \). Let

\[
 g_2(x_0, x_1, x_2, x_3) := (x_1 - x_3)^2 + 2x_2^2 - 3x_2x_0.
\]

Then \( C_1 \) is the conic defined by \( x_2 = x_3 \) and \( g_2(x_0, x_1, x_2, x_3) = 0 \). Note that \( Q_0 = (2: -2: 1: 0) \in C_1, P_1 \subset C_1, \) and \( C_1 \) is non-singular. Thus, \( \{ P_0 \} \notin \Delta(X_{4,4}^{0}) \).
Next, \( C_2 = \{ \Phi_{4,4}(t) | -1 < t < 1 \} = \Psi(\tilde{C}_2) \) is an open line segment \((P_1P_2)\) defined by \( x_0 = 2x_2, x_1 - 2x_2 - x_3 = 0 \) and \( x_1/x_0 \leq 3/2 \), whose ends are \( P_1 \) and \( P_2 \). Note that \( P_2 = (0:1:0:1) = (0:-1:0:-1) = \lim_{t \to -\infty} (a:t:c:t) = \Phi_{4,4}(0:1:-1:1:1) = \Psi(\tilde{P}_2) \).

In general, if \( D \in \Delta(X) \) has ruling structure, \( \mathcal{F}(D) \) cannot be a face component. Thus, \( \mathcal{F}(C_2) \) is not a face component of \( \mathcal{P}_{4,4}^0 \). By Theorem 1.17, \( \mathcal{F}(D_1) \) and \( \mathcal{F}(X_{4,4}^0) \) are not face components of \( \mathcal{P}_{4,4}^0 \). Thus we have.

**Lemma 5.5.** \( \Delta^0(X_{4,4}^0) = \{ P_2 \} \), \( \Delta^1(X_{4,4}^0) = \{ C_1, C_2 \} \), \( \Delta^2(X_{4,4}^0) = \{ D_1 \} \), \( \Delta^3(X_{4,4}^0) = \{ \text{Int}(X_{4,4}^0) \} \). \( \partial \mathcal{P}_{4,4}^0 = \mathcal{F}(C_1) \cup \mathcal{F}(P_2) \).

We regard \( \mathcal{H}_{4,3}^0 = \mathbb{R}^4 \), by identifying \( f = \sum_{i=0}^3 p_is_i \in \mathcal{H}_{4,3}^0 \) and \( (p_0, p_1, p_2, p_3) \in \mathbb{R}^4 \). We also use \((p_0, p_1, p_2, p_3)\) as a coordinate system of \( \mathcal{H}_{4,3}^0 = \mathbb{R}^4 \). We denote the local cone of \( \mathcal{P}_{4,4}^0 \) at \((t:1:1:1) \in \mathbb{P}_R^3 \) by \( \mathcal{L}_{t}^0 \). Note that if \( f \in \mathcal{F}(C_1) \), there exists \( t \in \mathbb{R} \) such that \( f(t, 1, 1, 1) = 0 \). Thus \( f \in \mathcal{L}_{t}^0 \). For \( t = \infty \in \mathbb{P}_R^3 \), we denote the local cone of \( \mathcal{P}_{4,4}^0 \) at \((1:0:0:0) \in \mathbb{P}_R^3 \) by \( \mathcal{L}_{\infty}^0 \).

**Lemma 5.6.** \( \mathcal{L}_{t}^0 = \mathbb{R}_+ \cdot g_t + \mathbb{R}_+ \cdot p \), and \( \mathcal{F}(C_1) = \bigcup_{t \in \mathbb{P}_R^1} \mathcal{L}_{t}^0 \). The discriminant of \( \mathcal{F}(C_1) \) and \( \mathcal{F}(P_2) \) are

\[
\begin{align*}
\text{disc}_{C_1}(p_0, p_1, p_2, p_3) &:= 8p_0^2 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3, \\
\text{disc}_{P_2}(p_0, p_1, p_2, p_3) &:= p_1 + p_3.
\end{align*}
\]

**Proof.** Since \( P_2 = (0:1:0:1) \), \( \text{disc}_{P_2}(p_0, p_1, p_2, p_3) = p_1 + p_3 \), by Algorithm 1.11(3).

By Corollary 1.3 of [27] or Corollary 2.1 of [28], we can choose

\[
\Omega := \{ (t, t, 1, 1) | t \in \mathbb{R} \} \cup \{ (t, 1, 1, 1) | t \in \mathbb{R} \} \cup \{ (1, 0, 0, 0) \}
\]

for a test set for \( f \in \mathcal{H}_{4,4}^0 \). That is, if \( f(a) \geq 0 \) for all \( a \in \Omega \), then \( f(a) \geq 0 \) for all \( a \in \mathbb{R}^4 \). For \( g_t(x, x, 1, 1) = (t - 1)^2(x^2 - 1)^2 \), \( g_t(x, 1, 1, 1) = 3(x - 1)^2(x - t)^2 \), \( g_t(1, 0, 0, 0) = 3 \).

Thus, \( g_t \in \mathcal{L}_{t}^0 \). Similarly, \( g_{\infty}(x, x, 1, 1) = (x^2 - 1)^2 \), \( g_{\infty}(x, 1, 1, 1) = 3(x - 1)^2(x - t)^2 \), \( g_{\infty}(1, 0, 0, 0) = 0 \), \( p(x, x, 1, 1) = (x - 1)^4 \), \( p(x, 1, 1, 1) = 0 \), \( p(1, 0, 0, 0) = 0 \).

Thus, \( p \in \mathcal{L}_{t}^0 \), and \( g_{\infty}, p \in \mathcal{L}_{\infty}^0 \). Thus \( \dim \mathcal{L}_{\infty}^0 \geq 2 \) and \( \dim \mathcal{L}_{\infty}^0 \geq 2 \). On the other hand, \( \dim \mathcal{L}_{t}^0 \leq \dim \mathcal{P}_{4,4}^0 = 3 \), we have \( \dim \mathcal{L}_{t}^0 = \dim \mathcal{L}_{\infty}^0 = 2 \).

Recall that \( P_2 = \Phi_{4,4}^0(-1:-1:1:1) \). Since \( g_t(-1, -1, 1, 1) = 0 \), we have \( g_t \in \mathcal{F}(P_2) \cap \mathcal{L}_{t}^0 \). Similarly, \( g_{\infty} \in \mathcal{F}(P_2) \cap \mathcal{L}_{\infty}^0 \). So, \( \dim(\mathcal{F}(P_2) \cap \mathcal{L}_{t}^0) = \dim(\mathcal{F}(P_2) \cap \mathcal{L}_{\infty}^0) = 1 \). This means that \( g_t \) and \( g_{\infty} \) are extremal elements of \( \mathcal{P}_{4,4}^0 \).

Since \( p \in \mathcal{L}_{t}^0 \) for all \( t \in \mathbb{P}_R^1 \), \( p \) is also an extremal element of \( \mathcal{P}_{4,4}^0 \). Thus \( \mathcal{L}_{t}^0 = \mathbb{R}_+ \cdot g_t + \mathbb{R}_+ \cdot p \) for all \( t \in \mathbb{P}_R^1 \).

It is easy to check that \( g_t (\forall t \in \mathbb{P}_R^1) \) and \( p \) exists on the hypersurface in \( \mathcal{H}_{4,4}^0 \) defined by \( 8p_0^2 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3 \). This equation is also the defining equation of the dual variety of \( C_1 \). So, this is \( \text{disc}_{C_1} \).

\[
\square
\]
Note that $g_1$ also satisfies $g_1(x, x, 1, 1) = 0$ for all $x \in \mathbb{P}_R^1$. Thus, $\mathcal{F}(C_2) = \mathbb{R}_+ \cdot g_1$. $g_{-3}$ satisfies $g_{-3}(a, b, c, d) = 0$ if $a + b + c + d = 0$.

**Proof of Theorem 5.4.** By the above lemma, we have

$$\mathcal{F}(P_2) = \left\{ \sum_{i=0}^{3} p_i s_i \in \mathcal{H}_{4,4}^{0} \mid p_1 + p_3 = 0, p_0 \geq 0, -9p_1^2 + 12p_0p_2 + 8p_0^2 \geq 0 \right\},$$

$$\mathcal{F}(C_1) = \left\{ \sum_{i=0}^{3} p_i s_i \in \mathcal{H}_{4,4}^{0} \mid p_1 + p_3 \geq 0, p_0 \geq 0, -9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3 + 8p_0^2 = 0 \right\}.$$ 

Thus, all the extremal elements of $\mathcal{P}_{4,4}^{0}$ are $g_t$ $(t \in \mathbb{P}_R^1)$ and $p$. Thus we have Theorem 0.3.

Thus, for $f = s_0 + ps_1 + qs_2 + rs_3 \in \mathcal{H}_{4,4}^{0}$, $f(a) \geq 0$ for all $a \in \mathbb{P}_R^3$ if and only if $p + r \geq 0$ and $-9p^2 + 12p + 12q + 12r + 8 \geq 0$.

(4) follow from $\partial \mathcal{P}_{4,4}^{0} = \mathcal{F}(C_1) \cup \mathcal{F}(P_2)$. $\square$

Theorem 0.2(1) and Theorem 0.5(1) follows from $S_4 = \sigma_1^2 - 4\sigma_1^3 \sigma_2 + 2\sigma_2^2 - 4\sigma_1 \sigma_3 - 4\sigma_4$, $T_{3,1} = \sigma_1^2 \sigma_2 - 2\sigma_2^2 - \sigma_1 \sigma_3 + 4\sigma_4$, $S_{2,2} = \sigma_1^2 - 2\sigma_1 \sigma_3 + 2\sigma_4$, $T_{2,1,1} = \sigma_1 \sigma_3 - 4\sigma_4$, and $U = \sigma_4$.

Conversely, $\sigma_1^2 = S_4 + 4T_{3,1} + 6S_{2,2} + 12T_{2,1,1} + 24U$, $\sigma_1 \sigma_3 = T_{3,1} + 2S_{2,2} + 5T_{2,1,1} + 12U$, $\sigma_2^3 = S_{2,2} + 2T_{2,1,1} + 6U$, $\sigma_1 \sigma_3 = T_{2,1,1} + 4U$. Note that $d_1 = \text{disc}(C_1)$ and $d_2 = \text{disc}(P_2)$.

### 5.3. The PSD cone $\mathcal{P}_{4,4}^{0+}$

In this subsection, we shall study $\mathcal{P}_{4,4}^{0+} := \mathcal{P}(\mathbb{P}_R^3, \mathcal{H}_{4,4}^{0})$. The aim of this subsection is to prove the following theorem.

**Theorem 5.7.** (I) For

$$f = S_4 + pT_{3,1} + qS_{2,2} + rT_{2,1,1} - (4 + 12p + 12q + 12r)U \in \tilde{\mathcal{H}}_{4,4}^{0},$$

$f(a_0, a_1, a_2, a_3) \geq 0$ for all $a_0 \geq 0, a_1 \geq 0, a_2 \geq 0, a_3 \geq 0$ if and only if the following “(1) or (2)” and “(3) or (4)” hold:

(1) $p \leq -4$ and $p^2 \leq 4q - 8$.
(2) $p \geq 4$ and $2p + q + 2 \geq 0$.
(3) $p \leq -2/3$ and $9p^2 \leq 12p + 12q + 12r + 8$.
(4) $p \geq -2/3$ and $3q + 3r \geq 1$.

(II) All the extremal elements of $\mathcal{P}_{4,4}^{0+}$ are positive multiples of $f_t^{ab}$ $(0 \leq t \leq 5)$, $f_t^a$ $(5 \leq t < \infty)$, $p := S_{2,2} + 6U - T_{2,1,1}, q_1 := T_{3,1} - 2S_{2,2},$ or $q_2 := T_{2,1,1} - 12U$, where

$$f_t^{ab}(a_0, a_1, a_2, a_3) := 3S_4 - 2(t + 1)T_{3,1}$$

$$+ 2(2t - 1)S_{2,2} + (t^2 + 3)T_{2,1,1} - 12(t^2 + 1)U,$$

$$f_t^a(a_0, a_1, a_2, a_3) := 9S_4 - 6(t + 1)T_{3,1}$$

$$+ (t^2 + 2t + 19)S_{2,2} + 2(t^2 + 5t - 8)T_{2,1,1} - 6(5t^2 + 10t - 19)U.$$ 

(III) The following set is a test set for $(\mathbb{P}_R^3, \mathcal{H}_{4,4}^{0+})$.

$$\left\{(t: 1: 1: 1) \in \mathbb{P}_R^3 \mid t \geq 0\right\} \cup \left\{(0: 0: t: 1) \in \mathbb{P}_R^3 \mid t \geq 0\right\}.$$ 

This theorem will be proved at the end of this subsection.

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Essentially, we use the same symbols as the previous subsection, but there are some changes. Let $A := \mathbb{P}^3_+ \setminus (a_0 : a_1 : a_2 : a_3)$, $X^{0,0}_{4,4} := \Phi_{4,4}^{0}(\mathbb{P}^3_+) = X(\mathbb{P}^3_+, \mathcal{Z}_{4,4}^0) \subset \mathbb{P}((\mathcal{Z}_{4,4}^0)^\vee)$. Moreover, let $D_0 := \Phi(\tilde{D}_0)$, $D_1 := \Phi(\tilde{D}_1) \subset D_1$, $P_1 := (2:3:1:1)$, $C_i^+ := \Phi(C_i^+)$ for $i = 3, 4$ and $P_j := \Phi(P_j)$ for $j = 3, 4, 5$. Note that
\[
\begin{align*}
P_3 &= (1:2:1:1) = \Phi_{4,4}^{0}(0,1,1,1), \\
P_4 &= (2:2:1:0) = \Phi_{4,4}^{0}(0,0,1,1), \\
P_5 &= (1:0:0:0) = \Phi_{4,4}^{0}(0,0,0,1).
\end{align*}
\]

We divide $C_1^+$ into three parts $0 < t < 1, 1 < t < 5, 5 < t$ and denote these by $C_1^a, C_1^b, C_1^c$. Put $C_1^{ab} := C_1^a \cup \{P_1\} \cup C_1^b$.

Every $D \in \Delta(X^{0,0}_{4,4})$ is obtained as $D = \Psi(\tilde{D})$ for a certain $\tilde{D} \in \Delta(\mathbb{P}^3_+ / \mathcal{S}_4)$. But $\mathcal{F}(D_0)$ and $\mathcal{F}(D_1)$ are not face components of $\mathcal{P}_{4,4}^{0,0}$ by Theorem 1.17. $P_1 \notin \Delta(X^{0,0}_{4,1})$, since $(1:1:1:1) \in \text{Bs} \mathcal{Z}_{4,4}^0$. As previous subsection, we have $C_2^+ = (P_1, P_3)$. Thus $\mathcal{F}(C_2^+)$ is not a face component of $\mathcal{P}_{4,4}^{0,0}$. Thus, we have
\[
\partial \mathcal{P}_{4,4}^{0,0} = \mathcal{F}(C_1^+) \cup \mathcal{F}(C_3) \cup \mathcal{F}(C_4) \cup \mathcal{F}(P_3) \cup \mathcal{F}(P_4) \cup \mathcal{F}(P_5).
\]

In fact, $\mathcal{F}(C_3)$ is not a face component. But it will be proved later. We summarize here what are $C_1^+, C_3$ and $C_4$.

**Lemma 5.8.**

1. Zar($C_1^+$) is a conic defined by $x_1^2 - 2x_1 x_2 - 3x_0 x_2 + 3x_3^2 = 0$, $x_2 - x_3 = 0$. Especially, Zar($C_1^+$) is nonsingular. The ends of $C_1^+$ are $P_3$ and $P_5$.
2. Zar($C_3$) has a cusp at $P_3$. The ends of $C_3$ are $P_4$ and $P_5$.
3. Zar($C_4$) is a conic defined by $x_1^2 - 2x_2^2 - x_0 x_2 = 0$ on the plane $V(x_3)$. The ends of $C_4$ are $P_4$ and $P_5$.

Next, we shall study $f_t^{ab}$ ($0 \leq t \leq 5$), $f_t^a$ ($5 \leq t < \infty$), $p = s_2 - s_3, q_1 = s_1 + 2s_2$, and $q_2 = s_3$. Put $f_\infty^a := s_2 + 2s_3$. Since $f_\infty^a = p + 3q_2, f_\infty^a$ is not extremal. For $u \geq 0$, let $b_u^a := 3u^2 s_0 - 6u(u^2 + 1)s_1 + 3(u^4 + 4u^2 + 1)s_2 + 2(3u^4 + 3u^2 + 2u^2 + 3u + 3)s_3$. If $t = (3u^2 - u + 3)/u$, then $b_u^a = \frac{u^2}{3} f_t^a$. So, $b_u^a$ is not a new polynomial, but it is convenient to study $\mathcal{F}(C_1)$ for the property $b_0^a(0, 0, u, 1) = 0$.

We shall denote the local cone of $\mathcal{P}_{4,4}^{0,0}$ at the point $(t:1:1:1) \in \mathbb{P}^3_+$ by $L_{t}^{C_1}$, and the local cone at the point $(0:0:1:1)$ by $L_{t}^{C_1}$.

**Lemma 5.9.** $f_t^{ab}$ ($0 \leq t \leq 5$), $f_t^a$ ($5 \leq t < \infty$), $p = s_2 - s_3, q_1$, and $q_2$ are extremal elements of $\mathcal{P}_{4,4}^{0,0}$. Moreover, they satisfy the following properties.

1. $f_t^{ab} \in \mathcal{F}(C_1) \cap \mathcal{F}(P_4)$ and $L_{t}^{C_1} = \mathbb{R}_+ \cdot f_t^{ab} + \mathbb{R}_+ \cdot p$ for $0 < t < 5$.
2. $f_t^a \in \mathcal{F}(C_1) \cap \mathcal{F}(C_4)$ and $L_{t}^{C_1} = \mathbb{R}_+ \cdot f_t^a + \mathbb{R}_+ \cdot p$ for $t > 5$. Moreover, $L_{t}^{C_1} = \mathbb{R}_+ \cdot b_u^a + \mathbb{R}_+ \cdot q_2$ for $u \geq 0$.
3. $f_0^{ab} \in \mathcal{F}(C_1) \cap \mathcal{F}(P_4) \cap \mathcal{F}(P_3)$.
4. $f_0^a \in \mathcal{F}(C_1) \cap \mathcal{F}(C_4) \cap \mathcal{F}(P_4)$.
5. $f_\infty^a \in \mathcal{F}(C_1) \cap \mathcal{F}(C_4) \cap \mathcal{F}(P_3)$.
6. $p \in \mathcal{F}(C_1) \cap \mathcal{F}(P_3) \cap \mathcal{F}(P_3)$.
7. $q_1 \in \mathcal{F}(P_3) \cap \mathcal{F}(P_4) \cap \mathcal{F}(P_5)$.

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Proof. First, we shall prove that $f_t^{ab}$, $f_t^s$, $q_1 = s_1 - 2s_2$, and $q_2 = s_3$ are positive semidefinite. $p \in P_{4,4}^0 \subset P_{4,4}^{0+}$ is already proved in §3.2. Let

$$A_1^+ := \{(t: 1: 1: 1) \in \mathbb{P}_4^+ \mid t \geq 0\},$$
$$A_2^+ := \{(t: t: 1: 1) \in \mathbb{P}_4^+ \mid 0 \leq t \leq 1\},$$
$$A_3 := \{(0: 0: 1) \in \mathbb{P}_4^+ \mid t \geq 0\},$$
$$A_4 := \{(0: 0: t: 1) \in \mathbb{P}_4^+ \mid t \geq 0\}.$$  

Note that $\Phi_4^0(A_1^+) \supset C_4^+ (i = 1, 2)$, and $\Phi_4^0(A_j) \supset C_4 (j = 3, 4)$. By Corollary 1.3 of [27] or Corollary 2.1 of [28], we can choose $A_1^+ \cup A_2^+ \cup A_3 \cup A_4$ as a test set for $(\mathbb{P}_4^+, \mathcal{G}_4^0)$. Since $\mathcal{F}(C_4^+)$ is not a face component of $\mathcal{P}_{4,4}^{0+}$ and $P_1 \in C_4^+$, $P_4 \in \text{Cls}(C_3) \cap \text{Cls}(C_4)$, we can omit $A_2^+$ from the test set. Thus, if $f \in \mathcal{G}_4^0$ satisfies $f(x, 1, 1, 1) \geq 0$, $f(0, x, 1, 1) \geq 0$ and $f(0, 0, x, 1) \geq 0$ for all $x \geq 0$, then $f \in \mathcal{P}_{4,4}^{0+}$. We shall start from $f_t^{ab} (0 \leq t \leq 5)$. Since

$$f_t^{ab}(0, x, 1, 1) = x(x + 2) \left( t - \frac{2(x - 1)^2}{(x + 2)} \right)^2 + \frac{x(16 - x)(x - 1)^2}{(x + 2)^2},$$
we have $f_t^{ab}(0, x, 1, 1) \geq 0$ if $x \leq 16$. On the other hand

$$f_t^{ab}(0, x, 1, 1) = x(16(25 - t)^2 + (t^2 + 120(5 - t) + 1575)(x - 16) + (4(5 - t) + 120(x - 16)^2 + 3(x - 16)^3),$$
we have $f_t^{ab}(0, x, 1, 1) \geq 0$ for $x \geq 16$. Similarly,

$$f_t^{ab}(x, 1, 1, 1) = 3(x - t)^2(x - 1)^2 \geq 0,$$

$$f_t^{ab}(0, 0, x, 1) = (x - 1)^2 \left( 3 x - \frac{t - 2}{3} \right)^2 + \frac{1}{3} (5 - t)(1 + t) \geq 0,$$

$$f_t^s(x, 1, 1, 1) = 9(x - t)^2(x - 1)^2 \geq 0,$$

$$f_t^s(0, x, 1, 1) = (2x + 1)^2 \left( t - \frac{(x - 1)^2(6x + 5)}{(2x + 1)^2} \right)^2 + \frac{24x(x - 1)^2(x + 2)(3x + 2)}{(2x + 1)^4} \geq 0,$$

$$f_t^s(0, 0, x, 1) = (3x^2 - (t + 1)x + 3)^2 \geq 0,$$

$$b_o^s(0, 0, x, 1) = 3(x - u)^2(ux - 1)^2 \geq 0,$$

$$q_1(x, 1, 1, 1) = 3x(x - 1)^2 \geq 0,$$

$$q_1(0, x, 1, 1) = 2(x - 1)^2(x + 1) \geq 0,$$

$$q_1(0, 0, x, 1) = x(x^2 + 1) \geq 0,$$

$$q_2(x, 1, 1, 1) = 3(x - 1)^2 \geq 0,$$

$$q_2(0, x, 1, 1) = x(x + 2) \geq 0,$$

$$q_2(0, 0, x, 1) = 0.$$ 

Thus $f_t^{ab}$, $f_t^s$, $q_1$, $q_2 \in \mathcal{P}_{4,4}^{0+}$.

If $\mathcal{F}(D)$ ($D \in \Delta(X_{4,4}^{0+})$) is a face component of $\mathcal{P}_{4,4}^{0+}$, then $\dim \mathcal{F}(D) = \dim(\partial \mathcal{P}_{4,4}^{0+}) = \dim \mathcal{P}_{4,4}^{0+} - 1 = 3$. So, if $D_1$, $D_2$, $D_3$ are distinct elements of $\Delta(X_{4,4}^{0+})$, and $\mathcal{F}(D_i)$ ($i = 1, 2, 3$) are face components, then $\dim(\mathcal{F}(D_1) \cap \mathcal{F}(D_2)) = 2$ and $\dim(\mathcal{F}(D_1) \cap \mathcal{F}(D_2) \cap \mathcal{F}(D_3)) = 1$. 

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If \( 0 \neq f \in \mathcal{F}(D_1) \cap \mathcal{F}(D_2) \cap \mathcal{F}(D_3) \) and \( \mathbb{R}_+ \cdot f \) is a common edge of \( \mathcal{F}(D_1), \mathcal{F}(D_2), \mathcal{F}(D_3) \) (i.e. not on a face), then \( f \) is an extremal element of \( \mathcal{P}_{a4}^+ \).

Now, we shall prove (1)—(8).

1) Since \( f^a_b(t, 1, 1, 1) = 0 \) and \( f^a_b(0, 0, 1, 1) = 0 \), we have \( f^a_b \in \mathcal{F}(C_1) \cap \mathcal{F}(P_4) \) for \( 0 \leq t \leq 3 \). Since \( p(t, 1, 1, 1) = 0 \), we have \( L^a_b(t) = \mathbb{R}_+ \cdot f^a_b + \mathbb{R}_+ \cdot p \subset \mathcal{F}(C_1) \). Especially, \( f^a_b \) and \( p \) are extremal.

2) Let \( u > 0 \) and \( t = (3a^2 - u + 3)/u \geq 5 \). Then \( f^a_b(t, 1, 1, 1) = 0 \) and \( f^a_b(0, 0, u, 1) = 0 \). Thus \( f^a_b \in \mathcal{F}(C_1) \cap \mathcal{F}(C_4) \). Since \( q_2(0, 0, x, 1) = 0 \) for all \( x \in \mathbb{R} \), we have \( L^a_b(0) = \mathbb{R}_+ \cdot f^a b + \mathbb{R}_+ \cdot q_2 \). As (1), we have \( L^a_b(0) = \mathbb{R}_+ \cdot f^a b + \mathbb{R}_+ \cdot p \). Thus, \( f^a_b \) (\( t \geq 5 \)) and \( q_2 \) are extremal.

3) follow from \( f^a_b(t, 1, 1, 1) = 0 \) and \( f^a_b(0, 0, 1, 1) = 0 \).

(4)—(8) can be proved similarly.

Since \( f^a_b(x, 1, 1, 1) = 4(t - 1)^2x(x - 1)^2x \), we have \( f^a_b(x, 1, 1, 1) = 0 \) for all \( x \in \mathbb{R} \).
This implies \( f^a_b \in \mathcal{F}(C_2^+) \). Let \( q_3 := s_0 - s_1 + s_3 \in \mathcal{F}(P_3) \). It is easy to see that \( \mathcal{F}(C_3^+) = \mathbb{R}_+ \cdot f^a b + \mathbb{R}_+ \cdot q_3 \subset \mathcal{F}(P_4) \).

Using the above lemma, we shall determine the structure of the face components \( \mathcal{F}(C_4^+), \mathcal{F}(C_4), \mathcal{F}(P_3), \mathcal{F}(P_4) \) and \( \mathcal{F}(P_5) \).

**Fig.5.1. \( \mathcal{P}_{a4}^+ \)**

**Lemma 5.10.** For \( f, g \in \mathcal{H}_{a4}^0 \), let \( \text{Fan}(f, g) := \mathbb{R}_+ \cdot f + \mathbb{R}_+ \cdot g \) be the fan whose edges are \( f \) and \( g \). Put
\[
W^{ab} := \mathbb{R}_+ \cdot \{ f^b_t \mid 0 \leq t \leq 5 \} \subset \mathcal{H}_{a4}^0, \quad W^c := \mathbb{R}_+ \cdot \{ f^c_t \mid t \geq 5 \} \cup \mathbb{R}_+ \cdot f^c_\infty.
\]

Then the following hold.

1) \( \partial \mathcal{F}(C_4^+) = W^{ab} \cup W^c \cup \text{Fan}(f^c_\infty, p) \cup \text{Fan}(p, f^0_0) \).

2) \( \partial \mathcal{F}(C_4) = W^c \cup \text{Fan}(f_3, q_2) \cup \text{Fan}(q_2, f^c_\infty) \).

3) \( \partial \mathcal{F}(P_3) = \text{Fan}(f^0_0, q_1) \cup \text{Fan}(q_1, p) \cup \text{Fan}(p, f^0_0) \). That is, \( \mathcal{F}(P_3) \) is a triangle cone with edges \( f^0_0, q_1 \) and \( p \).

4) \( \partial \mathcal{F}(P_4) = W^{ab} \cup \text{Fan}(f_5, q_2) \cup \text{Fan}(q_2, q_1) \cup \text{Fan}(q_1, f^0_0) \).

5) \( \mathcal{F}(P_5) \) is a triangle cone with edges \( p, q_1 \) and \( q_2 \). Note that \( f^c_\infty \in \text{Fan}(p, q_2) \), and \( \text{Fan}(p, f^c_\infty) = \mathcal{F}(P_5) \).
By the above lemma, we know that $\partial P^{0+}_{4,4}$ is enclosed by $\mathcal{F}(C_1^+), \mathcal{F}(C_4), \mathcal{F}(P_3), \mathcal{F}(P_4)$ and $\mathcal{F}(P_5)$. We don’t need $\mathcal{F}(C_3)$. Thus we have:

**Lemma 5.11.** $\partial P^{0+}_{4,4} = \mathcal{F}(C_1^+) \cup \mathcal{F}(C_4) \cup \mathcal{F}(P_3) \cup \mathcal{F}(P_4) \cup \mathcal{F}(P_5)$. Especially, $\mathcal{F}(C_3)$ is not a face component of $P^{0+}_{4,4}$. We can choose $A^+_1 \cup A_4$ as a test set for $(\mathbb{P}^3_+, \mathcal{K}^{0+}_{4,4})$.

Geometrically, $C_3 = \{P_3, P_4, P_5\}$ is included in the interior of the convex closure of $P^{0+}_{4,4}$. So, any $f \in P^{0+}_{4,4}$ cannot satisfy $f(0, x, 1, 1) = 0$ for $x > 0, x \neq 1$.

Theorem 0.4 is also proved from the above results.

Finally, we shall study discriminants $\text{disc}_D = \text{disc}(D)$ for $D = C_1^+, C_4, P_3, P_4$ and $P_5$.

We use $(p_0, p_1, p_2, p_3)$ as a coordinate system of $\mathcal{K}^{0+}_{4,4}$ as before. $(p_0, p_1, p_2, p_3)$ corresponds to $\sum_{i=0}^{3} p_i c_i$.

**Lemma 5.12.**

\[
\text{disc}(C_1^+) = 8p_0^3 - 9p_1^2 + 12p_0p_1 + 12p_0p_2 + 12p_0p_3, \\
\text{disc}(C_4) = -8p_0^2 - p_1^2 + 4p_0p_2, \\
\text{disc}(P_3) = p_0 + 2p_1 + p_2 + p_3, \\
\text{disc}(P_4) = 2p_0 + 2p_1 + p_2, \\
\text{disc}(P_5) = p_0.
\]

**Proof.** $\text{disc}(C_1^+) = \text{disc}(C_1)$, since Zar($C_1^+$) = Zar($C_1$).

If $P = (c_0; c_1; c_2; c_3) \in \Delta(P^{0+}_{4,4})$, then $\text{disc}(P) = \sum_{i=1}^{3} c_i p_i$. Thus we have $\text{disc}(P_i) (i = 3, 4, 5)$.

We shall study $\text{disc}(C_4)$. Take $f = h_u^c + vq \in \mathcal{F}(C_4) (u > 0, v \geq 0)$. The coefficients of $f$ are $p_0 = 3u^3, p_1 = -6u(u^2 + 1)s_1, p_2 = 3(u^4 + 4u^2 + 1), p_3 = 2(3u^4 + 3u^3 + 2u^2 + 3u + 3) + v$. Eliminate $u$ and $v$ from these relations. Then we have $\text{disc}(C_4) = -8p_0^2 - p_1^2 + 4p_0p_2 = 0$.

**Proof of Theorem 5.7.** Now, we have completed the proof of Theorem 5.7. What we should do is to observe the signature of discriminants on $P^{0+}_{4,4}$. Since $P^{0+}_{4,4}$ is not a basic semialgebraic set, we need to divide it as (1)—(4) of Theorem 5.7(I).

We can obtain Theorem 0.2(2) by the same method of the last part of §3.2.

**Section 6. Cubic Inequalities of Four Variables.**

**6.1. Structure of $P^{0+}_{4,3}$.**

In this section, we shall study $P^{0+}_{4,3} := \mathcal{P}(\mathbb{P}^3_+, \mathcal{K}^{0+}_{4,3})$. To state the main theorem of this section we need to fix some symbols. Put

\[
S_3 := \sum_{i \in \mathbb{Z}/4\mathbb{Z}} a_i^3, \quad S_{2,1,0} := \sum_{i \in \mathbb{Z}/4\mathbb{Z}} a_i^2 a_{i+1}, \quad S_{2,0,1} := \sum_{i \in \mathbb{Z}/4\mathbb{Z}} a_i^2 a_{i+2}, \\
S_{1,2,0} := \sum_{i \in \mathbb{Z}/4\mathbb{Z}} a_i^2 a_{i+3}, \quad S_{1,1,1} := \sum_{i \in \mathbb{Z}/4\mathbb{Z}} a_i a_{i+1} a_{i+2}.
\]
We need two discriminants $\text{disc}_C$ and $\text{disc}_S$.

$$\text{disc}_C(p_0, p_1, p_3) := 27p_0^3 + 4p_0p_1^3 + 4p_0p_3^3 - p_1^2p_3^2 - 18p_0^2p_1p_3 = \text{Disc}_3(p_0, p_1, p_3, p_0),$$

$$d_S(p_0, p_2, q, r) := (p_0 - p_2 - q)^2(13p_0^2 - 2p_0p_2 + p_2^2 + 2p_0q + 2pq)^2$$

$$(104p_0^3 + 100p_0^2p_2 - 4p_0p_2^2 + 36p_0^2q + 36p_0p_2q - p_0q^2 - p_2q^2 + 8q^3)$$

$$+ (17173p_0^7 - 121p_0^6p_2 - 563p_0^5p_2^2 + 7651p_0^4p_2^3 - 3489p_0^3p_2^4 + 469p_0^2p_2^5 - 45p_0p_2^6 + p_2^7)$$

$$+ 6250p_0^6q + 10028p_0^5p_2q + 3142p_0^4p_2^2q - 1368p_0^3p_2^3q - 746p_0^2p_2^4q - 20p_0p_2^5q - 6p_2^6q$$

$$+ 898p_0^5q^2 + 7230p_0^4p_2q^2 + 1748p_0^3p_2^2q^2 - 1572p_0^2p_2^3q^2 - 86p_0p_2^4q^2 - 26p_2^5q^2$$

$$+ 2780p_0^6q^3 - 368p_0^5p_2q^3 + 1448p_0^4p_2^2q^3 - 496p_0^3p_2^3q^3 + 28p_0^2p_2^4q^3 + 518p_0q^4$$

$$+ 1018p_0^2p_2q^4 - 190p_0p_2^2q^4 + 78p_2^3q^4 + 164p_0^2q^5 + 168p_0p_2q^5 + 4p_2^2q^5) + r^2$$

$$+ (2945p_0^6 - 317p_0^5p_2 - 1886p_0^4p_2^2 - 842p_0^3p_2^3 - 81p_0^2p_2^4 - 3p_2^5 - 1768p_0^3q + 4p_0p_2q$$

$$- 988p_0^2p_2q + 380p_0^3q - 12p_0^2q^2 + 291p_0q^2 + 897p_0p_2q^2 - 463p_0^2p_2q^2 + 83p_2^3q^2$$

$$+ 226p_0^3q^3 + 92p_0p_2q^3 - 38p_2^3q^3 - p_2^4q - p_2^4q^3 + 4p_2^4q^5) + r^4$$

$$+ (95p_0^3 + 65p_0^2p_2 - 43p_0p_2^2 + 3p_2^3 + 98p_0q - 20p_0p_2q - 6p_2^2q - 4p_2^2q^2) + r^6$$

$$+ (-3p_0 + p_2) + r^8,$$

$$\text{disc}_S(p_0, p_1, p_2, p_3) := \frac{1}{4} d_S(p_0, p_2, p_1 + p_3, p_1 - p_3).$$

Since $\text{disc}_C(p_0, p_1, p_3)$ has an obstacle branch in the first quadrant $p_1/p_0 > 0$, $p_3/p_0 > 0$, we put

$$d_C(x, z) := \begin{cases} 
\text{disc}_C(1, x, z) & \text{if } x < 0 \text{ or } z < 0 \\
1 & \text{if } x \geq 0 \text{ and } z \geq 0
\end{cases}$$

to avoid complexity. $d_C(x, z) \geq 0$ implies $\text{disc}_C(1, x, z) \geq 0$ or $x \geq 0$ and $z \geq 0$. $d_C(x, z) \geq 0$ defines a convex domain, but $\text{disc}_C(1, x, z) \geq 0$ does not. The following $\eta(x, y)$ is a nice separator whose property is explained in Lemma 6.8.

$$\eta(x, y) := 61 + 62x + 56y + 32x^2 + 30xy - 6y^2$$

$$+ 9x^3 + 4x^2y - 6xy^2 - 16y^3 + x^4 - 4x^2y^2 - 6xy^3 + y^4 - x^3 y^2.$$
(II) Let’s denote \( f = p_0s_3 + p_1s_{2,1,0} + p_2s_{2,0,1} + p_3s_{1,2,0} - (p_0 + p_1 + p_2 + p_3)s_{1,1,1} \). Then, all the discriminants of \( \Phi^{0+}_{4,3} \) are \( \text{disc}_S(p_0, p_1, p_2, p_3) \), \( \text{disc}_C(p_0, p_1, p_3) \), \( \text{disc}_P = p_0 \), and \( \text{disc}_{p_2} = p_0 + p_2 \).

This theorem will be proved at the end of this section.

We choose \( s_0 := s_3 - s_{1,1,1}, s_1 := s_{2,1,0} - s_{1,1,1}, s_2 := s_{2,0,1} - s_{1,1,1}, s_3 := s_{1,2,0} - s_{1,1,1} \) as a base of \( \mathcal{J}^{c0+}_{4,3} \), and define \( \Phi^{c0}_{4,3} : \mathbb{P}^3_+ \cdots \to \mathbb{P}^3_+ \) by \( \Phi^{c0}_{4,3}(a) = (s_0(a) : s_1(a) : s_2(a) : s_3(a)) \). Put \( X^{c0+}_{4,3} := \Phi^{c0}_{4,3}(\mathbb{P}^3_+) \). By AM-GM inequality, we have

\[
\begin{align*}
s_0(a_0, a_1, a_2, a_3) &= \frac{1}{3} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^3 + a_{i+1}^3 + a_{i+2}^3 - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
s_2(a_0, a_1, a_2, a_3) &= (a_0 - a_1 + a_2 - a_3)(a_0 a_2 - a_1 a_3), \\
s_0 - s_2 &= \frac{1}{3} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^3 + a_i^3 + a_{i+2}^3 - 3a_i^2 a_{i+2}) \geq 0, \\
s_0 + 2s_2 &= \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^2 a_{i+2} + a_{i+1}^3 + a_{i+2}^2 - 3a_i^2 a_{i+1} a_{i+2}) \geq 0, \\
2s_1 + s_2 &= \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^2 a_{i+1} + a_{i+1}^2 a_{i+2} + a_{i+2}^2 a_i - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
2s_3 + s_2 &= \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^2 a_{i+1}^2 + a_{i+1}^2 a_{i+2} + a_{i+2}^2 a_i - 3a_i a_{i+1} a_{i+2}) \geq 0, \\
s_0 - s_1 &= \frac{1}{3} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^3 + a_i^3 + a_{i+1} - 3a_i^2 a_{i+1}) \geq 0, \\
s_0 - s_3 &= \frac{1}{3} \sum_{i \in \mathbb{Z}/4\mathbb{Z}} (a_i^3 + a_{i+1}^3 + a_{i+1} - 3a_i a_{i+1}^2) \geq 0, \\
s_1 + s_3 &= (a_0 + a_2)(a_1 - a_3)^2 + (a_1 + a_3)(a_0 - a_2)^2 \geq 0.
\end{align*}
\]

Thus \( X^{c0+}_{4,3} \) is a subset of a cube defined by \(-1/2 \leq s_i/s_0 \leq 1, -1/2 \leq s_2/s_0 \leq 1, -1/2 \leq s_3/s_0 \leq 1\). Note that \( s_1 \) and \( s_3 \) are not PSD, for \( s_1(1/100, 1/2, 1/10, 1) = -229/20000 < 0 \). \( \Phi^{c0}_{4,3}(\mathbb{P}^3_+) : \mathbb{P}^3_+ \cdots \to X^{c0+}_{4,3} \) split as

\[
\Phi^{c0}_{4,3}(\mathbb{P}^3_+) : \mathbb{P}^3_+ \xrightarrow{\pi} \mathbb{P}^3_+/(\mathbb{Z}/4\mathbb{Z}) \xrightarrow{\Psi} X^{c0+}_{4,3}.
\]

It is easy to see that \( \Psi : \mathbb{P}^3_+/(\mathbb{Z}/4\mathbb{Z}) \cdots \to X^{c0+}_{4,3} \) is a birational map, but is not holomorphic at a singular point \( \pi(1:1:1:1) \). We shall provide more precise structure of \( X^{c0+}_{4,3} \). The following \( \epsilon_{s,t}(a_0, a_1, a_2, a_3) \in \mathcal{J}^{c0+}_{1,3} (s, t \in \mathbb{R}) \) has a possibility to be an extremal element. But there exists \( (s, t) \) such that \( \epsilon_{s,t} \) is not PSD.

\[
\begin{align*}
g_0(s, t) &= t(-s + 2s^2 - s^3 + t + st + s^2 t + s^3 t - st^2), \\
g_1(s, t) &= 1 - 2s^2 + s^4 + 3st - s^3 t - 2s^4 t + 2st^2 - 2s^2 t^2 - 2t^3 - s^3 + st^4, \\
g_2(s, t) &= t(1 - 3s + 4s^2 - 3s^3 + s^4 - t - st - s^2 t - s^3 t - 2t^2 - 3st^2 + 2s^2 t^2 + t^4), \\
g_3(s, t) &= s - 2s^3 + s^5 - 2t - st + 3s^3 t - 2st^2 + 2s^2 t^2 - 3t^3 - 2s^2 t^3 + t^4, \\
\epsilon_{s,t}(a) &= s_0(a) + \sum_{i=0}^3 \frac{g_i(s, t)}{g_0(s, t)} s_i(a).
\end{align*}
\]
Lemma 6.2. Let 

\[ B_0 := \{(0: s: t: 1) \in \mathbb{P}_+^3 \mid s > 0, t > 0\}, \]

\[ S := \Phi_{4,3}^0(B_0), \]

\[ E_1 := \{(a_0; a_1; a_2; a_0 - a_1 + a_2) \in \mathbb{P}_+^3 \mid a_1 \leq a_0 + a_2\}, \]

\[ E_2 := \{(a; b: a; b) \in \mathbb{P}_+^3 \mid a, b \in \mathbb{R}_+\}, \]

\[ f(0;x_1,x_2,x_3) := (x_1^3 - x_0x_1x_3 + x_0^3)\]

\[ \quad + x_2^3(x_1^3 - x_0x_1x_3 + x_0^3) + 2x_0x_1x_3(x_1 + x_3) \]

\[ \quad + 3x_1^2x_3^2 + 9x_1x_3^2 - 7x_1^2x_3^2 + x_3^4 \]

\[ \quad + x_0^2(2x_0x_1^2 - x_0x_1x_3 + 2x_0x_3^2) + (x_1 + x_3)(x_1^2 - 3x_1x_3 + 3x_3^2) \]

\[ + x_2^2(x_1^2 + x_1x_3 + x_3^2). \]

(1) \(B \Phi_{4,3} = \{(1: 1: 1: 1)\}\) and the exceptional sets of \(\Phi_{4,3}^0\) are \(E_1\) and \(E_2\). \(\Phi_{4,3}^0(E_1) = (2: 1: 0: 1), \) and \(\Phi_{4,3}^0(E_2) = (1: 0: 1: 0). \)

(2) \(\text{Zar}(\partial X_{4,3}^0) = V(f_{4,3}^0). \)

(3) \(\Phi_{4,3}^0(B_0) = \partial X_{4,3}^0 \) and \(S = \text{Reg}(\Phi_{4,3}^0(B_0)). \)

(4) Let \(L_{\{0:s:t:1\}}^{\partial+}\) be the local cone of \(\mathcal{I}_{4,3}^0\) at \((0: s: t: 1)\). Take \((0: s: t: 1) \in B_0\). If \(e_{s,t}\) is positive semidefinite then the local cone \(L_{\{0:s:t:1\}}^{\partial+}\) is a half line with the base \(e_{s,t}\) and \(e_{s,t}\) is an extremal element of \(\mathcal{I}_{4,3}^0\). If \(e_{s,t}\) is not positive semidefinite then \(L_{\{0:s:t:1\}}^{\partial+}\) is a hyperplane.

Proof. (1) follows from \(\Phi_{4,3}^0(a_0; a_1; a_2; a_0 - a_1 + a_2) = (2: 1: 0: 1), \) and \(\Phi_{4,3}^0(a; b; a; b) = (1: 0: 1: 0). \)

(2), (3) The Jacobian of \(\Phi_{4,3}^0\) is equal to

\[ (a_0 - a_1 + a_2 - a_3)^3((a_0 - a_2)^2 + (a_1 - a_3)^2)S_1s_0^2. \]

Note that \(V_{\mathbb{P}_+^3}(s_0) = \{(1: 1: 1: 1)\}\). On the other hand,

\[ f_{4,3}^0(\Phi_{4,3}^0(a_0,a_1,a_2,a_3)) \]

\[ = a_0a_1a_2a_3(a_0 - a_1 + a_2 - a_3)^4((a_0 - a_2)^2 + (a_1 - a_3)^2)^2 \geq 0. \]

By Corollary 1.14 and (1), we have the conclusion.

(4) We use Algorithm 1.11. Let \(f_i(x_0,x_1,x_2,x_3) := \frac{\partial}{\partial x_i}f_{4,3}^0(x_0,x_1,x_2,x_3)\)

\[ h_i(s,t) := f_i(\Phi_{4,3}^0(0,s,t,1)), \]

\[ g_i(s,t) := st(1 + s + t)^2(1 - s^2 - 2s + s^2 + t)^2. \]

Then \(h_i(s,t) = g_i(s,t)g_i(s,t)\) \((i = 0, 1, 2, 3). \)

It is easy to draw a graph of \(S\) using Mathematica. But it may present incorrect impression. It seems that \(X_{4,3}^{\partial+}\) is a convex set. But it is not true. The following observation show us that \(X_{4,3}^{\partial+}\) is not convex near \((1: 0: 0: 0)\). Cut \(\partial X_{4,3}^{\partial+}\) by the plane \(V(x_1 - x_3)\). Note that

\[ f_{4,3}^0(1,x,y,x) = x^2(2x - 3y - 1)(2x^2 + x^2y - y^3 - x^2 + 2y^2 - y). \]

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The graph of \( V(2x^3 + x^2y - y^3 - x^2 + 2y^2 - y) \) is not convex near \((x, y) = (0, 0)\). Thus \( X_{4,3}^{0+} \) is not convex. This also implies that \( c_{s,t} \not\in P_{4,3}^{0+} \) for some \((s,t) \in B_0\).

**Lemma 6.3.** Let \( P_1 := (1:0:0:0), P_2 := (1:0:1:0) = \Phi_{4,3}^0(E_2), \) and
\[
C := \{ \Phi_{4,3}^0(0:0:t:1) \in \mathbb{P}_+^3 \mid t > 0 \},
\]
\[
= \{ (x_0:x_1:0:x_3) \in \mathbb{P}_+^3 \mid x_1^2 - x_0x_1x_3 + x_3^2 = 0 \} - \{ P_1, P_2 \},
\]
\[
(P_1, P_2) := \{ \Phi_{4,3}^0(0:t:0:1) \in \mathbb{P}_+^3 \mid 0 < t < 1 \}.
\]

(1) \( \text{Sing}(\text{Zar}(\partial X_{4,3}^{0+})) \cap (\partial X_{4,3}^{0+}) = C \cup (P_1, P_2) \cup \{ P_1, P_2 \}. \) \( C \) and \( (P_1, P_2) \) are nodal double curves. \( P_1 \) and \( P_2 \) are normal crossing triple points.

(2) \( \Delta^3(X_{4,3}^{0+}) = \{ \text{Int}(X_{4,3}^{0+}) \}, \) \( \Delta^2(X_{4,3}^{0+}) = \{ S \}, \) \( \Delta^1(X_{4,3}^{0+}) = \{ C, (P_1, P_2) \}, \) \( \Delta^0(X_{4,3}^{0+}) = \{ P_1, P_2 \} \).

**Proof.** (1) is a result of basic but long calculation. Please observe the graph of \( \Phi_{4,3}^0(0, s, t, 1) \) and \( f_{4,3}(1, x, y, z) = 0 \) using Mathematica.

(2) follows from (1). \( \square \)

Remember that \( \partial P_{4,3}^{0+} = \bigcup_{D \in \Delta(X_{4,3}^{0+})} F(D) \). Let’s observe whether each \( F(D) \) is a face component of \( P_{4,3}^{0+} \) or not.

Since \( \text{Zar}(\text{Int}(X_{4,3}^{0+})) = \mathbb{P}_+^3, \) \( F(\text{Int}(X_{4,3}^{0+})) \) is not a face component.

\( \text{Zar}(F((P_1, P_2))) \) is two dimensional plane defined by \( p_0 = p_2 = 0 \). Thus, \( F((P_1, P_2)) \) is not a face component. Therefore, we have:

**Proposition 6.4.** \( \partial P_{4,3}^{0+} = F(P_1) \cup F(P_2) \cup F(S) \cup F(C) \).

**Lemma 6.5.** We regard as \( H_{4,3}^{0+} = \mathbb{R}^4 \) by identifying \( (p_0, p_1, p_2, p_3) \in \mathbb{R}^4 \) with \( \sum_{i=0}^3 p_i s_i \in H_{4,3}^{0+} \). Then,

(1) \( \text{Zar} (F(P_1)) = V(p_0). \) Thus \( F(P_1) = \{ f \in P_{4,3}^{0+} \mid f \text{ is at infinity} \}. \)

(2) \( \text{Zar} (F(P_2)) = V(p_0 + p_2). \)

(3) \( \text{disc}_{\Sigma}(g_0(s,t), g_1(s,t), g_2(s,t), g_3(s,t)) = 0 \) for all \( s, t \in \mathbb{R} \).

(4) \( \text{disc}_{\Sigma}(g_0(s,t), g_3(s,t), g_2(s,t), g_1(s,t)) = 0 \) for all \( s, t \in \mathbb{R} \).

(5) \( \text{Zar} (F(C)) = V(\text{disc}_C). \)

(6) \( \text{Zar} (F(S)) = V(\text{disc}_S). \)

**Proof.** (1) and (2) are trivial.

(3) and (4) follow from direct calculation.

(5) follows from study of \( P_{3,3}^{0+} \). See §3 of [1].

(6) follows from (3). \( \square \)

Let \( F_{P} := F(P_1), F_{C} := F(C), \) and \( F_{S} := F(S) \).

**Lemma 6.6.** \( F_{P} \cap F_{S} \subset V(p_0 + p_2, s p_0(p_1 + p_3) - (p_1 - p_3)^2) \).
Proof. Let $\Gamma := \mathbb{P}(V(p_0 + p_2, 8p_0(p_1 + p_3) - (p_1 - p_3)^2)) \subset \mathbb{P}(\mathfrak{H}_{4,3}^{0})$. Since
$$\text{disc}_s(p_0, p_1, -p_0, p_3) = 2p_0((p_0 - p_1)^2 + (p_0 - p_3)^2)(8p_0(p_1 + p_3) - (p_1 - p_3)^2)^3,$$
we have $\mathbb{P}(\mathfrak{P}_{p_2} \cap \mathfrak{F}_S) \subset \Gamma \cup \{(1: 1: -1: 1)\}$. Let $\mathcal{S}_0 := \{(0: x: y: 1) \in \mathbb{P}^3_+ | x, y \in \mathbb{R}_+\}$, and define a rational map $G^S : \mathcal{S}_0 \to \mathbb{P}(\mathfrak{H}_{4,3}^{0})$ by
$$G^S(0, x, y, 1) := (g_0(x, y), g_1(x, y), g_2(x, y), g_3(x, y)).$$
If $t \neq 0$, $G^S(0, s, t, 1) = e_{s,t} \in \mathbb{P}(\mathcal{L}_{0; s,t:1}) \subset \mathbb{P}(\mathfrak{H}_{4,3}^{0})$. Note that $(0: 1: 0: 1) \in \text{Bs} G$. Since $g_i(s, t) = f_i(\Phi^S_{p_2}(0, s, t, 1))/g_i(s, t)$ (see the proof of Lemma 6.2), we can extend $G$ to $G^S : \mathbb{P}^3_+ \to \mathbb{P}(\mathfrak{H}_{4,3}^{0})$ by $G^S(x : y : 1 : 0) = G^S(y : 1 : 0 : x) = G^S(0 : 1 : x : y) = G^S(0 : x : y : 1) = G^S(0, x, y, 1)$. For $0 < r < 1$, let
$$C'_r := \{(0 : 1 + r \cos \theta : r \sin \theta : 1) \in \mathcal{S}_0 | 0 \leq \theta \leq \pi\}.$$Then \( \lim_{r \to 0} G^S(C'_r) = \Gamma \). Thus, $\mathbb{P}(\mathfrak{P}_{p_2} \cap \mathfrak{F}_S) \supset \Gamma$.

Since $g_0(x, y) + g_2(x, y) = y((x - 1)^2 + y^2)^2$, we have $(1: 1: -1: 1) \notin \mathbb{P}(\mathfrak{P}_{p_2})$. Thus $\mathfrak{P}_{p_2} \cap \mathfrak{F}_S \subset V(p_0 + p_2, 8p_0(p_1 + p_3) - (p_1 - p_3)^2)$.

Remember that $\text{disc}_C$ is the edge discriminant of $X_{4,3}^{0+}$ and $X_{4,3}^{0+}$. Let
$$\mathfrak{D}_C := \{(x, y, z) \in \mathfrak{H}_{4,3}^{0+} | y \geq -1 \text{ and } d_C(x, z) \geq 0\}.$$Then $\mathfrak{D}_C$ is a closed convex set such that $\mathfrak{P}_{4,3}^{0+} \subset \mathfrak{D}_C$, and $(\partial \mathfrak{P}_{4,3}^{0+}) \cap \text{Int} (\mathfrak{D}_C) \subset V(\text{disc}_S)$ by Lemma 6.5. We need the following polynomial to describe the cusp loci of $V(\text{disc}_S)$.

$$f^c_{s}(x, y) = 26040379669 + 153581431744x + 10225553008x^2 + 57589066563x^3 + 2375407488x^4 + 298011968x^5 + 47223216x^6 - 115722240x^7 + 17307648x^8 - 438272x^9 + 4096x^{10} + 89440948796y + 32061417248xy + 8138124864x^2y - 1752885472x^3y - 2067065472x^4y - 828572544x^5y + 1188607488x^6y - 112318464x^7y - 15593472x^8y - 126976x^9y + 8192x^{10}y - 2230719728sy^2 - 1623138328x^2y^2 - 12833341936x^2y^2 + 40377065344x^3y^2 + 550524544x^4y^2 + 481918440x^5y^2 - 264563678x^6y^2 + 218927104x^7y^2 + 9482240x^8y^2 + 176128x^9y^2 + 4096x^{10}y^2 + 30713189004y^3 + 8690225536xy^3 + 17703049984x^2y^3 - 2170474624x^3y^3 - 705133440x^4y^3 - 4728214912x^5y^3 - 1856392192x^6y^3 - 112496640x^7y^3 - 3928064x^8y^3 - 135168x^9y^3 + 6122931323y^4 - 3267147270xy^4 - 1613541908x^2y^4 - 1936345784x^3y^4 + 2347438208x^4y^4 + 668450944x^5y^4 + 1133005568x^6y^4 + 47364096x^7y^4 + 1464320x^8y^4 - 4000452071y^5 + 1411479097xy^5 - 921252992x^2y^5 + 9081775296x^3y^5 + 71177344x^4y^5 + 679918976x^5y^5 - 122998496x^6y^5 - 68218888x^7y^5 + 1068843692y^6 - 139854880xy^6 + 3457102112x^2y^6 - 1135819904x^3y^6 + 55287936x^4y^6 - 134577536x^5y^6 - 18625280x^6y^6 - 87042932y^7 + 226903552x^2y^7 - 733186304x^2y^7 - 48610432x^3y^7 - 35363712x^4y^7 - 12108928x^5y^7 - 108565637y^8 - 133149760xy^8 + 1725104x^2y^8 + 664660x^3y^8 - 2811392x^4y^8 + 4147404y^9 + 9240992y^9.$
Lemma 6.7. Consider on $\tilde{\mathcal{F}}_{1,3}^0$ : $(1, x, y, z) \cong \mathbb{R}^3$. Then

Sing $(V(\text{disc}_S(1, x, y, z))) \cap \mathcal{P}^c_{4,3} = \{Q_0\} \cup L_s \cup C_{1}^{\text{cusp}} \cup C_2^{\text{cusp}} \cup C_3^{\text{cusp}} \cup C_4^{\text{cusp}} \subset \tilde{\mathcal{F}}_{4,3}^0$,

where $Q_0$, $L_s$ and $C_i^{\text{cusp}}$ are as the follows:

1. $Q_0$ is a point defined by $(x, y, z) = (-1, 3, -1)$. This lies on $\partial \mathcal{P}^c_{4,3}$.
2. $L_s$ is the half line defined by $x = z = (y - 1)/2$, $y \geq -1$. Note that $L_s \cap \partial \mathcal{P}^c_{4,3} = \{Q_0\}$.
3. The hyperbolic curve $C_s$ on the plane $V(x - z)$ defined by $x = z = -(y^2 - 2y + 13)/(4y + 4)$ is a singular locus of $V(\text{disc}_S(1, x, y, z))$. But $C_s \cap \partial \mathcal{P}^c_{4,3} = \{Q_0\}$.
4. Let $x = \alpha_1(y)$ be all the four real roots of $f_S^{\text{cusp}}(x, y) = 0$ when we regard $y$ to be a constant where $y \geq 3$ and $\alpha_1(y) \leq \alpha_2(y) \leq \alpha_3(y) \leq \alpha_4(y)$. Note that $\alpha_1(3) = \alpha_2(3) = \alpha_3(3) = \alpha_4(3) = 1$. Then, the following four branches are cusps of $S$.

$$
\begin{align*}
C_1^{\text{cusp}} &= \{(1: \alpha_1(y): y: \alpha_4(y)) \in \tilde{\mathcal{F}}_{4,3}^0 \mid y > 3\}, \\
C_2^{\text{cusp}} &= \{(1: \alpha_2(y): y: \alpha_3(y)) \in \tilde{\mathcal{F}}_{4,3}^0 \mid y > 3\}, \\
C_3^{\text{cusp}} &= \{(1: \alpha_3(y): y: \alpha_2(y)) \in \tilde{\mathcal{F}}_{4,3}^0 \mid y > 3\}, \\
C_4^{\text{cusp}} &= \{(1: \alpha_4(y): y: \alpha_1(y)) \in \tilde{\mathcal{F}}_{4,3}^0 \mid y > 3\}.
\end{align*}
$$

Proof. Let $f(x, y, z) := \text{disc}_S(1, x, y, z)$ and $f_w := \frac{\partial f}{\partial w}$ for $w = x, y, z$. Sing $(V(\text{disc}_S(1, x, y, z)))$ can be obtained by solve the system of equations $f(x, y, z) = f_x(x, y, z) = f_y(x, y, z) = f_z(x, y, z) = 0$. But it is next to impossible to proceed this calculation. Instead of it, we eliminate $z$ from $f_x(x, y, z) = 0$, $f_y(x, y, z) = 0$, and $f_z(x, y, z) = 0$. During this elimination process, we obtain some factors which include Sing $(V(\text{disc}_S(1, x, y, z)))$.

We take the section of $\mathcal{P}^c_{4,3}$ by the hyperplane

$$
H_r := \{(1: x: y: z) \in \mathcal{F}_{4,3}^0 \mid y = r\}.
$$

We regard $H_r$ as $(x, z)$-plane. Put

$$
\begin{align*}
D_r := & \{(x, z) \in H_r \mid (1: x: r: z) \in \mathcal{P}^c_{4,3}\}, \\
D_C := & D_C \cap H_r = \{(x, z) \in H_r \mid d_C(x, z) \geq 0\}, \\
V_C := & \partial D_C = \{(x, z) \in H_r \mid d_C(x, z) = 0\}, \\
V'_S := & \{(x, z) \in H_r \mid \text{disc}_S(1, x, r, z) = 0\} - (C_s \cup L_s) \cap H_r.
\end{align*}
$$

(O-1) If $r < -1$, then $D_r = \emptyset$, by Lemma 6.5(2).

(O-2) If $r = -1$, then the condition of (1) of Theorem 6.1 determines the set $\mathcal{P}^c_{4,3} \cap H_{-1}$, because of Lemma 6.6.
(I) When \(-1 < r < 3\), \(V_S^3\) is as Fig.6.1. Two points \(C_s^s \cap H_r\) and \(L^s \cap H_r\) are all the isolated singularities of \(V(\text{disc}_S) \cap H_3\). \(V_S^3\) is a smooth curve in \(D_C\) and enclose a convex set \(P_{4,3}^{\text{cusp}} \cap H_r\). Thus,

\[
D_r := \{(x, z) \in \mathbb{R}^2 \mid \text{disc}_S(1, x, r, z) \geq 0 \text{ and } d_C(x, z) \geq 0\}.
\]

Thus, the conditions of (2) of Theorem 6.1 determines \(\mathcal{P}_{4,3}^{\text{cusp}} \cap H_r\).

(II) Consider the case \(r = 3\). Let

\[
f_3^S(x, z) := x^6 - 4x^5z + 7x^4z^2 - 8x^3z^3 + 7x^2z^4 - 4xz^5 + z^6
\]

\[
- 174x^5 - 342x^4z - 508x^3z^2 - 508x^2z^3 - 342xz^4 - 174z^5
\]

\[
- 414x^4 - 712x^3z - 1332x^2z^2 - 712xz^3 - 414z^4
\]

\[
- 800x^3 - 4320x^2z - 4320xz^2 - 800z^3
\]

\[
- 6592x^2 - 16512xz - 6592z^2 - 16384x - 16384z - 11776.
\]

Then \(\text{disc}_S(1, x, 3, z) = 2(x + z + 2)^2 f_3^S(x, z)\). As Fig 6.2, \(V(f_3^S)\) tangents \(V_C\) at three points \(P_{3,1}^{\text{tan}}, P_{3,4}^{\text{tan}}\) and \(P_{3,2}^{\text{tan}} = P_{3,3}^{\text{tan}}\) (these symbols will be explained in (III)). Moreover \(V(f_3^S) \subset D_C\). Thus,

\[
D_3 := \{(x, z) \in \mathbb{R}^2 \mid \text{disc}_S(1, x, 3, z) \geq 0 \text{ and } d_C(x, z) \geq 0\}
\]

and the conditions of (2) of Theorem 6.1 determines \(\mathcal{P}_{4,3}^{\text{cusp}} \cap H_3\).

Note that \(V(f_3^S) \cap V(z + 11.851831 \cdots) = \emptyset\), and \(V(f_3^S) \cap V(z - z_0)\) is two points for \(z_0 > -11.851831\)…

(III) Consider the case \(r > 3\). Then, \(V_S^3\) has just four cusps \(P_{r,i}^{\text{cusp}} := C_i^{\text{cusp}} \cap H_r = (\alpha_i(r), \alpha_{i-1}(r)) (i = 1, 2, 3, 4)\). Since \(V_S^3\) is symmetric with respect to the line \(V(x - z)\), it is enough to consider the part \(z \geq 0\). As Fig. 6.3, we divide the part \(z > \alpha_4(y)\) of \(V_S^3\) into \(V_S^{r,a}\) and \(V_S^{r,b}\). Let \(V_S^{r,c}\) be the smooth interval between \(P_{r,2}^{\text{cusp}}\) and \(P_{r,3}^{\text{cusp}}\) of \(V_S^3\).

We observe \(\mathcal{F}_S \cap \mathcal{F}_C \cap \mathcal{H}_{4,3}^{\text{cusp}}\). Let

\[
L_x := \{(0: 0: w: 1) \in \mathbb{P}_3^+ \mid w \in [0, \infty)\},
\]

\[
L_y := \{(0: w: 0: 1) \in \mathbb{P}_3^+ \mid w \in [0, \infty)\},
\]

\[
L_z := \{(0: x: y: 0) \in \mathbb{P}_3^+ \mid (x: y) \in \mathbb{P}_1^+\}.
\]
Note that \( \partial \text{Cl}_{38}(\mathbf{B}_0) = L_x \cup L_y \cup L_z \). Since \( g_0(w, 0) = 0 = G^S(L_y) \cap \mathcal{F}_{1,3}^0 = \emptyset \). Since \( G^S(0: x: y: 0) = G^S(0: 0: x: y: 1) \), we have \( G^S(L_z) = G^S(L_x) \). Since

\[
disc_C \left( g_0(0, w), g_1(0, w), g_2(0, w) \right) = 0,
\]
we have \( G^S(L_z) \subset V(\text{disc}_C) \cap V(\text{disc}_S) \). Put \( C^z_{\text{tan}} := G^S(L_z) \).

Similarly, we define a rational map \( G^z : \mathbf{B}_0 \to \mathbb{P}(\mathcal{F}_{4,3}^0) \) by

\[
G^z(0: x: y: 1) := \left( g_0(x, y) : g_3(x, y) : g_2(x, y) : g_1(x, y) \right),
\]
and let \( C^z_{\text{tan}} := G^z(L_z) \). Then \( C^z_{\text{tan}} \cup C^z_{\text{tan}} \subset V(\text{disc}_C) \cap V(\text{disc}_S) \).

Put \( H_{\geq 3} := \{(1: x: r: z) \in \mathcal{F}_{4,3}^0 \mid r \geq 3\} \). We regard \( H_{\geq 3} \subset \mathcal{F}_{4,3}^0 \subset \mathbb{P}(\mathcal{F}_{4,3}^0) \). We shall determine \( C_{\text{tan}} \cap H_{\geq 3} \). Let \( \delta := 0.29559774252208 \) be the real root of \( t^3 + t^2 + 3 t - 1 = 0 \).

Then, all the real roots of \( g_2(0, t)/g_0(0, t) = 3 \) are \( t = 1, \delta \). We put

\[
C^1_{\text{tan}} := \left\{ G^z(0: 0: w: 1) \in \mathbb{P}(\mathcal{F}_{4,3}^0) \mid 0 < w \leq \delta \right\} \subset C^z_{\text{tan}}, \quad C^r_{\text{tan}} := C^1_{\text{tan}} \cap H_r \in \mathbb{P}(\mathcal{F}_{4,3}^0),
\]

\[
C^2_{\text{tan}} := \left\{ G^S(0: 0: w: 1) \in \mathbb{P}(\mathcal{F}_{4,3}^0) \mid w \geq 1 \right\} \subset C^z_{\text{tan}}, \quad C^r_{\text{tan}} := C^2_{\text{tan}} \cap H_r \in \mathbb{P}(\mathcal{F}_{4,3}^0),
\]

\[
C^3_{\text{tan}} := \left\{ G^z(0: 0: w: 1) \in \mathbb{P}(\mathcal{F}_{4,3}^0) \mid w \geq 1 \right\} \subset C^z_{\text{tan}}, \quad C^r_{\text{tan}} := C^3_{\text{tan}} \cap H_r \in \mathbb{P}(\mathcal{F}_{4,3}^0),
\]

\[
C^4_{\text{tan}} := \left\{ G^S(0: 0: w: 1) \in \mathbb{P}(\mathcal{F}_{4,3}^0) \mid 0 < w \leq \delta \right\} \subset C^z_{\text{tan}}, \quad C^r_{\text{tan}} := C^4_{\text{tan}} \cap H_r \in \mathbb{P}(\mathcal{F}_{4,3}^0).
\]

Then \( C_{\text{tan}} \cap H_{\geq 3} = C^1_{\text{tan}} \cup C^3_{\text{tan}} \) and \( C_{\text{tan}} \cap H_{\geq 3} = C^2_{\text{tan}} \cup C^4_{\text{tan}} \). Note that \( \mathcal{F}_S \cap \mathcal{F}_C \cap \left\{ G^S(0: 0: w: 1) \in \mathbb{P}(\mathcal{F}_{4,3}^0) \mid \delta < w < 1 \right\} = \emptyset \).

**Lemma 6.8.** \( C^1_{\text{tan}} \cup C^2_{\text{tan}} \cup C^3_{\text{tan}} \cup C^4_{\text{tan}} \subset \text{Zar}(\mathcal{F}_S \cap \mathcal{F}_C) \cap H_{\geq 3} \).

**Proof.** Clear.

\[
C^\text{cusp} := \text{Cl}_{38}(C^1_{\text{cusp}} \cup C^2_{\text{cusp}} \cup C^3_{\text{cusp}} \cup C^4_{\text{cusp}}).
\]

Let’s determine \( C_{\text{tan}} \cap C^\text{cusp} \). Since \( C_{\text{tan}} = G^S(L_x) \subset V(\text{disc}_C) \cap V(\text{disc}_S) \), and

\[
G^S(0: 0: w: 1) = \left( \frac{1}{w} : \frac{w^2}{w} : \frac{(w^2 + 1)^2}{w} : \frac{w^3 - 2}{w} \right),
\]
we put \( G^S_p(w) := (1 - 2 w^3)/w^2, G^S_p(w) := ((w^2 + 1)^2 - w)/w \) and \( G^S_p(w) := (w^3 - 2)/w = G^S_p(1/w) \).

**Lemma 6.9.** \( (x, y) = 61 + 62 x + 56 y + 32 x^2 + 30 y x - 6 y^2 + 9 x^3 + 4 x^2 y - 6 y^2 - 16 y^3 + x^4 - 4 x^2 y^2 - 6 y x y^2 + y^4 - x^3 y^2 \) has the following properties:

1. If \((1: x: y: z) \subset C^r_{\text{tan}} \cup C^z_{\text{tan}}\), then \( \eta(x + z, y) = 0 \).

2. Let \( r > 3 \). On a plane \( H_r \), the zero locus \( \eta(x + z, r) = 0 \) is the union of two lines. One is the line \( P^r_{1,1} \cap P^r_{1,3} \), and the other is the line \( P^r_{r,2} \cap P^r_{r,3} \). \( \eta(x + z, r) < 0 \) between these two lines, and \( \eta(x + z, r) \) outside outside.

**Proof.** (1) follows from \( \eta(G_p^S(w) + G_p^S(w), G_p^S(w)) = 0 \).

(2) \( \eta(x, r) = 0 \) has just two real roots for \( r > 3 \), and \( \eta(y - 3, y) < 0 \) for \( y < 3 \).

Note that

\[
f^\text{cusp}_S \left( G_p^S(w), G_p^S(w) \right) = \frac{(w - 1)^2(w^2 + 1)^2(w^4 - 6 w^2 - 8 w + 1)^2 f_{38}(w)}{w^2},
\]
here \( f_{38}(w) \) is a polynomial of degree 38 whose real roots are two negative numbers \( w = -652.3327 \cdots, -20.9069 \cdots \). Let \( \alpha := 0.1150 \cdots \) and \( \beta := 2.9343 \cdots \) are all the real roots of

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where eliminate $P$ 6.1, we put

Similarly, $C$ $\alpha$ $S$ \tan $P$ $c$ $43042537$ $r$ $\gamma$, $\beta$$\beta$, $3$ $= r$, $\alpha$ $\alpha$, $\gamma$ $w$. In the case Now, we shall complete the proof of Theorem 6.1. To prove (3), (4), (5) of Theorem

Note that $g := r$ $\alpha$, $\delta$ $\eta$ $G$ $\delta$ $G$ $\gamma$ $\alpha$ $\gamma$ $\alpha$. Then $\eta$ $s$, $\beta$, $\delta$ $\delta$. Let $\gamma$, $\delta$ be all the imaginal roots $w^4 - 6w^2 - 8w + 1 = 0$, and put $s_2 := \gamma + \delta$, $t_2 := \gamma\delta$. Then $s + s_2 = 0$, $tt_2 = 1$, $t + t_2 + ss_2 = -6$, $ts_2 + st_2 = 8$. When we eliminate $s$, $t$, $s_1$, $t_1$ from these ralations, we have

\[
\begin{align*}
817808203c_1^6 - 546807084c_1^4 + 129155640c_1^2 - 13342016c_1^4 + 556080c_1^2 - 10176c_1 + 64 &= 0, \\
43042537c_2^6 - 4514514c_2^5 - 188769c_2^4 - 38684c_2^3 + 4119c_2^2 - 114c_2 + 1 &= 0.
\end{align*}
\]

\[\square\]

Now, we shall complete the proof of Theorem 6.1. To prove (3), (4), (5) of Theorem
6.1, we put

\[
\begin{align*}
D_1^{(3)} &:= \{ (x, z) \in H_r \mid c_1(x + z) + c_2r \geq 1, \text{disc}_S(1, x, r, z) \geq 0, \text{d}_C(x, z) \geq 0 \}, \\
D_1^{(4)} &:= \{ (x, z) \in H_r \mid c_1(x + z) + c_2r < 1, \eta(x + z, r) > 0, \text{disc}_S(1, x, r, z) \geq 0, \text{d}_C(x, z) \geq 0 \}, \\
D_1^{(5)} &:= \{ (x, z) \in H_r \mid c_1(x + z) + c_2r < 1, \eta(x + z, r) \leq 0, \text{d}_C(x, z) \geq 0 \}.
\end{align*}
\]
(III-1) If $3 < r < r_1$, then $P_{cusp}^{r,1} = (\alpha_1(r), \alpha_4(r)) \in \text{Int}(D_{C})$, and $V_{S}^{r,a}$ tangents to $V_{C}$ at $P_{r,1}^{\tan}$, as Fig. 6.3. This implies that $P_{r,1}^{\tan} \in (\partial F_{C}) \cap (\partial F_{S})$. We divide the curve segment $V_{S}^{r,a}$ at the point $P_{r,1}^{\tan}$, and denote the upper part by

$$V_{S}^{r,1} := \{(x, z) \in H_r \mid \text{disc}_{S}(x, r, z) = 0, d_{C}(x, z) \geq 0, z \geq z(P_{r,1}^{\tan})\},$$

where $z(P)$ is the $z$-coordinate of the point $P \in H_r$. Then $V_{S}^{r,1} \subset F_{S}$, but every $P \in V_{S}^{r,a} - V_{S}^{r,1}$ is obtained as $P = G(0; s: t: 1)$ for a certain $(s, t) \in C^2 - B_0$.

Let $V_{S}^{r,2}$ be the symmetric set of $V_{S}^{r,1}$ with respect to the line $x = z$ on $H_r$.

Similarly, $(\alpha_2(r), \alpha_3(r)) \in \text{Int}(D_{C})$, and $V_{S}^{r,c}$ tangents to $V_{C}$ at $P_{r,2}^{\tan}$, as Fig. 6.4. Let

$$V_{S}^{r,3} := \{(x, z) \in H_r \mid \text{disc}_{S}(x, r, z) = 0, d_{C}(x, z) \geq 0, z(P_{r,2}^{\tan}) < z \leq z(P_{r,1}^{\tan})\}$$

be the interval of $V_{S}^{r,3}$ between $P_{r,2}^{\tan}$ and $P_{r,3}^{\tan}$. By Lemma 6.9,

$$V_{S}^{r,1} \cup V_{S}^{r,2} \cup V_{S}^{r,3} = \{(1; x: r: z) \in (\partial F)_{D}^{0+} \mid \text{disc}_{S}(1, x, r, z) = 0, d_{C}(x, z) \geq 0, \eta(x + z, r) \geq 0\}.$$

So, $D_{r} = D_{r}^{(3)} \cup D_{r}^{(4)} \cup D_{r}^{(5)}$.

(III-2) If $r = r_1$, then $P_{r,1}^{\tan} = (\alpha_1(r_1), \alpha_4(r_1))$, $P_{r,4}^{\tan} = (\alpha_1 4r_1, \alpha_1(r_1)) \in (\partial F_{C}) \cap (\partial F_{S})$. The line defined by $c_1(x + z) + c_2r_1 = 1$ agrees with the line $P_{r,1}^{\tan} P_{r,4}^{\tan}$. Others are similar as (III-1).

(III-3) Consider the case $r_1 < r < r_2$. About $V_{S}^{r,3}$ the situation is same as (III-1).

The situation of $V_{S}^{r,1}$ and $V_{S}^{r,2}$ changes. If $r > r_1$, then $(\alpha_1(r), \alpha_4(r)) \notin D_{C}$, and $P_{r,1}^{\tan} \notin D_{C}$ as Fig. 6.5. In this case, $V_{C}$ and $V_{S}^{r,a}$ intersect at a point $Q_{r}^{a}$ transversally. So, let

$$V_{S}^{r,1} := \{(x, z) \in H_r \mid \text{disc}_{S}(x, r, z) = 0, d_{C}(x, z) \geq 0, z \geq z(Q_{r}^{a})\},$$
be the interval of $V_S^{r,a}$ upper than $Q_r^a$. Let $V_S^{r,2}$ be the symmetric set of $V_S^{r,1}$ with respect to $V(x - z)$. Then,

$$V_S^{r,1} \cup V_S^{r,2} = \left\{ (1: x: r: z) \in \partial \mathcal{F}_{4,3}^0 \mid \begin{align*}
\text{disc}_S(1, x, r, z) &= 0, d_C(x, z) \geq 0, \\
c_1(x + z) + c_2r &= 1
\end{align*} \right\},$$

$$V_S^{r,3} = \left\{ (1: x: r: z) \in \partial \mathcal{F}_{4,3}^0 \mid \begin{align*}
\text{disc}_S(1, x, r, z) &= 0, d_C(x, z) \geq 0, \\
\eta(x + z, r) &= 0, c_1(x + z) + c_2r < 1
\end{align*} \right\}.$$ 

So, $D_r = D_r^{(3)} \cup D_r^{(4)} \cup D_r^{(5)}$.

(III-4) If $r = r_2$, then $p_{r,2,1}^{\text{tan}} = (\alpha_2(r_2), \alpha_3(r_2))$, $p_{r,2,3}^{\text{tan}} = (\alpha_2(r_3), \alpha_3(r_2)) \in (\partial \mathcal{F}_C) \cap (\partial S)$. Others are similar as (III-3).

(III-5) If $r > r_2$, then $(\alpha_2(r), \alpha_3(r)) \notin D_C$, and $p_{r,2}^{\text{tan}}, p_{r,3}^{\text{tan}} \notin D_C$ as Fig.6.6. In this case, $V_C$ and $V_S^{r,c}$ intersect at two points $Q_r^1, Q_r^2$ transversally. So, let

$$V_S^{r,3} := \left\{ (x, z) \in H_r \mid \text{disc}_S(x, r, z) = 0, d_C(x, z) \geq 0, z(Q_r^2) \leq z \leq z(Q_r^3) \right\}$$

be the the interval of $V_S^{r,c}$ between $Q_r^1$ and $Q_r^2$. Then

$$V_S^{r,1} \cup V_S^{r,2} \cup V_S^{r,3} = \left\{ (1: x: r: z) \in \partial \mathcal{F}_{4,3}^0 \mid \begin{align*}
\text{disc}_S(1, x, r, z) &= 0, d_C(x, z) \geq 0
\end{align*} \right\}.$$ 

If $r > r_2$, then $c_1(x + z) + c_2r \geq 1$ holds for any $(x, z) \in D_C$. Thus $D_r = D_r^{(3)}$ in this case.

By (III-1)–(III-5) and Lemma 6.10, we conclude that the conditions of (3), (4), (5) of Theorem 6.1 determine $\mathcal{F}_{4,3}^{c,0+}$ when $r > 3$.

Let’s determine

$$E := \left\{ (s, t) \in \mathbb{R}_+^2 \mid \epsilon_{s,t} \text{ is an extremal element of } \mathcal{F}_{4,3}^{c,0+} \right\}.$$
Note that if $\xi_{s,t} \in \mathcal{P}_{4,3}^{0+}$, then $\xi_{s,t} \in E$. We can use Theorem 6.1 to determine whether $\xi_{s,t} \in \mathcal{P}_{4,3}^{0+}$ or not. Let

$$d^c_e(s, t) := -1 - 2s + 3s^2 + 8s^3 - 2s^4 - 12s^5 - 2s^6 + 8s^7 + 3s^8 - 2s^9 - s^{10} + 8t + 8st - 32s^2t - 32s^3t + 48s^4t + 48s^5t - 32s^6t + 8s^7t + 8s^8t + 8s^9t + 6t^2 + 76st^2 + 112s^2t^2 - 76s^3t^2 - 236s^4t^2 - 76s^5t^2 + 112s^6t^2 + 76s^7t^2 + 6s^8t^2 - 2t^3 - 22st^3 + 78s^2t^3 - 54s^3t^3 - 54s^4t^3 + 78s^5t^3 - 22s^6t^3 - 2t^7 + 70st^4 + 401s^2t^4 + 756s^3t^4 + 401s^4t^4 + 70s^5t^4 + 15s^6t^4 + 12t^5 - 36st^5 + 24s^2t^5 + 24s^3t^5 - 36s^4t^5 + 12s^5t^5 - t^6 - 96s^6t^6 - 238s^5t^6 - 96s^4t^6 + 6t^7 - 6st^7 - 6s^2t^7 + 6s^3t^7 + 6t^8 + 24s^8t + 6s^2t^8 - t^{10},$$

Then

$$\text{disc}_C(g_0(s, t), g_1(s, t), g_3(s, t)) = s^2(t - s - 1)^2((s - 1)^2 + t^2)^2d^c_e(s, t).$$

If we check the conditions (1), (2), (3), (4), (5) of Theorem 6.1, we obtain

$$E = \{(s, t) \in \mathbb{R}_+^2 \mid d^c_e(s, t) \geq 0\}.$$

**Theorem 6.11.** Let $t_1 := 0.11508799467984865 \cdots$ and $t_2 := 2.934317165179855 \cdots$ be two real roots of $t^4 - 6t^2 - 8t + 1 = 0$. If $f$ is an extremal element of $\mathcal{P}_{4,3}^{0+}$, then $f$ is a positive multiple of one of the following polynomials:

(1) $\xi_{s,t}$ where $(s, t) \in E$.

(2) $h_t := t\xi_{0,1} + (1 - t^2)(1 + t^2)^2s_2$ where $t_1 \leq t \leq t_2$.

(3) $s_1 + s_3$.

Conversely, all the above polynomials are extremal element of $\mathcal{P}_{4,3}^{0+}$.

**Proof.** Let $f \in \mathcal{E}(\mathcal{P}_{4,3}^{0+})$. Then $f(r, s, t, 1) = 0$ for a certain $(r, s, t) \in \mathbb{R}_+^3$. Since $\mathcal{E}(\mathcal{P}_{4,3}^{0+}) \subset \mathcal{F}(P_1) \cup \mathcal{F}(P_2) \cup \mathcal{F}(S) \cup \mathcal{F}(C)$, we may assume $r = 0$.

(1) If $s > 0$ and $t > 0$, then $f \in \mathcal{L}_{(0:s:t:1)}^{0+}$. Thus $f$ is a positive multiple of $\xi_{s,t}$.

(2) We consider the case $r = s = 0$. In this case, $f \in \mathcal{F}(S) \cap \mathcal{F}(C)$. Solve the equation $\text{disc}_S(g_0(0, t), g_1(0, t), x, g_3(0, t)) = 0$, we have two solutions $x = g_2(0, t)$ and $x = g_0(0, t) + (1 - t^2)(1 + t^2)^2s_2$. By Theorem 6.1, only these two $x$ have possibilities to provide extremal elements. In the fact, $\xi_{0,t}(a, b, c, 1)$ is characterized by the condition $\frac{\partial \xi_{0,t}}{\partial b}(0, 0, t, 1) = 0$, besides $\frac{\partial \xi_{0,t}}{\partial b}(0, 0, t, 1) = (1 - t)^2(1 + t)(1 + t^2)^2$. On the other hand, $\frac{\partial \xi_{0,t}}{\partial a}(0, 0, t, 1) = 0$, besides $\frac{\partial \xi_{0,t}}{\partial a}(0, 0, t, 1) = (1 - t)^2(1 + t)(1 + t^2)^2$. Since $d^c_e(s, t) = -(1 + t)^2(1 - t^2)^2(1 - 8st - 6t^2 + t^4)$, $1 - 8st - 6t^2 + t^4 \leq 0$ is required,

(3) We consider the case $r = t = 0$. In this case, $f \in \mathcal{F}(S) \cap (\mathcal{F}(P_1) \cup \mathcal{F}(P_2))$. Note that $\xi_{s,0} = (1 - s^2)^2(s_1 + ss_3)$. Let $\mathcal{K} := \{g \in \mathcal{F}_{4,3}^0 \mid g(0, u, 0, 1) = 0 \text{ for all } u \in \mathbb{R}_+\}$. It is easy to see that $\mathcal{K} = \mathbb{R}_+ \cdot s_1 + \mathbb{R}_+ \cdot ss_3$. But $s_1 + ss_3$ is PSD if and only if $s = 1$ by Theorem 6.1. Thus $\mathcal{K} \cap \mathcal{P}_{4,3}^{0+} = \mathbb{R}_+ \cdot (s_1 + ss_3)$. So, $\xi_{s,0} \in \mathcal{P}_{4,3}^{0+}$ if and only if $s = 1$. Remember that $f := s_1 + ss_3$ satisfies $f(x, 1, x, 1) = 0$ for all $x \geq 0$, besides $f(0, u, 0, 1) = 0$ for all $u \geq 0$. $\square$

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Remark 6.12. As a special case, if $t = s + 1$,
\[ \epsilon_{s,s+1} = (1 + s)(1 + s^2)(s_0 - s_1 + 3s_2 - s_3) = (1 + s)(1 + s^2)(a - b + c - d)^2(a + b + c + d) \]
has extra zeros $\epsilon_{s,1+s}(x, y, z, x - y + z) = 0$.

6.2. Structure of $\mathcal{P}_{4,3}^+$

We have not complete any of (I1), (I2), (I3) for $\mathcal{P}_{4,3}^+$. But, we shall give (I4) and some information about $X_{4,3}^+$.

Proposition 6.13. We define $\Phi_{4,3}^+: \mathbb{P}^3_+ \cdots \to \mathbb{P}_+^4$ by
\[ \Phi_{4,3}^+(a) = (S_3(a) : S_{2,1,0}(a) : S_{2,0,1}(a) : S_{1,2,0}(a) : S_{1,1,1}(a)), \]
and let $X_{4,3}^+ := \Phi_{4,3}^+(\mathbb{P}_+^3)$. Then $\text{Zar}(X_{4,3}^+)$ is the hyper surface of $\mathbb{P}_+^3$ : $(t_0 : \cdots : t_4) = \mathbb{P}((\mathcal{H}_{4,3}^+)^\vee)$ defined by
\begin{align*}
 f_{4,3}^+(x_0, x_1, x_2, x_3, x_4) :=
 & x_3^3 - x_0 x_1 x_2 + x_1 x_2^2 - x_0 x_1 x_3 - x_0 x_2 x_3 - x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2 + x_3^3 \\
 & + x_2^3 x_4 + x_1^2 x_4 - x_2^2 x_4 - 4 x_1 x_3 x_4 + x_3^2 x_4 + x_0^2 x_4 - x_1 x_4^2 - x_2 x_4^2 - x_3 x_4^2 + 2x_4^3.
\end{align*}
This cubic hypersurface has an isolated singularity at $(1:1:1:1:1)$.

Proof. This follows from $f_{4,3}^+(S_3, S_{2,1,0}, S_{2,0,1}, S_{1,2,0}, S_{1,1,1}) = 0$.

Proposition 6.14. $E(\mathcal{P}_{4,3}^{0+}) = \mathcal{E}(\mathcal{P}_{4,3}^+) \cup \mathcal{E}_{4,3}^{0+}$.

Proposition 6.15. Assume that $f \in \mathcal{E}(\mathcal{P}_{4,3}^+) - \mathcal{E}(\mathcal{P}_{4,3}^{0+})$. Then $f(a) > 0$ for all $a \in \text{Int}(\mathbb{R}_+^4)$. Thus, $X_{4,3}^+$ does not have the main component.

Proof. Assume that $f(a) = 0$ for $a = (a:b:c:1) \in \text{Int}(\mathbb{P}_+^4)$. $(a, b, c) \neq (1, 1, 1)$, since $f \notin \mathcal{P}_{4,3}^{0+})$. Put $b := (b:c:1:a) \in \text{Int}(\mathbb{P}_+^4)$. Note that $a \neq b$. Then the line $ab$ is a bitangent line of the cubic surface $V_C(f) \in \mathbb{P}_C^3$. But a cubic surface has no bitangent line.

So, if $f \in \mathcal{E}(\mathcal{P}_{4,3}^+)$, then there exists $s, t \in \mathbb{R}_+$ such that $f(0, s, t, 1) = 0$, by Proposition 1.8. Let
\[ \mathcal{L}_{(s,t)} := \{ g \in \mathcal{P}_{4,3}^+ \ | \ g(0, s, t, 1) = 0 \} \]
be the local cone of $\mathcal{P}_{4,3}^+$ at $(0, s, t, 1)$. If $s > 0$ and $t > 0$, then $\dim \mathcal{L}_{(s,t)} \leq 2$. Thus $\mathcal{L}_{(s,t)}$ has at most two extremal rays. If $\dim \mathcal{L}_{(s,t)} = 2$, there exists $u \geq 0$ and $v \geq 0$ such that $f(0, u, v, 1) = 0$ and $(u, v) \neq (s, t)$.

Corollary 6.16. If $f \in \mathcal{H}_{4,3}$ satisfies $f(0, s, t, 1) \geq 0$ for all $s, t \in \mathbb{R}_+$, then $f \in \mathcal{P}_{4,3}^+$. 

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Section 7. Philosophy of Semialgebraic Variety.

7.1. Real algebraic quasi-variety.

Till §6, we used the notion of (quasi-) semialgebraic varieties without exact definition. In this section, we shall discuss how its definition should be, at least for theory of PDS cones. Before to give it, we must discuss what a real algebraic variety is.

Usually, we say $(X, \mathcal{O}_X)$ is an algebraic variety over $\mathbb{R}$ when $(X, \mathcal{O}_X)$ is an integral separated scheme of finite type over $\mathbb{R}$. $X(\mathbb{R})$ denotes the set of $\mathbb{R}$-rational points, and $X_{\mathbb{C}} := X \times_{\text{Spec} \mathbb{R}} \text{Spec} \mathbb{C}$. By this definition, $X$ and $X_{\mathbb{C}}$ are irreducible and reduced. To treat possibly reducible or non-reduced varieties, we shall call a separated scheme of finite type over $\text{Spec} \mathbb{R}$ to be an algebraic quasi-variety. This notion is not convenient for algebraic inequalities. For example, there exists infinitely many algebraic varieties $X$ over $\mathbb{R}$ such that $X(\mathbb{R}) = \mathbb{R}^2$. $X$ may not be affine even if $X(\mathbb{R}) = \mathbb{R}^2$.

The definition of a real algebraic variety is given in §3.2 in [6]. According to this definition, every real algebraic variety is reduced but may be reducible (i.e. not irreducible). To keep consistency with complex algebraic geometry, we shall add a restriction that real algebraic varieties must be irreducible and separated. To treat possibly non-reduced varieties, we shall give alternative definition of real algebraic quasi-varieties as the following:

**Definition 7.1.** (Real algebraic quasi-variety) (I) A locally ringed space $(X, \mathcal{R}_X)$ is called a real algebraic variety, if there exists a separated scheme $(Y, \mathcal{O}_Y)$ of finite type over $\text{Spec} \mathbb{R}$ which satisfies the following:

1. There exists an injective morphism $\iota: (X, \mathcal{R}_X) \rightarrow (Y, \mathcal{O}_Y)$ as locally ringed spaces, and $\iota$ induces a homeomorphism $X \rightarrow Y(\mathbb{R})$ as topological spaces with respect to Zariski topology and Euclidean topology.

2. Take any affine open subset $V \subset Y$. Let $n_P$ be the maximal ideal of $\mathcal{O}_Y(V)$ corresponding to a closed point $P \in Y$. For an arbitral non-empty subset $U \subset V \cap \iota(X)$, we put

$$S_U := \bigcap_{P \in U} \left( \mathcal{O}_Y(V) - n_P \right).$$

If $U$ is an Euclidean open set, then $\iota^*: S_U^{-1}\mathcal{O}_Y(V) \rightarrow \mathcal{R}_X(\iota^{-1}(U))$ is an isomorphism of $\mathbb{R}$-algebra. Thus, each maximal ideal $m \subset \mathcal{R}_X(\iota^{-1}(V))$ corresponds to a point $P \in \iota^{-1}(V) \subset X$.

3. Take an arbitral affine open subset $V \subset Y$. Then

$$\{ f \in \mathcal{O}_Y(V) \mid f(P) = 0 \text{ for all } P \in V(\mathbb{R}) \}$$

is a nilpotent ideal of $\mathcal{O}_Y(V)$.

In this case, $Y$ is said to be a $\mathbb{R}$-scheme which represents $X$. If we can choose $Y$ such that $Y_{\mathbb{C}}$ is irreducible and reduced, then we shall call $X$ to be a real algebraic variety (See Notation 0.1 of [21]).

$U \subset X$ is called an affine open subset of $X$, if there exists an affine open subset $U_Y \subset Y$ such that $U = \iota^{-1}(U_Y(\mathbb{R}))$. Zariski open (resp. closed) subsets are defined similarly. The Euclidean topology of $X$ is the topology induced from the analytic topology of $Y_{\mathbb{C}}$. $Y(\mathbb{R})$ is also denoted as $Y_{\mathbb{C}}(\mathbb{R})$. When $V \subset Y$ is an affine open subset and $B \subset V(\mathbb{R})$ is a subset such that $\text{Cl}_{Y(\mathbb{R})}(\text{Int}(B)) = \text{Cl}_{Y(\mathbb{R})}(B)$, we put

$$S_B := \bigcap_{P \in B} \left( \mathcal{O}_Y(V) - n_P \right),$$

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and \( \mathcal{R}_X(\iota^{-1}(B)) := \iota^*(S_B^{-1}\mathcal{O}_Y(V)) \). By this definition, \((X, \mathcal{R}_X)\) can be also regarded as a locally ringed space with respect to the Zariski topology and the Euclidean topology. We usually omit to write \( \iota \). For example, we write \( X = Y(\mathbb{R}) \).

Note that if \((X, \mathcal{R}_X)\) is a (possibly reducible) separated real algebraic variety in the sense of [6], there exists a reduced scheme \((Y, \mathcal{O}_Y)\) which satisfies the above conditions. Contrary, if \((X, \mathcal{R}_X)\) is a reduced real algebraic quasi-variety as Definition 7.1, then \((X, \mathcal{R}_X)\) is a real algebraic variety in the sense of [6]. Definition 7.1 may not be so clear, the author wishes someone will give more nice definition.

### 7.2. Semialgebraic quasi-variety.

**Definition 7.2.** (Semialgebraic quasi-variety) A locally ringed space \((A, \mathcal{R}_A)\) is called semialgebraic quasi-variety, if there exists a real algebraic quasi-variety \((X, \mathcal{R}_X)\) and a finite affine open covering \(\{V_i\}_{i=1}^r\) of \(X\) which satisfies the following:

1. There exists an injective morphism \(\iota: (A, \mathcal{R}_A) \rightarrow (X, \mathcal{R}_X)\) as locally ringed spaces, and \(\iota\) induces a homeomorphism \(A \rightarrow \iota(A)\) as Euclidean spaces. Moreover, \(\iota(A)\) is a semialgebraic subset of \(X\), i.e. \(\iota(A) \cap V_i\) is a semialgebraic subset of \(V_i\) for each \(i = 1, \ldots, r\).

2. \(\text{Zar}_X(A) = X\).

3. Take an arbitral \(i \in \{1, 2, \ldots, r\}\), and take any Euclidean open subset \(U \subset \iota^{-1}(V_i)\). Put \(R_i := \mathcal{R}_V(V_i)\). For a point \(P \in \iota(U)\), let \(m_P\) be the maximal ideal of \(R_i\) corresponding to \(P\), and let

\[
S_U := \bigcap_{P \in U} (R_i - m_P) \subset R_i.
\]

Then \(\iota^* : S_U^{-1}R_i \rightarrow \mathcal{R}_A(U)\) is an isomorphism of \(\mathbb{R}\)-algebra.

Moreover, if \(X\) is a real algebraic variety, then \(A\) is said to be an semialgebraic variety. In this case, the field of fractions \(Q(\mathcal{R}_A(U_i))\) is called the field of rational functions, and is denoted by \(\text{Rat}(A) := Q(\mathcal{R}_A(U_i))\).

The Zariski topology and the Euclidean topology on \(A\) are defined naturally. A semialgebraic quasi-variety \(A\) is called irreducible if it is irreducible with respect to the Zariski topology. \(A\) is said to be reduced if \(\mathcal{R}_{A,P}\) has no nilpotent elements except 0 for every \(P \in A\). \(\dim A\) is defined by \(\dim A = \min_{P \in A} \dim \text{Krull dim } \mathcal{R}_{A,P}\). \(A\) is called connected if it is connected with respect to Euclidean topology. Note that \(A\) may not be connected even if \(A\) is irreducible. \(A\) is called affine, if we can choose \(X\) to be isomorphic to a closed Zariski subset of \(\mathbb{R}^n\) for a certain \(n\).

Notions about singularities of \(A\) are defined using \(\mathcal{R}_{A,P}\). Note that if \(Y\) is a \(\mathbb{R}\)-scheme which represents \(X\), then \(\mathcal{R}_{A,P} \cong \mathcal{O}_{Y,P}\). We denote

\[
\begin{align*}
\text{Sing}(A) &:= \{P \in A \mid \mathcal{R}_{A,P} \text{ is not a regular local ring}\}, \\
\text{Reg}(A) &:= \text{Int}(A) - \text{Sing}(A).
\end{align*}
\]

A regular map or holomorphic map (resp. isomorphism) between semialgebraic quasi-varieties is defined as a morphism (resp. isomorphism) of locally ringed space.

For a subset \(B \subset A\), the minimum Zariski closed subset of \(A\) which includes \(B\) is called the Zariski closure of \(B\) in \(A\) and is denoted by \(\text{Zar}_A(B)\) or \(\text{Zar}(B)\).
We can choose a real algebraic quasi-variety $X$ and a separated scheme $Y$ of finite type over $\mathbb{R}$ so that $Y_\mathbb{C}$ is complete and $Y$ represents $X$. Then, we say $X$ is a real envelope of $A$, and $Y_\mathbb{R}$ is a complex envelope of $A$.

Let $X$ be a real envelope of $A$. The interior of $B$ with respect to the Euclidean topology of $X$ is denoted as $\text{Int}(B)$. $\partial B := B - \text{Int}(B)$ is called the absolute boundary of $B$. Note that $\text{Int}(B)$ and $\partial B$ do not depend on the choice of $X$. $A$ is called open if $\partial A = \emptyset$. $A$ is called closed if $A$ is compact with respect to the Euclidean topology.

$X$ and $Y_\mathbb{C}$ are not unique for $A$, but it is easy to see that:

**Proposition 7.3.** Let $A$ be a semialgebraic quasi-variety, $Y_\mathbb{C}$ and $Y'_\mathbb{C}$ be complex envelopes of $A$. Then $Y_\mathbb{C}$ and $Y'_\mathbb{C}$ are birational equivalent. If $A$ is a semialgebraic variety, then $\text{Rat}(A) \otimes_{\mathbb{R}} \mathbb{C} = \text{Rat}(Y_\mathbb{C})$.

This follows from Proposition 7.10 given later.

By this proposition, if $\nu(Y_\mathbb{C})$ is a certain birational invariant of complex algebraic varieties, then we can define $\nu(X) := \nu(Y_\mathbb{C})$ be an invariant of $X$. Especially, when $A$ is non-singular semialgebraic variety, we can choose $Y$ to be non-singular projective, and we can define $h^i(A) := \dim_{\mathbb{C}} H^i(Y_\mathbb{C}, O_{Y_\mathbb{C}})$ and $P_m(A) := \dim_{\mathbb{C}} H^0(Y_\mathbb{C}, O_{Y_\mathbb{C}}(mKY_\mathbb{C}))$ for $m \geq 0$. Using $P_m(A)$, we can define the Kodaira dimension $\kappa(A)$,

**Remark 7.4.** (1) $\text{Reg}(A) \neq \emptyset$ if $A$ is reduced.

(2) $\text{Reg}(A)$ is not always dense in $A$ with respect to the Euclidean topology. For example, consider the case that $A$ has an isolated singularity as a connected component.

(3) If $P \in \text{Reg}(A) \cap \text{Int}(A)$ and $\dim A = n$, then there exists an Euclidean open neighbourhood $P \in U \subset A$ such that $U$ is homeomorphic to an open subset of $\mathbb{R}^n$.

(4) By our definition, an isolated singular locus of $A$ is included in $\text{Int}(A)$. But $\text{Sing}(A)$ sometimes acts as if it is a boundary. So it will be safe to discuss $\text{Int}(A) \cap \text{Reg}(A)$.

In complex algebraic geometry, a subscheme is a closed subscheme of an open subscheme. But to define semialgebraic subvarieties, we must be careful. For example, any semialgebraic subset $B$ of a real algebraic variety $A$, must be able to be treated as semialgebraic quasi-subvariety of $A$.

**Definition 7.5.** (Image of a regular map) Let $A$, $B$ be semialgebraic quasi-varieties, and $\varphi: A \to B$ be a regular map. Let $C := \varphi(B)$. By Tarski-Seidenberg theorem, $C$ is a semialgebraic subset of $B$. We define $\mathcal{R}_C$ as the following:

We may assume $A$ and $B$ are affine, since definition of $\mathcal{R}_C$ is local. Let $R_A := \mathcal{R}_A(A)$, $R_B := \mathcal{R}_B(B)$, and $\varphi^*: R_B \to R_A$ be the homomorphism induced by $\varphi$. We put $R := R_B/\text{Ker} \varphi^*$. Note that $R$ defines $\text{Zar}_B(C)$. For a point $P \in C$, there exists the unique maximal ideal $m_P \subset R$ corresponding to $P$. Put $S := \bigcap_{P \in C} (R - m_P)$, and $R_C := S^{-1}R$. Note that $R_C$ is a $R_B$-module. The structure sheaf of $C$ is defined by $\mathcal{R}_C := \widetilde{R_C}$ which is the coherent $\mathcal{R}_B$-module defined by $R_C$.

$(C, \mathcal{R}_C)$ is called the image of $\varphi$, and simply denoted by $C = \varphi(A)$.

**Definition 7.6.** (Semialgebraic quasi-subvariety) Let $A$, $B$ be semialgebraic quasi-varieties. A morphism $\varphi : (B, \mathcal{R}_B) \to (A, \mathcal{R}_A)$ is called an immersion, if $\varphi$ induces an isomorphism $B \to \varphi(B)$.
If $B$ is a semialgebraic subset of $A$, and the inclusion map $B \to A$ is an immersion, then $B$ is called a semialgebraic \textit{quasi-subvariety} of $A$.

If $A$ is a semialgebraic quasi-variety, and $B \subset A$ be a semialgebraic subset. Then, there exists a unique sheaf of rings $\mathcal{R}_B$ such that $(B, \mathcal{R}_B)$ is a semialgebraic quasi-subvariety of $(A, \mathcal{R}_A)$ and $(B, \mathcal{R}_B)$ is reduced. $\mathcal{R}_B$ is called the \textit{reduced structure} of $B \subset A$.

Assume that $A$, $B$, $C$ are non-singular semialgebraic varieties such that $A = B \cup C$, and $P \in B \cap C$. It may happen that $\mathcal{R}_{B,P} \neq \mathcal{R}_{C,P}$. It is easy to see that $\mathcal{R}_{A,P}$ agree with one of $\mathcal{R}_{B,P}$ and $\mathcal{R}_{C,P}$.

\textbf{Definition 7.7} (Fibre product) Let $A$, $B$, $C$ be semialgebraic quasi-varieties, and $f: A \to C$, $g: B \to C$ be regular maps. The \text{fiber product} $A \times_C B$ is a semialgebraic set

\[ A \times_C B = \{(a,b) \in A \times B \mid f(a) = g(b)\} \]

with a structure sheaf $\mathcal{R}_A \otimes_{\mathcal{R}_C} \mathcal{R}_B$.

\textbf{Definition 7.8} (Inverse image) Let $A$, $B$ be semialgebraic quasi-varieties, and $\varphi: A \to B$ be a regular map. Let $C \subset B$ be a semialgebraic quasi-subvariety. The \text{inverse image} $\varphi^{-1}(C)$ is defined as the fiber product $\varphi^{-1}(C) := A \times_B C$.

\textbf{Definition 7.9} (Birational map) Let $A$, $B$ be semialgebraic quasi-varieties. If there exists Zariski open subsets $U \subset A$ and $W \subset B$ such that $\text{Zar}_A(U) = A$, $\text{Zar}_B(W) = B$ and there exists a regular map $\varphi: U \to W$, then we say that there exists a \textit{rational map} $\varphi: A \cdots \to B$. Moreover, if $\varphi: U \to W$ is an isomorphism, we say that $\varphi: A \cdots \to B$ is a \textit{birational map}, and $A$ and $B$ are \textit{birational equivalent}.

\textbf{Proposition 7.10}. Let $A$, $B$ be semialgebraic quasi-varieties, and let $X$, $Y$ be complex envelopes of $A$, $B$.

(1) If there exists a rational map $\varphi: A \cdots \to B$, then there exists a rational map $\Phi: X_\mathbb{C} \cdots \to Y_\mathbb{C}$ such that $\Phi|_A = \varphi$.

(2) In (1), if $\varphi$ is a birational map, then $\Phi$ is a birational map.

\textit{Proof.} (1) We may assume $\varphi$ is a regular map. Take a point $P \in \text{Int}(A)$ such that $Q := \varphi(P) \in \text{Int}(B)$, and take an affine open subset $W \subset Y$ such that $Q \subset W$.

We can choose $f_1, \ldots, f_r \in \mathcal{R}_{Y,Q}$ such that we can regard $f_i \in \mathcal{O}_Y(W)$ and $\mathcal{O}_Y(W) = \mathbb{C}[f_1, \ldots, f_r]$. Put $g_j := \varphi^*(f_j) \in \mathcal{R}_{A,P}$. We can find an affine open subset $U \subset X_\mathbb{C}$ such that $g_1, \ldots, g_r$ are holomorphic (regular) on $U$, and that $U \cap X$ is dense in $X$ and $U \cap A$ is dense in $A$. Then, $\psi^*: \mathcal{R}_B \to \mathcal{R}_A$ induces $\Psi^*: \mathcal{O}_Y(W) \to \mathcal{O}_X(U)$. $\Psi^*$ induces a rational map $\Phi: X_\cdots \to Y_\cdots$.

(2) is easy.

\section*{7.3. Some properties of semialgebraic quasi-varieties.}

A notion of semialgebraic quasi-varieties brings some merits to Real Algebraic Geometry.

\textbf{Theorem 7.11}. Every semialgebraic quasi-variety is affine. In other words, if $A$ is a semialgebraic quasi-variety, then there exists $n \in \mathbb{N}$ and an immersion $\iota: A \to \mathbb{R}^n$.  

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Proof. Let $A$ be a semialgebraic quasi-variety. We can take a real envelope $X$ of $A$. Take an affine open covering $\{V_i, \ldots, V_r\}$ of $X$. Fix a $1 \leq j \leq r$. We may assume $V_j$ is a closed subset of $\mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ be the coordinate system of $\mathbb{R}^n$, and $s_i := 1/(x_i^2 + 1)$, $t_i := x_i/(x_i^2 + 1)$. For $P \in X - V_j$, we put $s_i(P) = 0$ and $t_i(P) = 0$. Then $s_i$ and $t_i$ are regular functions on $X$. The set of functions $F_j := \{s_i, t_i \mid 1 \leq i \leq n\}$ defines a map $\Phi_j: X \to \mathbb{R}^2n$. This $\Phi_j$ is a regular map as semialgebraic quasi-varieties, and $\Phi_j|_{V_j}: V_j \to \mathbb{R}^2n$ is an immersion. Note that $\Phi_j(X)$ is a semialgebraic quasi-variety but is not always algebraic quasi-variety. Put $F := F_1 \cup \cdots \cup F_r$ and $N := \# F$. $F$ defines a regular map $\Phi: X \to \mathbb{R}^N$, and $F$ is an immersion as semialgebraic quasi-varieties. \hfill \qed

Remark 7.12. A real algebraic variety is an affine semialgebraic variety, but is not always a real affine variety. For example, $\mathbb{R}^2 - \{(0,0)\}$ is not a real affine variety.

Corollary 7.13. Let $A$ be a semialgebraic quasi-variety (or a real algebraic quasi-variety) and put $R_A := \mathcal{R}_A(A)$. Then, $\mathcal{R}_A$ is the sheaf obtained as $\tilde{R}_A$.

Note that $R_A$ is a Noetherian ring, but is not finitely generated over $\mathbb{R}$ if $\dim A \geq 1$. Each maximal ideal of $R_A$ corresponds to a certain point of $A$.

Corollary 7.14. Let $A$ be a semialgebraic quasi-variety (or a real algebraic quasi-variety) and $\mathcal{F}$ be a quasi-coherent $\mathcal{R}_A$-module. Then, $H^i(A, \mathcal{F}) = 0$ for all $i > 0$.

Proof. There exists an immersion $\iota: A \to \mathbb{R}^n$. As Definition 7.5, there exists a closed real algebraic quasi-subvariety $X \subset \mathbb{R}^n$ such that $X$ is real envelope of $A$. Let $R_X := \mathcal{R}_X(X)$ and $R_A := \mathcal{R}_A(A)$. We can present as $R_A = S_A^{-1} R_X$ by a certain multiplicatively closed set $S_A$. Since $R_A$ is an $R_X$-module, $\mathcal{F}$ is a quasi-coherent $\mathcal{R}_X$-module. Thus, $\mathcal{F}$ is a quasi-coherent $\mathcal{R}_{\mathbb{R}^m}$-module. We have

$$H^i(A, \mathcal{F}) \cong H^i(\mathbb{R}^m, \mathcal{F}) = 0$$

(see [17] Chap.III, Theorem 3.5). \hfill \qed

The author should apologize that the Definition 1.7 of [1] is incorrect. It should be replaced by:

Definition 7.15. (Signed linear system) Let $A$ be a semialgebraic quasi-variety, and $\mathcal{R}_A^{an}$ be the sheaf the germs of real analytic functions on $A$. For a subset $B \subset A$ such that $\text{Cls}_A(\text{Int}(B)) = \text{Cls}_A(B)$, $\mathcal{R}_A^{an}(B)$ is defined as the set of all real continuous functions $f$ on $B$ such that $f$ is real analytic on $\text{Int}(B)$. (For example, if $A = \mathbb{R}_+$, then $\sqrt{x} \in \mathcal{R}_A^{an}(\mathbb{R}_+)$.) Assume that there exists an invertible $\mathcal{R}_A$-sheaf $\mathcal{J}$ and an invertible $\mathcal{R}_A^{an}$-sheaf $\mathcal{J}$ such that $\mathcal{J}\otimes \mathcal{R}_A, \mathcal{J}^{an} = \mathcal{J}\otimes \mathcal{R}_A^{an} \mathcal{J}$. For any point $a \in A$, we assume that we can take an affine open subset $a \in U \subset A$ such that $\mathcal{J}|_U = \mathcal{R}_A|_U \cdot e_U^2$ by a certain $e_U \in H^0(U, \mathcal{J})$. Then, for $f \in H^0(A, \mathcal{K})$, there exists $g_U \in H^0(U, \mathcal{R}_A)$ such that $f|_U = g_U e_U^2$. We define $\text{sign}(f(a)) \in \{0, \pm 1\}$ by $\text{sign}(f(a)) = \text{sign}(g_U(a))$. $\mathcal{J}$ is called a signed invertible sheaf on $A$. A finite dimensional subspace $\mathcal{K} \subset H^0(A, \mathcal{J})$ is called a signed linear system on $A$.

Let $f_1, \ldots, f_n \in \mathcal{K}$ be a base of $\mathcal{K}$, and $a \in A$. Since $\mathcal{L}$ is invertible, there exists an open set $U \subset A$ and a function $e \in \mathcal{L}(U)$ such that $\mathcal{L}(U) = \mathcal{R}_A(U) \cdot e$. Thus $f_i|_U = g_i e$ by a certain $g_i \in \mathcal{R}_A(U)$. $\text{sign}(f_i(a)) = \text{sign}(g_i(a)) \text{sign}(e(a))$ by $(*)$. 79
Note that if $\mathcal{L}(U) = \mathcal{R}_A(U) \cdot e'$ by another $e' \in \mathcal{L}(U)$, then there exists $h \in (\mathcal{R}_A)^\times$ such that $e = he'$. Assume that $a \notin \text{Bs}\mathcal{H}$ and $n \geq 2$. Since $g_i(a) \in \mathbb{R}$, we can define a rational map $\Phi_{\mathcal{H}} : A \to \mathbb{P}_{\mathbb{R}}^{n-1}$ by $\Phi_{\mathcal{H}}(a) := (g_1(a) : \cdots : g_n(a)) \in \mathbb{P}_{\mathbb{R}}^{n-1}$ even if $\mathcal{R}_A(U)$ is not an integral domain. Since $e = he'$, $\Phi_{\mathcal{H}}$ does not depend on the choice of $e$.

By the way, birational geometries of complex and real algebraic varieties are very different. In a complete complex algebraic varieties, exceptional subsets are special subsets. This is not true for complete real algebraic varieties.

**Theorem 7.16.** Let $A$ be a semialgebraic quasi-variety, $E \subseteq A$ be a closed semialgebraic subset such that $E = \text{Zar}_A(E) \subsetneq A$. Then there exists a semialgebraic quasi-variety $B$ and a regular surjective morphism $\varphi : A \to B$ such that $P := \varphi(E)$ is a point and that $\varphi|_{A-P} : (A-E) \to (B-P)$ is a contraction of $E$ to a point $P$.

**Proof.** We may assume $A \subseteq \mathbb{R}^n$. Let $f_1, \ldots, f_r$ be defining polynomials of $\text{Zar}_{\mathbb{R}^n}(E)$ in $\mathbb{R}[x_1, \ldots, x_n]$. Consider a map $\Phi : \mathbb{R}^n \to \mathbb{R}^m$ defined by linear system with the base $\{x_i f_j \mid 1 \leq i \leq n, 1 \leq j \leq r\}$. $\Phi$ is a regular map. Put $B := \Phi(A)$ and $\varphi := \Phi|_A : A \to B$. Then, $B$ and $\varphi$ satisfy the conclusion of the Proposition.

### 7.4. Non-singular semialgebraic surface.

For an example, when $A = \mathbb{P}_{\mathbb{R}}^2$ and $E$ is a line in $A$, there exists a contraction $\varphi : A \to B$ such that $B$ is a non-singular complete real algebraic variety. An intersection number $(C_1 \cdot C_2)_A$ of closed curves $C_1, C_2 \subseteq A = \mathbb{P}_{\mathbb{R}}^2$ has sense only in $(C_1 \cdot C_2)_A \in \mathbb{Z}/2\mathbb{Z}$. In fact, for any $n \in \mathbb{Z}$, there exists a non-singular complex envelope $X$ of $A$ such that $(\text{Zar}_X(E))^2 \subseteq X = 2n+1$. For some special $A$, we can define divisors and intersection numbers.

**Proposition 7.17.** Let $A$ be a non-singular semialgebraic surface whose Kodaira dimension satisfies $\kappa(A) \geq 0$. Then there exists a minimal non-singular complex envelope $X$ of $A$. That is, if $Y$ a non-singular complex envelope of $A$, then there exists the unique regular map $\varphi : Y \to X$ such that $\varphi|_A = \text{id}|_A$.

**Proof.** Let $X_0$ be a non-singular complex envelope of $A$. Assume that $X_0$ includes a $(-1)$-curve $C$ such that $\dim(C \cap A) \leq 0$. There exists a complex conjugate map $J : X_0 \to X_0$ such that $J|_A = \text{id}|_A$. If $C = J(C)$, then let $\pi_0 : X_0 \to X_1$ be the smooth contraction of $C$. If $C \neq J(C)$, then let $\pi_0 : X_0 \to X_1$ be the smooth contraction of $C$ and $J(C)$. Inductively, we repeat this process. After some contractions, we obtain a complex envelope $X$ of $A$ such that $X$ includes no $(-1)$-curve $C$ such that $\dim(C \cap A) \leq 0$.

We shall show that $X$ is unique. Let $\varphi : X \to Z$ be the minimal model of $X$. It is well known that $Z$ is unique. $\varphi$ can be decomposed as a composite of contractions of a $(-1)$-curve: $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} X_n = Z$. Consider $f_1 : X_0 \to X_1$ which is the contraction of $C_0$. Note that $\dim(C_0 \cap A) = 1$. Let $C_1 \subseteq X_1$ be a $(-1)$-curve, and let $C'_1$ be the strict transform of $C_1$ to $X_0$. If $C_1 \cap A = \emptyset$, then $C'_1$ is a $(-1)$-curve such that $\dim(C'_1 \cap A) \leq 0$. This is impossible for $X$. Thus $C_1 \cap A \neq \emptyset$. Then $C'_1 = J(C'_1)$ by Lemma 7.18.

So, every $f_i : X_{i-1} \to X_i$ is the contraction of a $(-1)$-curve $C_{i-1} \subset X_{i-1}$ such that $C_{i-1} = J(C_{i-1})$ and $\dim(C_{i-1} \cap A) = 1$.

This implies that $X$ is unique.
Lemma 7.18. Let $A$ be a non-singular semialgebraic surface, and $X$ be a non-singular complex envelope of $A$. Assume that $E \subset X$ is a $(-1)$-curve. Let $E := J(E)$. Moreover, we assume that $(E \cdot E)_X \geq 1$. Then, $\kappa(X) = -\infty$.

Proof. Let $\pi: X \to Y$ be the contraction of $E$, $m := (E \cdot E)_X$, $C_X := E$, and $C_Y := \pi(C_X) \subset Y$.

(1) We consider the case $m = 1$. Then $C_Y$ is a smooth rational curve with $(C_Y^2)_Y = 0$. Then, $Y$ is a ruled surface. Thus $\kappa(X) = \kappa(Y) = -\infty$. (See [3] Cap. V, Prop. 4.3.)

(2) We consider the case $m \geq 2$. Then $C_Y$ is a singular rational curve with $(C_Y^2)_Y = m - 1 \geq 1$. Note that $(C_Y \cdot K_Y)_Y = (C_X \cdot \pi^* K_Y)_X = (C_X \cdot (K_X - E))_X = -1 - m$. Take non-singular points $P_1, \ldots, P_{m-1} \in L$. Let $\varphi: Z \to Y$ be the blowing up at $P_1, \ldots, P_{m-1}$, and let $C_Z \subset Z$ be the proper transform of $C_Y$. Then $(C_Z^2)_Z = 0$. Put $E_i = \varphi^{-1}(P_i)$. Then

$$(C_Z \cdot K_Z)_Z = (C_Z \cdot \pi^* K_Y)_Z + \sum_{i=1}^{m-1} (C_Z \cdot E_i)_Z = (C_Y \cdot K_Y)_Y + (m - 1) = (-1 - m) + (m - 1) = -2.$$  

(2-1) Consider the case $H^1(Z, \mathcal{O}_Z) = 0$. Then, since $0 \to H^0(\mathcal{O}_Z) \to H^0(\mathcal{O}_Z(C_Z)) \to H^0(\mathcal{O}_CZ(C_Z)) \to 0$ is exact, we have $h^0(\mathcal{O}_Z(C_Z)) = 2$. The divisor $C_Z$ define a regular map $\Phi: Z \to \mathbb{P}^1_C$.

Since $Z$ is non-singular, general fibres of $\Phi$ are smooth curves. Let $F := \Phi^{-1}(Q)$ be a smooth fibre. If $F \cong \mathbb{P}^1$, then $Z$ is rational surface and $\kappa(X) = \kappa(Z) = -\infty$. Assume that $g(F) \geq 1$. Since $F \sim C_Z$, we have $-2 = (C_Z \cdot K_Z)_Z = (F \cdot K_Z)_Z = (F \cdot (K_Z + F))_Z = \deg K_F \geq 0$. A contradiction.

(2-2) Consider the case $q := h^1(Z, \mathcal{O}_Z) \geq 1$. Let $S$ be the Albanese variety of $Z$ and $\alpha: Z \to S$ be the Albanese map. Take the Stein factorization $\alpha: Z \xrightarrow{f} T \xrightarrow{g} S$. Note that $q(T) = q(S) = q$. Since any complex torus does not include a rational curve, and since $g: T \to S$ is a finite map, we have $f(C_Z)$ is a point. If $T$ is a surface, then $(C_Z^2)_Z < 0$. Thus, $\dim T = 1$. Since general fibres of $f: Z \to T$ are smooth curves, $C_Z$ is included in a singular fibre $f^{-1}(P)$. Put $F := f^*P$ as a divisor. Since $(C_Z \cdot F)_Z = 0$ and $(C_Z^2)_Z = 0$, $F$ must be irreducible. Thus $F = rC_Z$ for a certain $r \in \mathbb{N}$. Take a general fibre $F_1 := f^{-1}(Q)$. Then

$$-2r = r(C_Z \cdot K_Z)_Z = (F \cdot K_Z)_Z = (F_1 \cdot K_Z)_Z = \deg K_{F_1} \geq -2.$$  

Thus, $r = 1$ and $F_1 \cong \mathbb{P}^1_C$. Since $f: Z \to T$ is a ruled surface, we have $\kappa(X) = \kappa(Z) = -\infty$. \hfill $\Box$

Assume that $A$, $X$ and $Y$ are as Proposition 7.17. Take closed semialgebraic curves $C_1$, $C_2 \subset A$. Put $(C_1 \cdot C_2)_Y := (\text{Zar}_Y(C_1) \cdot \text{Zar}_Y(C_2))_Y$. Then $(C_1 \cdot C_2)_Y \leq (C_1 \cdot C_2)_X$. Thus, we can define the intersection number of $C_1$ and $C_2$ on $A$ by $(C_1 \cdot C_2)_A := (C_1 \cdot C_2)_X$. By natural way, we can define Weil divisors on $A$, and their intersection numbers.

List of symbols

\[\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}\]
\[\mathbb{P}^n_\mathbb{R} := \{(x_0 : \cdots : x_n) \in \mathbb{P}^n_\mathbb{R} \mid x_0 \geq 0, \ldots, x_n \geq 0\}\]
\[V_K(f) := \{(x_1 : \cdots : x_n) \in \mathbb{P}^{n-1}_\mathbb{R} \mid f(x_1, \ldots, x_n) = 0\}\]

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\( V_+(f) := V_2(f) \cap \mathbb{P}^{n-1}_+ \)
\[ \text{Zar}_V(B) \] Zariski closure of \( B \) in \( V \).
\[ \text{Cls}_V(B), \overline{B} \] Euclidian closure of \( B \) in \( V \).
\[ \text{Int}(B) = V - \text{Cls}_V(V - B) \] Interior of \( B \).
\[ \partial B = B - \text{Int}(B) \] Boundary of \( B \).
\[ \Delta(X) \] Critical decomposition of \( X \) (see Definition 1.2).
\[ \mathcal{F}(D) \] Face Component (see Definition 1.5).
\[ \text{disc}_D, \text{disc}(D) \] Discriminant (see Definition 1.5).
\[ \text{Disc}_{n}(c_n, \ldots, c_0) \] The discriminant of \( c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 \).
\[ \mathcal{H}_{n,d} := (\mathbb{R}[x_1, \ldots, x_n])_d \] (The vector space of all the homogeneous polynomials of degree \( d \)).
\[ \mathcal{H}_{n,d}^c := \mathcal{H}_{n,d}^s = \{ f \in \mathcal{H}_{n,d} \mid \text{is cyclic.} \} \]
\[ \mathcal{H}_{n,d}^s := \mathcal{H}_{n,d}^c = \{ f \in \mathcal{H}_{n,d} \mid \text{is symmetric.} \} \]
\[ \mathcal{H}_{n,d}^0 := \{ f \in \mathcal{H}_{n,d} \mid f(1,1,\ldots,1) = 0 \} \]
\[ \mathcal{H}_{n,2d} := \mathcal{H}_{n,2d} \cap \mathbb{R}[x_1^2, x_2^2, \ldots, x_n^2] \]
\[ \mathcal{H}_{n,d}^{0,0} := \mathcal{H}_{n,d}^{0} \cap \mathcal{H}_{n,d}^{0} \]
\[ \mathcal{H}_{n,d} := \mathcal{H}_{n,d} \cap \mathcal{H}_{n,d} \]
\[ \mathcal{H} := \{ f \in \mathcal{H} \mid \text{is monic} \} \]
\[ \mathcal{P}(A, \mathcal{H}) := \{ f \in \mathcal{H} \mid f(a) \geq 0 \text{ for all } a \in A \} \] PSD cone
\[ \mathcal{E}(\mathcal{P}) := \{ f \in \mathcal{P} \mid \text{f is extremal} \} \]
\[ \mathcal{P}_{n,d} := \{ f \in \mathcal{P} \mid f \in \mathcal{H}_{n,d} \} \]
\[ \mathcal{P}_{n,2d} := \mathcal{P}(\mathbb{P}^{n-1}_+, \mathcal{H}_{n,2d}) \]
\[ \mathcal{P}_{n,2d}^c := \mathcal{P}(\mathbb{P}^{n-1}_+, \mathcal{H}_{n,2d}^c) \cap \mathcal{H}_{n,2d}^c \]
\[ \mathcal{P}_{n,2d}^s := \mathcal{P}(\mathbb{P}^{n-1}_+, \mathcal{H}_{n,2d}^s) \cap \mathcal{H}_{n,2d}^s \]
\[ \Sigma_{n,2d} := \{ f_1^2 + \cdots + f_k^2 \in \mathcal{P}_{n,2d} \mid k \in \mathbb{N}, f_1, \ldots, f_k \in \mathcal{H}_{n,d} \} \]
\[ \Sigma_{n,2d}^s := \Sigma_{n,2d} \cap \mathcal{P}_{n,2d}^s \]
\[ \Phi_{\mathcal{H}} : A \cdots \to \mathcal{P}(\mathcal{H}^c) \] Rational map defined by \( \mathcal{H} \).
\[ \Phi_{n,d} := \Phi_{\mathcal{H}_{n,d}}, \Phi_{n,d}^c := \Phi_{\mathcal{H}_{n,d}^c}, \Phi_{n,d}^s := \Phi_{\mathcal{H}_{n,d}^s}, \text{and so on.} \]
\( X(A, \mathcal{H}) := \text{Cls}_{\mathbb{P}(\mathcal{H}^\vee)}(\Phi_{\mathcal{H}}(A)) \) \hspace{1cm} \text{Characteristic variety}

\[ X_{n,d} := X(\mathbb{P}_d^{n-1}, \mathcal{H}_{n,d}), \quad X_{n,d}^s := X(\mathbb{P}_d^{n-1}, \mathcal{H}_{n,d}^s), \quad X_{n,d}^{+} := X(\mathbb{P}_d^{n-1}, \mathcal{H}_{n,d}^+), \quad X_{n,d}^{e0} := X(\mathbb{P}_d^{n-1}, \mathcal{H}_{n,d}^{e0}), \] and so on.

References

[18] W. R. Harris, Real Even Symmetric Ternary Forms, J. Alg., 222 (1999), 204-245.